The Post-Newtonian Maclaurin Spheroids to Arbitrary Order

David Petroff *

Sept. 3 2003

Abstract

In this paper, we develop an iterative scheme to enable the explicit calculation of an arbitrary post-Newtonian order for a relativistic body that reduces to the Maclaurin spheroid in the appropriate limit. This scheme allows for an analysis of the structure of the solution in the vicinity of bifurcation points along the Maclaurin sequence. The post-Newtonian expansion is solved explicitly to the fourth order and its accuracy and convergence are studied by comparing it to highly accurate numerical results.

1 Introduction

At the latest upon the discovery of pulsars in 1968 and their identification as neutron stars, it became apparent that a relativistic description of rotating, compact stars was needed. Early work in this direction dealt with simplified models for the matter making up these objects. In particular, Chandrasekhar [?] looked at stars of constant density and calculated the first post-Newtonian correction to the Maclaurin spheroids. Bardeen [?] reexamined this work using a modified approach and gained new insight regarding, foremost, the points of onset of secular, axisymmetric instability along a one parameter Maclaurin curve.

Given the amount of work that has been done since then to study stars with more realistic equations of state, the return to a model of constant density can hardly be motivated by astrophysical considerations. Many other good arguments however, suggest that precisely this model deserves closer attention: For one, it allows, as we shall see, for the development of an iterative scheme to calculate explicitly any order of the post-Newtonian expansion, limited in practice only by computer algebra programs and

* D.Petroff@tpi.uni-jena.de
the machines running them. Furthermore, by considering an arbitrary order, one can study properties of the full relativistic solution and carry out Bardeen’s task of testing conjectures “by going to higher orders in the relativistic expansion”. Finally, due to the fortuitous circumstance of being in possession of highly accurate numerical values, one can go even further. We are in the rare position of being able to examine the behaviour of the post-Newtonian expansion itself, providing, by analogy, a testbed for the most widely used analytic approximation within the field of General Relativity.

In section 2 of this paper, we motivate the method used here by briefly describing Bardeen’s approach [?] for the first order of the expansion and explaining why modifications are necessary when going to higher orders. Section 3 presents the line element and the Einstein equations to be solved iteratively. An iterative scheme allowing for the explicit calculation of an arbitrary order is presented in section 4 and some properties of the solution are discussed in section 5. After providing by way of example the explicit calculation of a few expressions and introducing various physical quantities, the PN approximation up to the fourth order is compared with highly accurate numerical results in section 6.

2 Preliminary Remarks

For a given mass-density, \( Q \), the Maclaurin spheroids and the relativistic model both depend on two parameters.\(^1\) Since the post-Newtonian approximation describes the relativistic model in terms of Newtonian parameters, some convention is needed to determine which relativistic parameters are implied by the specification of the Newtonian ones. Bardeen argued that the “most appropriate choice” compares Newtonian and relativistic bodies of the same rest mass \( M_0 \) and angular momentum \( J \) since these quantities (together with \( Q \)) are coordinate independent and “play the primary role in the Hartle-Sharp \([?]\) variation principle”. In this paper we take a somewhat different approach since our purpose is less the comparison of Newtonian and relativistic configurations, than the development of a method for calculating an arbitrary order of the expansion. Therefore we use the freedom that one has in defining the PN approximation in order to simplify the mathematical structure of the equations. The remaining freedom regarding the choice of a constant is left unspecified as long as possible. What effect the specification of this constant then has, will be studied in section ?? of the paper.

At this point, a brief description of the method that Bardeen used in [?] will provide us with the basis for understanding the motivation for the methods used in this paper.\(^2\) Up to the first order of the PN approximation, one has to determine two metric functions

\(^1\)In the Newtonian case, one of these is a mere scaling parameter.

