Comment on ‘Vector potential of the Coulomb gauge’∗

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Abstract. The expression for the Coulomb-gauge vector potential in terms of the ‘instantaneous’ magnetic field derived by Stewart [2003 Eur. J. Phys. 24 519] by employing Jefimenko’s equation for the magnetic field and Jackson’s formula for the Coulomb-gauge vector potential can be obtained immediately by just using the Helmholtz theorem.

In a recent article [1], Stewart has derived the following expression for the Coulomb-gauge vector potential \( A_C \) in terms of the ‘instantaneous’ magnetic field \( B \):

\[
A_C(r, t) = \frac{\nabla \times}{4\pi} \int d^3r' \frac{B(r', t)}{|r - r'|}.
\]  

(1)

His derivation consists of substituting in (1) Jefimenko’s expression for the magnetic field in terms of the retarded current density and its partial time derivative [2], and then obtaining, after some non-trivial algebra, an expression for \( A_C \) in terms of the current density derived recently by Jackson [3].

Stewart has used the Helmholtz theorem as a starting point of his derivation, to provide a ‘suggestion’ that (1) is true. In this comment, we show that there is no need to go beyond a simple application of this theorem in order to prove formula (1).

According to the Helmholtz theorem [4], an arbitrary-gauge vector potential \( A \), as any three-dimensional vector field whose divergence and curl vanish at infinity, can be decomposed uniquely into a longitudinal part \( A_\parallel \), whose curl vanishes, and a transverse part \( A_\perp \), whose divergence vanishes:

\[
A(r, t) = A_\parallel(r, t) + A_\perp(r, t) \quad \nabla \times A_\parallel(r, t) = 0 \quad \nabla \cdot A_\perp(r, t) = 0.
\]  

(2)

The longitudinal and transverse parts in (2) are given explicitly by

\[
A_\parallel(r, t) = -\frac{\nabla}{4\pi} \int d^3r' \frac{\nabla' \cdot A(r', t)}{|r - r'|} \quad A_\perp(r, t) = \frac{\nabla \times}{4\pi} \int d^3r' \frac{\nabla' \times A(r', t)}{|r - r'|}.
\]  

(3)

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Let us now decompose the vector potential \( \mathbf{A} \) in terms of the Coulomb-gauge vector potential \( \mathbf{A}_C \) as follows:

\[
\mathbf{A}(\mathbf{r}, t) = [\mathbf{A}(\mathbf{r}, t) - \mathbf{A}_C(\mathbf{r}, t)] + \mathbf{A}_C(\mathbf{r}, t).
\]

If the curl of \([\mathbf{A} - \mathbf{A}_C]\) vanishes, then, according to equation (2) and the fact that the Coulomb-gauge vector potential is by definition divergenceless, the Coulomb-gauge vector potential \( \mathbf{A}_C \) is the transverse part \( \mathbf{A}_\perp \) of the vector potential \( \mathbf{A} \). But because the two vector potentials must yield the same magnetic field, the curl of \([\mathbf{A} - \mathbf{A}_C]\) does vanish:

\[
\nabla \times [\mathbf{A}(\mathbf{r}, t) - \mathbf{A}_C(\mathbf{r}, t)] = \nabla \times \mathbf{A}(\mathbf{r}, t) - \nabla \times \mathbf{A}_C(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t) - \mathbf{B}(\mathbf{r}, t) = 0. \tag{5}
\]

Thus the Coulomb-gauge vector potential is indeed the transverse part of the vector potential \( \mathbf{A} \) of any gauge. Therefore, it can be expressed according to the second part of (3) and the fact that \( \nabla \times \mathbf{A} = \mathbf{B} \) as

\[
\mathbf{A}_C(\mathbf{r}, t) = \mathbf{A}_\perp(\mathbf{r}, t) = \frac{1}{4\pi} \int d^3r' \frac{\nabla' \times \mathbf{A}(r', t)}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi} \int d^3r' \frac{\mathbf{B}(r', t)}{|\mathbf{r} - \mathbf{r}'|}. \tag{6}
\]

The right-hand side of (6) is the expression (1) derived by Stewart.

There is an expression for the Coulomb-gauge scalar potential \( V_C \) in terms of the ‘instantaneous’ electric field \( \mathbf{E} \) that is analogous to the expression (6) for the Coulomb-gauge vector potential:

\[
V_C(\mathbf{r}, t) = \frac{1}{4\pi} \int d^3r' \frac{\nabla' \cdot \mathbf{E}(r', t)}{|\mathbf{r} - \mathbf{r}'|}. \tag{7}
\]

This follows directly from the definition \( V_C(\mathbf{r}, t) = \int d^3r' \rho(r', t)/|\mathbf{r} - \mathbf{r}'| \) of the Coulomb-gauge scalar potential and the Maxwell equation \( \nabla \cdot \mathbf{E} = 4\pi \rho \). The expressions (6) and (7) may be regarded as a ‘totally instantaneous gauge’, but it would seem more appropriate to view them as the solution to a problem that is inverse to that of calculating the electric and magnetic fields from given Coulomb-gauge potentials \( \mathbf{A}_C \) and \( V_C \) according to

\[
\mathbf{E} = -\nabla V_C - \frac{\partial \mathbf{A}_C}{c \partial t} \quad \mathbf{B} = \nabla \times \mathbf{A}_C. \tag{8}
\]

In closing, we note that the first equation of (8) gives directly the longitudinal part \( \mathbf{E}_\parallel \) and transverse part \( \mathbf{E}_\perp \) of an electric field \( \mathbf{E} \) in terms of the Coulomb-gauge potentials \( V_C \) and \( \mathbf{A}_C \) as \( \mathbf{E}_\parallel = -\nabla V_C \) and \( \mathbf{E}_\perp = -\partial \mathbf{A}_C/c \partial t \) (the apparent paradox that the longitudinal part \( \mathbf{E}_\parallel \) of a retarded electric field \( \mathbf{E} \) is thus an instantaneous field has been discussed recently in [5]).

\[2\] Jefimenko O D 1989 Electricity and Magnetism 2nd edn (Star City, WV: Electret Scientific)
\[3\] Jackson J D 1999 Classical Electrodynamics 3rd edn (New York: Wiley)
    Jefimenko O D 2002 Comment on ‘Causality, the Coulomb field, and Newton’s law of gravitation’ Am. J. Phys. 70 964
    Rohrlich F 2002 Reply to “Comment on ‘Causality, the Coulomb field, and Newton’s law of gravitation’” Am. J. Phys. 70 964