Gauge Field Theory on the $E_q(2)$–covariant Plane

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Abstract

Gauge theory on the $q$–deformed two-dimensional Euclidean plane $\mathbb{R}^2_q$ is studied using two different approaches. We first formulate the theory using the natural algebraic structures on $\mathbb{R}^2_q$, such as a covariant differential calculus, a frame of one-forms and invariant integration. We then consider a suitable star product, and introduce a natural way to implement the Seiberg-Witten map. In both approaches, gauge invariance requires a suitable “measure” in the action, breaking the $E_q(2)$-invariance. Some possibilities to avoid this conclusion using additional terms in the action are proposed.

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1 Introduction

Gauge theories provide the best known descriptions of the fundamental forces in nature. At very short distances however, physics is not known, and it is plausible that spacetime is quantized below some scale. This idea has been contemplated for quite some time, and gauge theory on noncommutative spaces has been the subject of much research activity, see e.g. [1] for a review.

There are several different approaches to gauge theories on noncommutative spaces: First, one can formulate the theory in terms of the algebraic structures which define the noncommutative space, such as the noncommutative algebra of functions, its modules, and differential calculi. Gauge transformations can then be defined by unitary elements of the algebra of functions. Examples of noncommutative gauge theories using this formulation can be found in [2,3,4]. While it is certainly very natural, this approach seems to be restricted to unitary gauge groups, and the set of admissible representations of the associated matter fields is also quite restricted.

Another approach has been developed following the discovery that string theory leads to noncommutative gauge theories under suitable conditions, as explained in [5]. This lead to a technique expressing the noncommutative fields in terms of commutative ones, and writing the Lagrangians in terms of ordinary (commutative) fields and star products. It allows to formulate models with general gauge groups and representations, including the standard model [6]. However, the Lagrangians become increasingly complicated at each order in the deformation parameter, and there is generally a large amount of arbitrariness in these actions. Moreover, the formulation of gauge theories on general noncommutative spaces with non-constant Poisson structure is less clear. In particular, no satisfactory formulation of gauge theory on spaces with quantum group symmetry has been given; see e.g. [7] for a clear manifestation of this problem. It seems that in general, a satisfactory implementation of generalized symmetries (quantum group symmetries) in noncommutative field theory is yet to be found.

In the present paper, we apply these different approaches to gauge theory on the Euclidean quantum plane $\mathbb{R}^2_q$, which is covariant under the $q$-deformed two-dimensional Euclidean group $E_q(2)$. This is one of the simplest quantum spaces with a non-trivial quantum group symmetry, and scalar field theory on $\mathbb{R}^2_q$ has already been studied in [8]. It seems therefore well suited to gain some insights into gauge theory on spaces with quantum group symmetry.

We first try to formulate (abelian) gauge theory on $\mathbb{R}^2_q$ using an algebraic approach, taking advantage of the covariant differential calculus on $\mathbb{R}^2_q$. This leads very naturally to a definition of gauge fields and their field strength, with gauge transformations being the unitaries of the algebra of functions. This field strength reduces to the usual one in the commutative limit. However, the definition of an invariant action turns out to be less clear: if one uses the natural invariant integral on $\mathbb{R}^2_q$, one must add a nontrivial “measure function” in order to obtain a gauge invariant action. This measure function explicitly breaks translation invariance, which seems to be a generic feature of gauge theory on spaces with quantum group symmetry. Hence gauge invariance appears to be in conflict with quantum group symmetry. However, we point out some ways to
avoid this conclusion. We propose a model with an additional scalar (“Higgs”) field with a suitable potential, which is manifestly gauge invariant and restores the formal $E_q(2)$-invariance while spontaneously breaking gauge invariance.