\(^2\)The reader who is interested in studying Bardeen’s paper [?] may benefit from the following list of errata. The coefficient of \( P_2(\eta) \) in eq. (21) should read \( \frac{1}{\rho(\xi)} \left[ \frac{1}{2} p_2(\xi) C_2 + \frac{\xi^2 - \xi^2}{\xi(1+\xi)^2} \right] \). The left hand side of eq. (36) should read \( v_2^2(G) \xi \) and on the right hand side \( D_2 W_2(\eta) \). The \( P_1(\eta) \) of eq. (43)
from Poisson-like equations as well as the unknown boundary of the star. To solve for the metric functions, Bardeen used a Poisson-integral in spheroidal coordinates $\xi$ and $\eta$, which represents potentials as an expansion in terms of orthogonal polynomials in $\eta$. An iterative scheme for the calculation of higher orders is only feasible if the sum over these polynomials terminates. The conditions for the termination of the sum are that the source remain a polynomial in $\eta$ and that the boundary of the star remain a constant in $\xi$. Neither of these conditions is met with in Bardeen’s approach, which is why it is only appropriate up to the first order. To that order it was possible to determine the metric functions in an elegant way, because they can be decomposed into one piece containing the new (post-Newtonian) source within the old boundary and another piece containing the old source within the new boundary. For higher orders, such a procedure can no longer be used and one has to devise a modified approach.

The approach used in this paper relies on the fact that an extended version of the Poisson-integral is valid for Poisson-like equations even in modified coordinates. Here coordinates will be introduced that are tailored to the unknown boundary of the star and satisfy the condition that the boundary be a constant in this coordinate. Furthermore we require that the unknown boundary of the star when written as a function of the old coordinate $\xi$ be given as a polynomial in $\eta$, a requirement that can be shown to be compatible with the condition that the pressure vanish at the surface of the star. This requirement ensures that the sources in the Poisson-like equations remain polynomials in $\eta$. Thus we have to deal only with terminating sums to any order of the PN approximation and the recursive method proposed here can be applied indefinitely.

### 3 Basic Equations

The line element for an axially symmetric, stationary, asymptotically flat space-time describing a perfect fluid with purely azimuthal motion can be written in Lewis-Papapetrou coordinates as

$$ds^2 = e^{2\mu}(d\varrho^2 + d\zeta^2) + \varrho^2 e^{2\lambda}(d\varphi - \omega dt)^2 - c^2 e^{2\nu} dt^2.$$  

The metric functions $\mu$, $\lambda$, $\omega$ and $\nu$ depend only on $\varrho$ and $\zeta$ and vanish at spatial infinity. The energy-momentum tensor for the pressure $P$ and the mass density $Q$, which is merely the energy density divided by $c^2$, is then given by

$$T_{\alpha\beta} = (Qc^2 + P) u_{\alpha} u_{\beta} + g_{\alpha\beta} P,$$

where $Q$ is a constant up to the surface of the star. The matter of the star rotates uniformly with an angular velocity $\Omega$. We introduce the spheroidal coordinates

$$\varrho^2 = a_0^2(1 + \xi^2)(1 - \eta^2) \quad \text{and} \quad \zeta = a_0 \xi \eta, \quad \eta \in [-1, 1], \quad \xi \in [0, \infty),$$

is to be replaced by $(P_l(\eta) - 1)$ and the second term in the first line of eq. (55) by $+ \frac{\Delta P}{c^2} \frac{\partial U}{\partial \xi}(\xi_s, 1)$.  


and obtain from various combinations of the Einstein equations the following partial differential equations for the metric functions (with $G = 1$ for the gravitational constant):

\[
\Delta_2 \nu = \frac{4\pi e^{2\mu}}{c^4} \left[ \frac{1 + \tilde{v}^2}{1 - \tilde{v}^2} (Qc^2 + P) + 2P \right] - \mathcal{L}(\nu, \nu + \lambda) \\
+ \frac{1}{2} \tilde{\Omega}^2 (1 + \tilde{v}^2)(1 - \eta^2)e^{2\lambda - 2\nu} \mathcal{L}(\tilde{\omega}, \tilde{\omega}),
\]

\[
\Delta_3 (\lambda + \nu) = \frac{16\pi e^{2\mu}}{c^4} P - \mathcal{L}(\nu + \lambda, \nu + \lambda),
\]

\[
\Delta_4 \tilde{\omega} = -\frac{16\pi (1 - \tilde{\omega}) e^{2\mu}}{c^4(1 - \tilde{v}^2)} (Qc^2 + P) - \mathcal{L}(\tilde{\omega}, 3\lambda - \nu),
\]

\[
\Delta_1 \mu = -\frac{4\pi e^{2\mu}}{c^4} (Qc^2 + P) + \mathcal{L}(\nu, \lambda) + \frac{1}{4} (1 + \xi^2)(1 - \eta^2)e^{2\lambda - 2\nu} \mathcal{L}(\tilde{\omega}, \tilde{\omega}) \\
+ \frac{1}{a_\delta^2 (\xi^2 + \eta^2)} \left( \xi \nu, \xi - \nu, \eta \right).
\]

