From short to long scales in the QCD vacuum

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We study approximate decimations in SU(N) LGT that connect the short to long distance regimes, and provide both upper and lower bounds on the exact partition function. This leads to a representation of the exact partition function in terms of successive decimations. The implications for a derivation of confinement from first principles are discussed.

A very large body of numerical and analytical work has been done by the lattice community in recent years on the types of configurations responsible for confinement. A great deal of information concerning the confinement mechanism has been obtained from these investigations (for recent review, see [1]). However, the goal of a direct derivation of confinement from first principles has remained elusive for the last thirty years.

The origin of the difficulty is clear. One is faced with a multi-scale problem: passage from the short-distance weakly coupled, ordered regime to the long distance strongly coupled, disordered, confining regime. Such variable multi-scale behavior can only be addressed by some nonperturbative block-spinning or decimation procedure. Exact decimation schemes appear analytically hopeless, and numerically very difficult. Here we will consider simple ‘bond moving’ decimations and show that they can provide bounds on the exact theory, allowing statements about its behavior and the question of an actual derivation of confinement in LGT.

Starting with some plaquette action, e.g the Wilson action $A_p(U) = \frac{\beta}{N} \text{Re} \text{tr} U_p$, at lattice spacing $a$, we consider the character expansion:

$$ F(U, a) = e^{A_p(U)} = \sum_j F_j(\beta, a) d_j \chi_j(U) \tag{1} $$

with Fourier coefficients:

$$ F_j = \int dU \ F(U, a) \frac{1}{d_j} \chi_j(U) . $$

E.g, for SU(2), $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, $d_j = (2j + 1)$.

In terms of normalized coefficients:

$$ F(U, a) = F_0 \left[ 1 + \sum_{j \neq 0} c_j(\beta) d_j \chi_j(U) \right] $$

$$ \equiv F_0 f(U, a) \tag{2} $$

For a reflection positive action one has:

$$ F_j \geq 0 \quad \text{hence} \quad 1 \geq c_j \geq 0 \quad \text{all} \quad j . \tag{3} $$

The partition function (PF) on lattice $\Lambda$ is then

$$ Z_\Lambda(\beta) = F_0^{[\Lambda]} \int dU_\Lambda \prod_p f_p(U, a) $$

We now consider RG decimation transformations $a \to \lambda a$. Simple approximate transformations of the ‘bond moving’ type are implemented by ‘weakening’, i.e. decreasing the $c_j$’s of interior plaquettes, and ‘strengthening’, i.e. increasing $c_j$’s of boundary plaquettes in every decimation cell of side length $\lambda$. The simplest scheme [2], which is adopted in the following, implements complete removal, $c_j = 0$, of interior plaquettes.

Under successive decimations

$$ a \to \lambda a \to \lambda^2 a \to \cdots \to \lambda^n a $$

$$ \Lambda \to \Lambda^{(1)} \to \Lambda^{(2)} \to \cdots \to \Lambda^{(n)} $$

the RG transformation rule is then:

$$ f(U, n-1) \to f(U, n) = \left[ 1 + \sum_{j \neq 0} c_j(n) d_j \chi_j(U) \right] \tag{4} $$

with:

$$ c_j(n) = F_j(n)/F_0(n) , \quad F_j(n) = \left[ \hat{F}_j(n) \right]^{\lambda^2} \tag{5} $$

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\( F_j(n) = \int dU \left[ f(U, n - 1) \right]^{\nu} \frac{1}{d_j} \chi_j(U) \). \hspace{1cm} (6)

The parameter \( \nu \) controls by how much the remaining plaquettes have been strengthened to compensate for the removed plaquettes. The resulting PF after \( n \) decimation steps is:

\[
Z_\Lambda(\beta, n) = \prod_{m=0}^{n} F_0(m)^{|\Lambda|/\lambda^{md}} \int dU_{\Lambda(n)} \prod_p f_p(U, n)
\]

It is important to note that after each decimation step the resulting action retains the original one-plaquette form but will, in general, contain all representations:

\[
A_p(U, n) = \sum_j \tilde{\beta}_j(\beta) \chi_j(U) .
\]

Also, both positive and negative effective couplings \( \tilde{\beta}_j \) will occur. This is the case even after a single decimation step \( a \to \lambda a \) starting with the Wilson action.