In the second part of this paper, we apply the star product approach to gauge theory on $\mathbb{R}^2_q$, expressing all fields in terms of commutative ones. We first construct a suitable star product, and study its properties and the relation with the integral. The gauge theory is then formulated using this star product in close analogy to the algebraic approach. In particular, the noncommutative calculus suggests a definition of the field strength in terms of a “frame”, which ensure the correct classical limit. This is somewhat different from other approaches proposed in the literature. The corresponding Seiberg-Witten maps are solved up to first order. The formulation of a gauge invariant action requires again a nontrivial measure function, which is essentially the same as in the algebraic approach. While it cannot be canceled as in the algebraic approach by introducing a Higgs field, we show how the action can be modified in order to obtain the correct commutative limit.

2 The $q$–deformed two-dimensional Euclidean Group and Plane

2.1 The dual symmetry algebras $E_q(2)$ and $U_q(e(2))$

We start by reviewing the quantum group $E_q(2)$, which is a deformation of the (Hopf) algebra of functions on the two-dimensional Euclidean Group $E(2)$. It is generated by the “functions” $n, v, \bar{n}, \bar{v}$ with the following relations and structure maps:

$$
\begin{align*}
v\bar{v} &= \bar{v}v = 1 & n\bar{n} &= \bar{n}n & vn &= qnv \\
n\bar{v} &= q\bar{v}n & \bar{v}n &= q\bar{n}v & \bar{n}\bar{v} &= q\bar{v}\bar{n} \\
\Delta(n) &= n \otimes \bar{v} + v \otimes n & \Delta(v) &= v \otimes v & \Delta(\bar{n}) &= \bar{n} \otimes v + \bar{v} \otimes \bar{n} \\
\Delta(\bar{v}) &= \bar{v} \otimes \bar{v} & \varepsilon(n) &= \varepsilon(\bar{n}) = 0 & \varepsilon(v) &= \varepsilon(\bar{v}) = 1 \\
S(n) &= -q^{-1}n & S(v) &= \bar{v} & S(\bar{n}) &= -q\bar{n} & S(\bar{v}) &= v
\end{align*}
$$

(1)

where $q \in \mathbb{R}$. This is a star-Hopf algebra with the conjugation

$$
n^* = \bar{n}, \quad v^* = \bar{v}.
$$

(2)

In terms of the operators $\theta, t, \bar{t}$ defined by

$$
v = e^{\frac{1}{2}\theta} \quad t = nv \quad \bar{t} = \bar{v}\bar{n}
$$

(3)

(note that $v$ is unitary and can therefore be parametrized by a hermitian element $\theta^* = \theta$), the coproduct of $t$ and $\bar{t}$ reads

$$
\Delta(t) = t \otimes 1 + e^{i\theta} \otimes t \quad \Delta(\bar{t}) = \bar{t} \otimes 1 + e^{-i\theta} \otimes \bar{t}.
$$

(4)
It is often convenient to consider also the dual quantum group. The dual Hopf algebra
\( U_q(e(2)) \) of \( E_q(2) \) is generated by \( T, \overline{T}, J \) with the following commutation relations and
structure maps\(^3\)\(^4\)
\[
T \overline{T} = q^2 T \overline{T} \quad [J, T] = i T \quad [J, \overline{T}] = -i \overline{T}
\]
\[
\Delta(T) = T \otimes q^{2iJ} + 1 \otimes T \quad \Delta(\overline{T}) = \overline{T} \otimes q^{2iJ} + 1 \otimes \overline{T}
\]
\[
\Delta(J) = J \otimes 1 + 1 \otimes J \quad \varepsilon(T) = \varepsilon(\overline{T}) = \varepsilon(J) = 0
\]
\[
S(T) = -T q^{-2iJ} \quad S(\overline{T}) = -\overline{T} q^{-2iJ} \quad S(J) = -J,
\]
where the dual pairing on the generators is given by
\[
\langle T, \theta^i \theta^j \rangle = \delta_0 \delta_1 \delta_0, \quad \langle \overline{T}, \theta^i \theta^j \rangle = -q^2 \delta_0 \delta_0 \delta_1 \delta_k, \quad \langle J, \theta^i \theta^j \rangle = \delta_1 \delta_0 \delta_0.
\]
This is again a star-Hopf algebra with the conjugation
\( J^* = -J, \quad T^* = \overline{T}. \)