The differential operators in the above equations are defined by

\[
\mathcal{L}(\phi, \chi) := [(1 + \xi^2) \phi, \xi \chi, \xi + (1 - \eta^2) \phi, \eta \chi, \eta] / a_\delta^2 (\xi^2 + \eta^2)
\]

and

\[
\Delta_m := \left[ (1 + \xi^2) \frac{\partial^2}{\partial \xi^2} + (1 - \eta^2) \frac{\partial^2}{\partial \eta^2} + m\xi \frac{\partial}{\partial \xi} - m\eta \frac{\partial}{\partial \eta} \right] / a_\delta^2 (\xi^2 + \eta^2)
\]

and the dimensionless function in eq. (1c) by

\[
\tilde{\omega} := \frac{\omega}{\tilde{\Omega}}.
\]

Note that the operator $\Delta_2$ is simply the Laplace operator in a flat three-dimensional space. The dimensionless pressure $\tilde{P} := P/Qc^2$ is related to the metric functions by

\[
\sqrt{1 + \tilde{v}^2} (1 + \tilde{P}) e^\nu = \text{const.} = 1 - \gamma
\]

with

\[
\tilde{v} := \tilde{g} \tilde{\Omega}(1 - \tilde{\omega}) e^{\lambda - \nu} / a_0 \quad \text{and} \quad \tilde{\Omega} := a_0 \Omega / c.
\]
4 The Iterative Scheme

4.1 The Expansion

The system of partial differential equations (1) is simplified by expanding the relevant quantities in terms of a dimensionless relativistic parameter. Here we choose the square root of the parameter\(^3\) used in \([?]\) defined by

\[
\varepsilon^2 := 8\pi Qa_0^2 \xi s \sqrt{1 + \xi^2 s^2 / 3c^2}.
\]

The three variables entering into this definition completely specify the Newtonian Maclaurin spheroid. \(Q\) is the mass density, \(a_0\) the focus of the ellipse describing the surface of the star in cross-section and \(\xi_s\) the value of the surface’s \(\xi\) coordinate. These are the same quantities which will enter into the PN expansion, but the latter two lose their simple geometrical meaning. The parameter \(\varepsilon\) remains finite in both the spherical limit, given by \(\xi_s \to \infty\) and \(a_0 \propto 1/\xi_s\) and the disc limit, which is given by \(\xi_s \to 0\) and \(Q \propto 1/\xi_s\) for non-vanishing mass.

The expansion of the dimensionless metric functions and the constants reads as follows:

\[
\begin{align*}
\nu &= \sum_{n=2}^{\infty} \nu_n \varepsilon^n \lambda &= \sum_{n=2}^{\infty} \lambda_n \varepsilon^n \tilde{\omega} = \sum_{n=2}^{\infty} \tilde{\omega}_n \varepsilon^n \\
\mu &= \sum_{n=2}^{\infty} \mu_n \varepsilon^n \gamma &= \sum_{n=2}^{\infty} \gamma_n \varepsilon^n \tilde{\Omega} = \sum_{n=1}^{\infty} \tilde{\Omega}_n \varepsilon^n.
\end{align*}
\]

As was already mentioned, \(Q\) is held constant to any order of the approximation, which is why it does not appear in eq. (4) and any other quantities of interest, such as \(\tilde{v}\) or \(\tilde{P}\) can be expressed in terms of these six quantities.

If these expansions are substituted into eqs (1), then comparing coefficients of \(\varepsilon\) yields differential equations for the metric functions of the form \(\Delta_m \phi_i = F\), where \(\phi = \nu, \lambda, \tilde{\omega}, \mu\). Because the right hand side of eqs (1a–1c) depends only on \(\phi_{i-j}, j > 0\), one can solve for \(\phi_i\) if the lower order functions are already known. In the case of \(\mu_i\), one can calculate it from eq. (1d) after having determined the other three functions to this order, or one can compute it from an integral over \(\eta\).