However, for integer \( \nu \), the property \( F_0(n) \geq 0 \), \( c_j(n) \geq 0 \) and hence reflection positivity are maintained at each decimation step.

The subsequent development hinges on the following two basic statements that can now be proven:

(I) With \( \nu = \lambda^{d-2} \):

\[
Z_\Lambda(\beta, n) \leq Z_\Lambda(\beta, n + 1) .
\]

(II) With \( \nu = 1 \):

\[
Z_\Lambda(\beta, n + 1) \leq Z_\Lambda(\beta, n) .
\]

In fact, in both (I), (II) one has strict inequality.

(II) says that decimating plaquettes while leaving the couplings of the remaining plaquettes unaffected results in a lower bound on the P.F. Reflection positivity (positivity of Fourier coefficients) is crucial for this to hold.

(I) says that modifying the couplings of the remaining plaquettes after decimation by taking \( \nu = \lambda^{d-2} \) (standard MK choice [2]) results into overcompensation (upper bound on the P.F.).

Consider now the, say, \((n-1)\)-th decimation step with Fourier coefficients \( c_j(n-1) \), which we relabel \( \tilde{c}_j(n-1) = \tilde{c}_j(n-1) \). Given these \( \tilde{c}_j(n-1) \), we proceed to compute the coefficients \( F_0(n) \), \( c_j(n) \) of the next decimation step according to (4)-(6) above with \( \nu = \lambda^{d-2} \).

Then introducing a parameter \( \alpha \), \((0 \leq \alpha)\), define the interpolating coefficients:

\[
\tilde{c}_j(n, \alpha) = \tilde{c}_j(n-1) \lambda^{2(1-\alpha)} c_j(n)^\alpha . \hspace{1cm} (7)
\]

Then,

\[
\tilde{c}_j(n, \alpha) = \begin{cases} c_j(n) & : \alpha = 1 \\ \tilde{c}_j(n-1) \lambda^{2} & : \alpha = 0 \end{cases}
\]

The \( \alpha = 0 \) value is that of the \( n \)-th step coefficients resulting from (4)-(6) with \( \nu = 1 \).

Thus defining

\[
Z_\Lambda(\beta, \alpha, n) = \left( \prod_{m=0}^{n-1} F_0(m)^{|\Lambda|/\lambda^{md}} \right) F_0(n)^\alpha \cdot \int dU_{\Lambda(n)} \prod_p f_p(U, n, \alpha)
\]

where

\[
f_p(U, n, \alpha) = \left[ 1 + \sum_{j \neq 0} \tilde{c}_j(n, \alpha) \chi_j(U) \right] ,
\]

we have from (I), (II) above:

\[
Z_\Lambda(\beta, 0, n) \leq Z_\Lambda(\beta, n - 1) \leq Z_\Lambda(\beta, 1, n) .
\]

But then, by continuity, there exist a value \( 0 < \alpha = \alpha(n) < 1 \) such that

\[
Z_\Lambda(\beta, \alpha(n), n) = Z_\Lambda(\beta, n - 1) .
\]

So starting at original spacing \( a \), at every decimation step \( m \), \((m = 0, 1, \ldots, n)\), there is a value \( 0 < \alpha(m) < 1 \) such that

\[
Z_\Lambda(\beta, \alpha(m+1), m + 1) = Z_\Lambda(\beta, \alpha(m), m) . \hspace{1cm} (8)
\]

This then gives an exact representation of the original PF in the form:

\[
Z_\Lambda(\beta) = F_0^{|\Lambda|} \int dU_{\Lambda} \prod_p f_p(U, a)
\]

\[
= \prod_{m=0}^{n} F_0(m)^{\alpha(m)} |\Lambda|/\lambda^{md}
\]

\[
\cdot \int dU_{\Lambda(n)} \prod_p f_p(U, n, \alpha(n)) . \hspace{1cm} (9)
\]
i.e. in terms of the successive bulk free energy contributions from the $a \to \lambda \to \cdots \to \lambda^n a$ decimations and a single plaquette effective action on the resulting lattice $\Lambda^{(n)}$.