### 2.2 The \( E_q(2) \)-covariant Euclidean plane \( \mathbb{R}^2_q. \)

Hopf algebras can be used to define generalized symmetries. There are two equivalent, dual notions. A Hopf algebra \( \mathcal{H} \) coacts on an algebra \( \mathcal{A} \) if \( \mathcal{A} \) is a left (or right) \( \mathcal{H} \)-comodule algebra (see Appendix A) via a left coaction \( \rho : \mathcal{A} \to \mathcal{H} \otimes \mathcal{A} \). In particular, every Hopf algebra \( \mathcal{H} \) admits a comodule structure on itself in virtue of the comultiplication
\[
\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}.
\]
Observing that the subalgebra of \( E_q(2) \) generated by \( t, \bar{t} \) is an \( E_q(2) \)-module subalgebra, we can obtain the \( E_q(2) \)-symmetric plane by renaming \( t \to z, \bar{t} \to \bar{z} \). Hence \( \mathbb{R}^2_q \) is the \( E_q(2) \)-comodule algebra with generators \( z, \bar{z} \) and commutation relations
\[
z \bar{z} = q^2 \bar{z} z.
\]
We will also allow formal power series, and define the algebra of functions on the \( E_q(2) \)-covariant plane \( \mathbb{R}^2_q \)
\[
\mathbb{R}^2_q := \mathbb{R}\langle \langle z, \bar{z} \rangle \rangle / (z \bar{z} - q^2 \bar{z} z).
\]
By construction, it is covariant under the following left \( E_q(2) \)-coaction
\[
\rho(z) = e^{i \theta} \otimes z + t \otimes 1
\]
\[
\rho(\bar{z}) = e^{-i \theta} \otimes \bar{z} + \bar{t} \otimes 1.
\]
More formally, we have a coaction \( \rho : \mathbb{R}^2_q \to E_q(2) \otimes \mathbb{R}^2_q \). From now on, functions are considered to be elements of this algebra.

\(^3\)Our generators are related to the generators \( \mu, \nu, \xi \) in \( \mathbb{B} \) by \( \mu \equiv T, -q^2 \nu \equiv \overline{T}, \xi \equiv J \)
In general, a left comodule algebra \( \mathcal{A} \) under \( \mathcal{H} \) is also a right \( \mathcal{H}' \)-module algebra, using the dual pairing between \( \mathcal{H} \) and its dual \( \mathcal{H}' \). Explicitly, the right action \( \alpha : \mathcal{A} \otimes \mathcal{H}' \rightarrow \mathcal{A} \) of \( \mathcal{H}' \) on \( \mathcal{A} \) is given by
\[
 f \triangleleft X := ((X, \cdot) \otimes \text{id}) \circ \rho(f) = \langle X, f(-1) \rangle f(0), \quad X \in \mathcal{H}, \; f \in \mathcal{A}.
\] (11)

Applied to the present situation using the coaction \( \Delta \) and the dual pairing \( \langle , \rangle \), we obtain an action of \( U_q(e(2)) \) on \( \mathbb{R}^2_q \). It is compatible with the conjugation \( z^* = \bar{z} \) in the sense
\[
 (f \triangleleft X)^* = f^* \triangleleft S^{-1}(X^*)
\] (12)
for any \( f \in \mathbb{R}^2_q \) and \( X \in U_q(e(2)) \). To calculate the action of \( U_q(e(2)) \) on formal power series in \( z, \overline{z} \), it is useful to note that any formal power series \( f(z, \overline{z}) \) can be written as
\[
 f(z, \overline{z}) = \sum_{m \in \mathbb{Z}} z^m f_m(z\overline{z}).
\] (13)