Because an analytic solution, the Maclaurin solution, for the first step is known, these equations would provide an iterative process for the determination of the metric functions to any order if the shape of the star were known. The boundary of the star also has to be determined iteratively however.

\(^3\)The square root was chosen in order to enable a more convenient indexing of the expansion coefficients.
We represent the surface of the star by the equation,

$$\xi = \xi_{\text{B}(\eta)} = \xi_{s} \left( 1 + \sum_{j=0}^{2} \sum_{k=2}^{k} s_{j k} C_{j/2}^{1/2}(\eta)^{e^{k}} \right) \equiv \xi_{s} \left( 1 + \sum_{k=2}^{k} B_{k}(\eta)^{e^{k}} \right).$$

(5)

where we have already taken into account that the boundary is an ellipsoid $\xi = \xi_{s}$ in the Newtonian order. We also require that the sum over the Legendre polynomials $C_{1/2}^{j}(\eta)$, a special case of the Gegenbauer polynomials discussed in section 4.2, terminate and show in section 4.4 that this leads to a consistent solution.

### 4.2 Solving the Poisson-like Equations

In the last section an iterative scheme was proposed for the determination of the metric functions in which an equation of the form $\Delta_{m} \phi = F$ need be solved for a known function $F = F(\xi, \eta)$. The regular and asymptotically flat solution of this equation is given by

$$\phi(\xi, \eta) = a_{0}^{2} \sum_{l=0}^{\infty} K_{l}^{m} C_{l}^{m-1/2}(\eta) \times$$

$$\left[ h_{l}^{m}(\xi) \int_{0}^{1} \int_{-1}^{1} g_{l}^{m}(\xi') C_{l}^{m-1/2}(\eta') F(\xi', \eta') k_{m}(\xi', \eta') d\eta' d\xi' + \right. \left. g_{l}^{m}(\xi) \int_{1}^{\infty} \int_{-1}^{1} h_{l}^{m}(\xi') C_{l}^{m-1/2}(\eta') F(\xi', \eta') k_{m}(\xi', \eta') d\eta' d\xi' \right].$$

(6)

In the above equation $C_{l}^{j}$ are the Gegenbauer polynomials, $g_{l}^{j}$ and $h_{l}^{j}$ are two linearly independent solutions of the (homogeneous) Gegenbauer equation defined by

$$g_{l}^{m}(\xi) := C_{l}^{m-1/2}(i\xi)$$

$$h_{l}^{m}(\xi) := g_{l}^{m}(\xi) \int_{\xi}^{1} \frac{d\xi'}{(g_{l}^{m}(\xi'))^{2} E(\xi')} \quad (l, m) \neq (0, 1) \quad \text{and}$$

(7)

$$h_{0}^{1}(\xi) := \text{arcsinh}(\xi)$$

with

$$E(\xi) := \exp \left( \int_{0}^{\xi} \frac{m\xi' d\xi'}{1 + \xi'^{2}} \right) = \left( 1 + \xi^{2} \right)^{m/2}.$$
The term
\[ k_m(\xi, \eta) \, d\eta \, d\xi := \left[ (1 + \xi^2) \left( 1 - \eta^2 \right) \right]^{m-1} \frac{m}{2} \, (\xi^2 + \eta^2) \, d\eta \, d\xi \] (8)
is a product of the volume element and the appropriate weight function for the Gegenbauer polynomials and
\[ K_l^m = \frac{l! (l + m - \frac{1}{2}) \left[ \Gamma \left( \frac{m}{2} - \frac{1}{2} \right) \right]^2}{\pi^{2-l-1} \Gamma(m - 1 + l)} \quad \text{for } m > 1 \]
\[ K_l^1 = \frac{l^2}{2\pi} \quad \text{for } l > 0 \quad \text{and} \]
\[ K_0^1 = \frac{1}{\pi} \]
are normalizing constants.