Consider now the coefficients at the, say, $n$-th step in this representation: $\tilde{c}_j(n, \alpha = \alpha^{(n)})$. Compare them to those evaluated at $\alpha = 1$: $\tilde{c}_j(n, \alpha = 1) \equiv c_j(n)$, which will be referred to as the MK coefficients ($\alpha = 1 \iff \nu = \lambda^{d-2}$, the standard MK choice), and which, according to (I) give an upper bound.

One can then prove that

$$\tilde{c}_j(n, \alpha) \leq c_j(n) \quad \text{for any} \quad 0 \leq \alpha \leq 1 \, . \quad (10)$$

This has the following important consequence.

Assume we are in a dimension $d$ such that under successive decimations the MK coefficients ($\alpha = 1$) are non-increasing. Then, $(10)$ implies:

$$\tilde{c}_j(n, \alpha^{(n)}) \geq c_j(n+1) \geq \tilde{c}_j(n+1, \alpha^{(n+1)}) \geq c_j(n+2) \geq \tilde{c}_j(n+2, \alpha^{(n+2)}) \geq \cdots$$

Thus, if the $c_j(n)$’s are non-increasing, so are the $\tilde{c}_j(n, \alpha)$. The $c_j(n)$’s must then approach a fixed point, and hence so must the $\tilde{c}_j(n, \alpha)$’s, since $c_j(n), \tilde{c}_j(n, \alpha) \geq 0$. Note the fact that this conclusion is independent of the specific value of the $\alpha$’s at every decimation step.

In particular, if the $c_j(n)$’s approach the strong coupling fixed point, i.e. $F_0 \to 1, c_j(n) \to 0$ as $n \to \infty$, so must the $\tilde{c}_j(n, \alpha)$’s of the exact representation. If the MK decimations confine, so do those in the exact representation (9).

As it is well-known by explicit numerical evaluation, the MK decimations for $SU(2)$ and $SU(3)$ indeed confine for all $\beta < \infty$ and $d \leq 4$.

Does this imply a proof of confinement for the exact theory?

Though strongly suggestive of confinement for all $\beta$ in the exact theory, the above does not yet constitute an actual proof. The statement concerns the long distance action part in the representation (9) which also includes the (dominant) bulk contributions from integration over all scales from $a$ to $\lambda^n a$. The complete representation provides an equality to the value of the exact free energy. This, just by itself, does not suffice to rigorously determine, at least in any direct way, the actual behavior of long distance order parameters characterizing phases in the exact theory. To do this one needs to carry through the above derivation for the partition function also for the case of the appropriate order parameters.

Now the derivation of the basic two statements (I) and (II) above assumes translation invariance and reflection positivity. In the presence of observables such as a Wilson loop, translation invariance is broken and reflection positivity is reduced to hold only in planes bisecting the loop. This does not allow the above derivations to be carried through in any obvious way. Fortunately, there are other order parameters that maintain translational invariance, and are indeed the natural candidates in the present context since they are constructed out of partition functions. These are the vortex free energy, and its $Z(N)$ Fourier transform (electric flux free energy). Recall that the vortex free energy is defined by

$$e^{-F_v} = \frac{Z_\Lambda(\tau)}{Z_\Lambda} \, .$$

Here, $Z_\Lambda(\tau)$ denotes the PF with action modified by the ‘twist’ $\tau \in Z(N)$ representing a discontinuous gauge transformation which introduces $\pi_1(SU(N)/Z(N))$ vortex flux winding through the periodic lattice in $(d-2)$ directions. The vortex free energy is then the ratio of the PF in the presence of this external flux to the PF in the absence of the flux (the latter is what was considered above). The above development, in particular the derivation of (9), should then be repeated also for $Z_\Lambda(\tau)$. There is a technical complication. The presence of the flux, though translation invariant, reduces reflection positivity to hold only in planes perpendicular to the directions in which the flux winds through the lattice. Hopefully, this will suffice to allow a generalization of the previous derivation to go through. This is currently under investigation.

REFERENCES

1. J. Greensite, hep-lat/0301023