The action on terms of this form is calculated in Appendix A
\[
z^k f(z\overline{z}) \triangleleft T = \frac{z^{k-1}}{1 - q^{-2}} (f(q^2 z\overline{z}) - q^{-2k} f(z\overline{z}))
\]
\[
z^k f(z\overline{z}) \triangleleft \overline{T} = \frac{q^k}{1 - q^2} z^{k+1} \frac{f(z\overline{z}) - f(q^{-2} z\overline{z})}{z\overline{z}}
\]
\[
z^k f(z\overline{z}) \triangleleft J = z^k f(z\overline{z}),
\] (14)
which has again the above form.

### 2.3 Covariant differential calculus on \( \mathbb{R}^2_q \)

A differential calculus is useful to write down Lagrangians. A covariant differential calculus over \( \mathbb{R}^2_q \) is a graded bimodule \( \Omega_q^* = \oplus_n \Omega_q^n \) over \( \mathbb{R}^2_q \) which is a \( U_q(e(2)) \) module algebra, together with an exterior derivative \( d \) which satisfies \( d^2 = 0 \) and the usual graded Leibniz rule. Its construction \( [10, 11] \) is reviewed here for convenience, in order to establish the notation. We start by introducing variables \( dz \) and \( d\overline{z} \), which are the \( q \)-differentials of \( z \) and \( \overline{z} \). These are noncommutative differentials which do not commute with the space coordinates \( z, \overline{z} \). Covariance and \( d(1) = 0 \) implies the coaction
\[
 \rho(dz) = e^{id} \otimes dz
\]
\[
 \rho(d\overline{z}) = e^{-id} \otimes d\overline{z},
\] (15)
and the commutation relations between coordinates and their differentials must be
\[
z dz = q^{-2} dzz \quad \overline{z} dz = q^{-2} dz\overline{z}
\]
\[
z d\overline{z} = q^2 d\overline{z}z \quad \overline{z} d\overline{z} = q^2 d\overline{z}\overline{z}.
\] (16)

\footnote{Similarly one gets a left action via a dual pairing from a right coaction.}
To see that $d : \mathbb{R}^2_q \rightarrow \Omega^1_q$ is well-defined, we have to verify that it respects the commutation relations of the algebra, i.e.

$$d(z\zbar - q^2\zbar z) = 0$$

(17)

which is easy to see. To obtain a higher order differential calculus, we apply $d$ on the commutation relations (16), which gives

$$dzd\zbar = -q^2d\zbar dz$$

(18)

and

$$(dz)^2 = (d\zbar)^2 = 0.$$  

This defines a star-calculus (i.e. with a reality structure), where the star of forms and derivatives is defined in the obvious way. One can now introduce $q$--deformed partial derivatives by

$$d = dz^i\partial_i = dz\partial_z + d\zbar\partial_{\zbar},$$

(19)

as in the commutative case. This defines the action of $\partial_z$ and $\partial_{\zbar}$ on functions. One can also introduce the algebra of differential operators with generators $\partial_z, \partial_{\zbar}, z, \zbar$. In order to distinguish the generators $\partial_z, \partial_{\zbar}$ in this algebra from their action on a function, we denote the latter by

$$\partial_z(f) \quad \text{and} \quad \partial_{\zbar}(f),$$

whereas we will not use brackets if $\partial_z, \partial_{\zbar}$ are interpreted as part of the algebra of differential operators.