In eq. (6) the integrands jump at the surface of the star because of the jump in the mass density. It is therefore necessary to split them into integrals over the interior and exterior of the star. Clearly if the surface of the star is given, as with the leading order, by a constant \( \xi = \xi_s \), then this division is trivial. If the boundary depends on \( \eta \), then matters are complicated considerably. As of the second order in the expansion, the \( \eta \)-integrals no longer run over the interval \( \eta \in [-1, 1] \) meaning that one can no longer make use of the orthogonality of the Gegenbauer polynomials and one is faced with non-terminating sums. We hence introduce new coordinates in order to circumvent these difficulties.

4.3 New Coordinates

We introduce the coordinates

\[ \psi_n = \frac{\xi}{\xi_B(\eta)}, \quad \eta = \eta \] (9)
such that \( \psi_n = \xi_s \) is the boundary of the star. The index \( n \) indicates that the coordinate transformation depends on the order of the approximation through \( \xi_B(\eta) \), cf. eq. (5).

In what follows, the index will be omitted from the equations with the understanding that when considering the \( j^{th} \) order of the approximation, the coordinate \( \psi_j \) is always meant. The new coordinate \( \psi \) is a function of both \( \eta \) and \( \varepsilon \) and contains the unknown coefficients \( S_{jk} \). We rewrite eqs (1) in terms of the new coordinates and manipulate them such that the left hand side has the same form as beforehand, but with \( \psi \) replacing \( \xi \). For example, the equation for \( \nu = \nu(\psi, \eta) \) now reads

\[ \left[ (1 + \psi^2) \frac{\partial^2 \nu}{\partial \psi^2} + (1 - \eta^2) \frac{\partial^2 \nu}{\partial \eta^2} + 2\psi \frac{\partial \nu}{\partial \psi} - 2\eta \frac{\partial \nu}{\partial \eta} \right] / a_0^2 (\psi^2 + \eta^2) = \bar{F}. \]
These new field equations are again expanded\textsuperscript{4} in terms of $\varepsilon$ in order to obtain a system of equations for $\phi_i$ as was explained in section 4.1. Since $\psi = \xi + \mathcal{O}(\varepsilon^2)$ holds, the new equations for $\phi_i(\psi, \eta)$ also depend only on known functions, thereby enabling their recursive determination.

The derivation of eq. (6) relies on the fact that in the coordinates $(\xi, \eta)$, the line $(0, \eta)$ is identical to the line $(0, -\eta)$ and that at spatial infinity we have $\xi \to \infty$. These properties hold for $\psi$ as well and an analysis of the derivation shows that we are free to use eq. (6) as it stands, only replacing $\xi$ by $\psi$.

In changing coordinates we have mapped the star onto the rectangle $[0, \xi_s] \times [-1,1]$, which means that the division of the integrals into inner and outer domains is trivial. The price that one pays for the simplicity in the structure of the integrals is that the sources of the Poisson-like equations become quite unwieldy. But the exchange of a conceptual for a mechanical difficulty can be termed a good deal, and all the more so when its result is the facilitation of the whole scheme.

### 4.4 Determining the Shape of the Star

Due to the factor $c^2$ in $g_{tt}$ of the line element, it is necessary to determine the function $\nu_{i+2}$ in order to calculate the $i^{th}$ order of the PN approximation. This is the only metric function that depends on the unknown coefficients $S_{jk}$ of the star’s boundary.\textsuperscript{5} Using eq. (2) one can arrive at the integral equation

$$\tilde{\Omega}_{1/2}^2(1 - \eta^2) \xi_s^2 B_i(\eta) = \sum_{l=0}^{\infty} K_l^2 C_l^{1/2} \left( \eta \right) f_l(\tilde{\eta}) B_i(\tilde{\eta}) d\tilde{\eta} + b_i(\eta), \quad (10)$$

to solve for these coefficients. The function $B_i$ is defined in eq. (5), $b_i(\eta)$ is a known polynomial of order $i + 2$ and $f_l(\tilde{\eta})$ is given by

$$f_l(\tilde{\eta}) := g_l^2(\xi_s) \int_{-1}^{1} \left[ 2\tilde{\eta}^2 \left( \frac{2\tilde{\eta}^2 - (1 - \eta^2)(\tilde{\eta}^2 + 2\eta^2)}{1 + \varepsilon^2} \right) \right] d\psi$$

$$+ h_l^2(\xi_s) \int_{0}^{\infty} \left[ 2\tilde{\eta}^2 \left( \frac{2\tilde{\eta}^2 - (1 - \eta^2)(\tilde{\eta}^2 + 2\eta^2)}{1 + \varepsilon^2} \right) \right] d\psi,$$

\textsuperscript{4}Note that the coefficients $\phi_i(\psi, \eta)$ are not mere transformations of $\phi_i(\xi, \eta)$ since $\psi$ depends on $\varepsilon$.

\textsuperscript{5}The other metric functions depend only on $S_{jk}$, with $j < i$. 

8
where a dot and prime indicate partial derivatives with respect to $\psi$ and $\tilde{\eta}$ respectively and the superscripts ‘i’ and ‘o’ refer to the regions inside and outside the star. Since $f_i(\eta)$ is a polynomial of second order, the sum in eq. (10) terminates for polynomial $B_i(\eta)$. Indeed, for the form of $B_i$ chosen in eq. (5), one arrives at a system of $i + 2$ algebraic equations for $i + 3$ unknowns (there are $i + 1$ $S_{ij}$ to determine as well as $\tilde{\Omega}_{i+1}$ and $\gamma_{i+2}$).\footnote{We shall see in section 5.1 that $S_{ij} = 0$ for odd $j$.} We choose to use this system to determine all these constants but for $\gamma_{i+2}$. As mentioned in section 2, this last constant can be chosen arbitrarily, which amounts to specifying “which” PN-approximation one wishes to have, i.e. which Maclaurin spheroid is to be associated with a given relativistic body. The choice of $\gamma_{i+2}$ will be discussed further in section ??.

We have shown that the form chosen for the surface of the star is consistent with the Einstein equations to any order of the PN expansion. This is not to say that this choice is unique. One can easily see that the form chosen in [?] is incompatible with that chosen here, since it is not a polynomial in $\eta$. There the surface was derived having stipulated that the ‘generating’ Maclaurin spheroid should have the same rest mass and angular momentum as the PN star, a condition that cannot be satisfied with the approach chosen here. In lieu of the freedom to choose two constants, we have chosen a form for the boundary of the star most appropriate to our goal of devising an iterative scheme and can choose only one further constant.

5 Properties of the Solution

5.1 Reflectional Symmetry

In Newtonian physics, it is known, that stationary, axisymmetric bodies are necessarily symmetric with respect to a reflection through the $\zeta = 0$ plane (see e.g. [?]). Although authors (e.g. [?]) have speculated that the same holds in General Relativity, it has not yet been proved true. In the case considered here, this symmetry arises automatically. A function $f$ exhibits reflectional symmetry in $\xi-\eta$ (or $\psi-\eta$) coordinates precisely when it is an even function of $\eta$. Because of the orthogonality of the Gegenbauer polynomials, the terms in the sum of eq. (6) for odd $l$ are zero if $F$ is a polynomial in $\eta^2$, a condition which turns out to be satisfied. The odd terms, which are provided by the unknown boundary coefficients $S_{li}$ must be zero for the boundary condition to be fulfilled. Thus we have shown that any axially symmetric, stationary relativistic solution that is continuously connected to the Maclaurin spheroids is symmetric with respect to reflections through the $\zeta = 0$ plane.
5.2 Powers of the Relativistic Parameter

Consideration of the field equations together with the knowledge of the Newtonian behaviour of the dimensionless metric functions, shows that their expansion coefficients $\phi_i$ begin with $i = 2$ and are non-zero only for even $i$. The same holds naturally for $\gamma_i$, whereas $\tilde{\Omega}_j$ begins with $j = 1$ and appears only with odd powers of $j$. Because of the choice to work with the dimensionless functions introduced here, it is most appropriate to refer to the $n^{th}$ order of the PN approximation and not the half orders in between. What we mean by the $n^{th}$ order is that the quantities $\lambda, \tilde{\omega}, \mu$ and $\xi_B$ are expanded up to and including the order $O(\varepsilon^{2n})$, $\tilde{\Omega}$ up to $O(\varepsilon^{2n+1})$ and $\nu$ and $\gamma$ to $O(\varepsilon^{2n+2})$.