The derivatives $\partial_z, \partial_{\zbar}$ satisfy a modified Leibniz rule. It can be derived from the Leibniz rule of the exterior differential together with the commutation relations of differentials and coordinates as follows: On the one hand, we have

$$d(fg) = (df)g + f(dg)$$

$$= dz^i\partial_i(f)g + f dz^i\partial_i(g)$$

$$= dz^i\partial_i(f)g + dzf(q^{-2}z, q^{-2}\zbar)\partial_z(g) + d\zbar f(q^2z, q^2\zbar)\partial_{\zbar}(g)$$

(20)

using the commutation relations

$$f(z, \zbar) \ dz = dz f(q^{-2}z, q^{-2}\zbar)$$

$$f(z, \zbar) \ d\zbar = d\zbar f(q^2z, q^2\zbar),$$

which follow from (16). On the other hand, we have

$$d(fg) = dz\partial_z(fg) + d\zbar\partial_{\zbar}(fg),$$

and together with (20) we obtain the $q$--Leibniz rule

$$\partial_z(fg) = \partial_z(f)g + f(q^{-2}z, q^{-2}\zbar)\partial_z(g)$$

(22)

$$\partial_{\zbar}(fg) = \partial_{\zbar}(f)g + f(q^2z, q^2\zbar)\partial_{\zbar}(g).$$

(23)
Applying this to the functions \( zf \) resp. \( \overline{z}f \), one obtains the following commutation relations:

\[
\begin{align*}
\partial_z z &= 1 + q^{-2}z \partial_z \\
\partial_z \overline{z} &= q^{-2}z \partial_z \\
\partial_{\overline{z}} z &= q^2 \overline{z} \partial_{\overline{z}} \\
\partial_{\overline{z}} \overline{z} &= 1 + q^2 \overline{z} \partial_{\overline{z}}.
\end{align*}
\] (24)

Furthermore, applying \( \partial_z \partial_{\overline{z}} \) on the function \( z\overline{z} \) we find

\[
\partial_z \partial_{\overline{z}} = q^2 \partial_{\overline{z}} \partial_z.
\] (25)

For completeness we also give the commutation relations for differentials and derivatives:

\[
\begin{align*}
\partial_z dz &= q^2 dz \partial_z \\
\partial_{\overline{z}} dz &= q^2 dz \partial_{\overline{z}} \\
\partial_z d\overline{z} &= q^{-2} d\overline{z} \partial_z \\
\partial_{\overline{z}} d\overline{z} &= q^{-2} d\overline{z} \partial_{\overline{z}}.
\end{align*}
\] (26)

Clearly, the \( q \)-differentials and \( q \)-derivatives become the classical differentials resp. derivatives in the limit \( q \to 1 \).

### 2.3.1 The frame

On many noncommutative spaces \([12, 13]\), there exists a particularly convenient basis of one forms (a “frame”) \( \theta^a \in \Omega^1 \), which commute with all functions. They are easy to find here: consider the elements

\[
\begin{align*}
\theta &\equiv \theta^z &:= z^{-1} \overline{z} dz \\
\overline{\theta} &\equiv \theta^{\overline{z}} &:= d\overline{z} \overline{z}^{-1}.
\end{align*}
\] (27)

Then the following holds:

**Lemma 1.**

\[
[\theta, f] = [\overline{\theta}, f] = 0
\] (28)

for all functions \( f \in \mathbb{R}^2_q \), and

\[
\theta \overline{\theta} = -q^2 \overline{\theta} \theta.
\] (29)

**Proof.** Easy verification using the above commutation relations. \( \square \)

It is even possible to find a one-form \( \Theta \) which generates the exterior differential: consider the following “duals” of the frame,

\[
\begin{align*}
\lambda_z &:= \frac{1}{1 - q^{-2}z^{-1}} \\
\lambda_{\overline{z}} &:= -\frac{1}{1 - q^2 \overline{z}^{-1}}
\end{align*}
\] (30)

and define

\[
\Theta := \theta^i \lambda_i.
\]