5.3 Singularities in Parameter Space

By comparing the highest coefficient of $\eta$ in eq. (10) one can arrive at the equation

$$S_{ii} = \frac{t_i}{\xi_s^2 \Omega_1^2 - a_{i+2}}$$

(11)

with

$$t_i \propto \int_{-1}^{1} C_{i+2}^{1/2}(\bar{\eta}) b_i(\bar{\eta}) d\bar{\eta}$$

and

$$C_{i+2}^{1/2}(\eta) f_{i+2}(\eta) =: \sum_{n=0}^{1} a_{2n,i+2} \eta^{2n}, \quad a_{i+2} := a_{2,i+2}$$

and where $f_{i+2}$ and $b_i$ are defined in eq.(10). It can be shown that the denominator of eq. (11) is proportional to the expression

$$g_{i+2}^2(\xi_s) h_{i+2}^2(\xi_s) = g_i(\xi_s) = C_i(\xi_s).$$

(12)

For a given (even) $i$, this expression vanishes for precisely one value of $\xi_s$, let us say for $\xi_s = \xi^*_i$. These values, beginning with $i = 2$, are the points of onset of axisymmetric, secular instability and the bifurcation points of new axisymmetric solutions, see [?], [?], [?], and numerical values for the first few of them can be found in Table 1. Since $t_i$ of eq. (11) is not zero at the point $\xi_s = \xi^*_i$, these bifurcation points are singularities in the two dimensional parameter space $(\xi_s, a_0)$ or $(\xi_s, \varepsilon)$. For values of $\xi_s$ differing
Table 1: Numerical values for $\xi_{2l}^*$ and the corresponding Newtonian eccentricities and ratios of polar to equatorial radii given by $e = 1/\sqrt{1 + \xi_s^2}$ and $r_p/r_e = \xi_s/\sqrt{1 + \xi_s^2}$.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$\xi_{2l}^*$</th>
<th>$e_{2l}^*$</th>
<th>$r_p/r_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.17383011</td>
<td>0.98522554</td>
<td>0.17126187</td>
</tr>
<tr>
<td>3</td>
<td>0.11230482</td>
<td>0.99375285</td>
<td>0.11160323</td>
</tr>
<tr>
<td>4</td>
<td>0.08303471</td>
<td>0.99657034</td>
<td>0.08274493</td>
</tr>
<tr>
<td>5</td>
<td>0.06588682</td>
<td>0.99783651</td>
<td>0.06574427</td>
</tr>
</tbody>
</table>

only slightly from $\xi_{i+2}^*$, the PN configurations have properties similar to those of the Newtonian configurations that branch off from the Maclaurin sequence at these points. The Maclaurin configuration itself cannot be reached for bodies with non-zero mass however, and even neighbouring configurations have strict mass limitations, since $\varepsilon$ must be made very small in order that the PN series converge. This mass limitation can be inferred, for example, by referring to the tables in Appendix ???. Because the $n^{th}$ PN order possesses a pole of order $2n - 1$ at the point $\xi = \xi_n^*$, we expect the coefficients for the expansion to grow large in the vicinity of this point. $^7$ This is indeed the case as can be seen in these tables by referring to the row with $\xi_s = 0.17$. The series containing these coefficients converge only for sufficiently small $\varepsilon$ as indicated above.

6 Explicit Solution to the Fourth Order

6.1 The Metric Functions and the Constants

Using the iterative scheme described above, the four metric functions and the constant $\tilde{\Omega}$ were explicitly solved up to the fourth post-Newtonian order. These calculations could in principle be carried out ad infinitum, but the lengthiness of the expressions (the fourth order functions would fill several hundred pages) puts a practical limit on the order that can be determined. Here we will merely carry out, by way of example, the calculation of the first few terms.

The expansion of eq. (1a) with respect to the relativistic parameter $\varepsilon$ yields the Newtonian equations

$$\Delta_2 v^{i}_2 = \frac{4\pi}{\varepsilon^2 c^2} Q = \frac{3}{2a_0^2 \xi_s \sqrt{1 + \xi_s^2}}$$

for the interior region ($\xi < \xi_s$) and

$$\Delta_2 v^{o}_2 = 0.$$  

$^7$A lengthier discussion regarding the order of the poles at $\xi_{2l+2}^*$, $i > 1$ can be found in [?].