Then we have
Lemma 2. The anti-hermitian one-form $\Theta^* = -\Theta$ generates the exterior differential by
\[ df = [\Theta, f] = [\lambda_i, f] \theta^i \] (32)
for all $f \in \mathbb{R}^2_q$. Similarly,
\[ d\alpha = \{\Theta, \alpha\} \] (33)
for any one-form $\alpha$. Here $\{\cdot, \cdot\}$ denotes the anti-commutator. Furthermore,
\[ d\Theta = \Theta^2 = 0 \] (34)

Proof. Equations (32) and (34) are shown in Appendix A.3. Equation (33) then follows easily noting that $\{\Theta, \alpha f\} = \{\Theta, \alpha\} f - \alpha [\Theta, f]$ and $\{\Theta, f \alpha\} = [\Theta, f] \alpha + f \{\Theta, \alpha\}$ for arbitrary functions $f$ and one-forms $\alpha$.

\[ \square \]

2.4 Invariant metric

A relation between the algebra, the differential calculus and the geometry on noncommutative spaces was proposed in [12]. We briefly address this issue here, arguing that $\mathbb{R}^2_q$ is flat. This can be seen as follows: According to [12], “local” line elements must have the form
\[ ds^2 = \theta^i \otimes \theta^j g_{ij} \] (35)
where $g_{ij}$ must be a central (i.e. numerical, here) tensor, and $\theta^i$ is the frame introduced above. The symmetry of $g_{ij}$ is expressed in the equation
\[ g_{ij} P^{(-)}_{_{kl}} = 0, \] (36)
where $P^{(-)}_{_{kl}}$ is the antisymmetrizer defined by the calculus
\[ \theta^k \theta^l P^{(-)}_{_{ij}} = \theta^i \theta^j. \] (37)

If we require furthermore that $ds^2$ be invariant under $E_q(2)$, it follows that
\[ ds^2 = \theta \otimes \theta + q^2 \theta \otimes \theta = q^{-2} dz \otimes d\bar{z} + q^4 d\bar{z} \otimes dz. \] (38)
This is certainly a flat metric, and for $q \rightarrow 1$ reduces to the usual Euclidean metric on $\mathbb{R}^2$. 

7
2.5 Representations of $\mathbb{R}^2_q$

In the following we will only need representations of the algebra $\mathbb{R}^2_q$, not including derivatives or forms. They are easy to find [9]: Since $r^2 = z\bar{z}$ is formally hermitian, we assume that it can be diagonalized. The commutation relations then imply that $z$ and $\bar{z}$ are rising resp. lowering operators which are invertible,

$$
\begin{align*}
    r^2 | n\rangle_{r_0} &= r_0^2 q^{2n} | n\rangle_{r_0}, \\
    \bar{z} | n\rangle_{r_0} &= r_0 q^n | n+1\rangle_{r_0}, \\
    z | n\rangle_{r_0} &= r_0 q^{n-1} | n-1\rangle_{r_0}.
\end{align*}
$$

(39)

We will denote this irreducible representation with $L_{r_0}$, where $r_0$ can be either positive or negative. The representations with $r_0$ and $-r_0$ are equivalent. The irreducible representations are labeled by $r_0 \in [1, q)$. It follows that $z^{-1}$ and $\bar{z}^{-1}$ are well-defined on $L_{r_0}$ unless $r_0 = 0$.

3 Invariant Integration

3.1 Integral of functions

In order to define an invariant action, we need an integral on $\mathbb{R}^2_q$ which is invariant under $E_q(2)$. In general, an integral (i.e. a linear functional) is called invariant with respect to the right action of $U_q(e(2))$ if it satisfies the following invariance condition

$$
\int^q f(z, \bar{z}) \triangleleft X = \varepsilon(X) \int^q f(z, \bar{z})
$$

(40)

for all $f \in \mathbb{R}^2_q$ and $X \in U_q(e(2))$. Here $\varepsilon(X)$ is the counit. Such an integral was found in [13]; however, we want to determine the most general invariant integral here. Since $\varepsilon$ is an algebra homomorphism, it is sufficient to check the condition [10] for the generators $T, \overline{T}$ and $J$. Let us first consider functions of the type

$$
z^m f(z\bar{z})
$$

(41)

where $f(r^2), r^2 = z\bar{z}$ can be considered as a classical function in one variable. We can choose it such that the integral will be well-defined. Invariance under the action [13] of $J$ implies

$$
\int^q z^m f(z\bar{z}) = \delta_{m,0} \langle f(r^2) \rangle_r
$$

(42)

where $\langle f(r^2) \rangle_r$ is a “radial” integral to be determined. Invariance under the action of $T$ and $\overline{T}$ then leads to the following algebraic condition

$$
\langle f(q^2 r^2) - q^{-2} f(r^2) \rangle_r = 0
$$

(43)
on the radial part of the integral. This condition is satisfied for
\[
\langle f(r^2) \rangle_{r_0} := r_0^2(q^2 - 1) \sum_{k=-\infty}^{\infty} q^{2k} f(q^{2k}r_0^2),
\]
for any \( r_0 \in \mathbb{R} \). Notice that the integral can then be written as “quantum trace” (or Jackson-sum) over the irreducible representation \( L_{r_0} \) defined in (39):
\[
\int_{q,(r_0)} f(z, \overline{z}) := (q^2 - 1) \text{Tr}_{r_0}(r^2 f(z, \overline{z})),
\]
where \( \text{Tr}_{r_0} \) is the ordinary trace on \( L_{r_0} \); note that \( \text{Tr}_{r_0}(z^m f(r^2)) = 0 \) for \( m \neq 0 \). If we allow superpositions of this basic integral (resp. direct sums of irreps of \( \mathbb{R}_q^2 \)), then we can take an arbitrary superposition of the form
\[
\langle f(r^2) \rangle_r = \int_1^q dr_0 \mu(r_0) \langle f(r^2) \rangle_{r_0}
\]
with arbitrary (positive) "weight" function \( \mu(r) > 0 \). If \( \mu(r) \) is a delta-function, this is simply the above Jackson-sum. For \( \mu(r_0) = \frac{1}{r_0(q^2 - 1)} \), one obtains the classical radial integral
\[
\int_{q} f(z, \overline{z}) = \int_1^q dr_0 \frac{1}{r_0(q^2 - 1)} \int_{q,(r_0)} f_0(z, \overline{z}) = \int_0^\infty dr f_0(r^2)
\]
for \( f(z, \overline{z}) = \sum_m z^m f_m(r^2) \), assuming \( q > 1 \). Any of these integrals reduces to the usual (Riemann) integral on \( \mathbb{R}^2 \) for \( q \to 1 \), using the obvious mapping from \( \mathbb{R}_q^2 \) to \( \mathbb{R}^2 \) induced by (13).

It is quite remarkable that the classical radial integral is indeed invariant, cp. [15]. This will be useful in the star-product approach in Section 5. Nevertheless, the invariant integrals are not cyclic in the ordinary sense:

**Lemma 3.** For any invariant integral [40] the following cyclic property holds:

i) For any functions \( f, g \), we have
\[
\int_{q} fg = \int_{q} g\mathcal{D}(f)
\]
where \( \mathcal{D} \) is the algebra homomorphism defined by
\[
\mathcal{D}(z^m) := q^{-2m} z^m \quad \mathcal{D}(\overline{z}^m) := q^{2m} \overline{z}^m.
\]

ii) \( \mathcal{D} \) is an inner automorphism:
\[
\mathcal{D}(f(z, \overline{z})) = z\overline{z} f(z, \overline{z})^{-1} z^{-1}.
\]

**Proof.** Easy verification using the commutation relation [3].

A similar cyclic property for invariant integrals on a \( SO_q(N) \)-covariant space was found in [15].
3.2 Integral of forms

Since any 2-form $\alpha^{(2)} \in \Omega^2_q$ can be written as $\alpha^{(2)} = f \theta \bar{\theta}$ and $\theta \bar{\theta}$ is invariant, we define

$$\int^q \alpha^{(2)} = \int^q f \theta \bar{\theta} := \int^q f. \tag{51}$$

For one-forms $\alpha, \beta$ we then obtain the following cyclic property:

$$\int^q \alpha \beta = - \int^q \beta \mathcal{D}(\alpha) \tag{52}$$

where $\mathcal{D}$ is defined on forms as above. Noting that $\mathcal{D}(\Theta) = \Theta$, this immediately yields Stokes theorem:

**Theorem 1.** Let $\alpha$ be a one-form. Then

$$\int^q d\alpha = 0. \tag{53}$$

**Proof.** Since $d\alpha = \{\Theta, \alpha\}$ due to (33), we get with (52)

$$\int^q d\alpha = \int^q \{\Theta, \alpha\} = 0. \tag{53}$$

\[ \Box \]

4 Gauge Transformations, Field Strength and Action

We consider matter fields as functions in $\mathbb{R}^2_q$. An infinitesimal noncommutative gauge transformation of a matter field $\psi$ is defined as

$$\delta \psi = i \Lambda \psi \tag{54}$$

while of course $\delta z^i = 0$. We introduce the “covariant derivative” (or rather a covariant one-form)

$$D := \Theta - iA \tag{55}$$

which should be an anti-hermitian one-form. Requiring that $D\psi(x)$ transforms covariantly, i.e.

$$\delta D\psi = i \Lambda D\psi$$

leads to

$$\delta D = i [\Lambda, D], \tag{56}$$

which using (32) implies the following gauge transformation property for the gauge field $A$

$$\delta A = [\Theta, \Lambda] + i [\Lambda, A] = d \Lambda + i [\Lambda, A]. \tag{57}$$
This suggests to define the noncommutative field strength $F$ as
\[ F := iD^2 = F_{ij} \theta^i \theta^j, \]
which is a 2-form transforming as
\[ \delta_{\Lambda} F = i[\Lambda, F]. \]  
(58)
Since $\Theta^2 = 0$ and $\{\Theta, A\} = dA$ we obtain the familiar form
\[ F = dA - iA^2, \]  
(59)
which shows that $F$ reduces to the classical field strength in the limit $q \to 1$. To write it in terms of components, it is most natural to expand the 1-forms in the frame basis $\theta^i = (\theta, \bar{\theta})$, because then no ordering prescription is needed. Hence we can write
\[ A = A_i \theta^i = \theta^i A_i, \]  
(60)
and the field strength is
\[
F = (\lambda_i A_j + A_i \lambda_j - iA_i A_j)\theta^i \theta^j = (\lambda_1 A_2 - q^{-2} A_1 \lambda_1 - q^{-2} \lambda_2 A_1 + A_1 \lambda_2 - iA_1 A_2 + iq^{-2} A_2 A_1)\theta \bar{\theta}
\]  
(61)
where $\lambda_i = (\lambda_z, \lambda_{\bar{z}})$. Notice that this is written in terms of the components of the frame, not of the differentials $dz, d\bar{z}$. In order to understand its classical limit, it is better to write\(^5\)
\[ A = \tilde{A}_z dz + \tilde{A}_{\bar{z}} d\bar{z}, \]  
(62)
and we recover from (59) the classical field strength
\[ F \xrightarrow{q \to 1} (\partial_z \tilde{A}_{\bar{z}} - \partial_{\bar{z}} \tilde{A}_z) dz d\bar{z}. \]  
(63)
In order to write down a Lagrangian for a Yang-Mills theory, we also need the Hodge dual $*_{H} F$ of $F$. This is easy to find: since any two-form $F$ can be written as
\[ F = f \theta \bar{\theta} = q^{-2} f dz d\bar{z} \]
for some function $f$, we define $*_{H}$ on two-forms as
\[ *_{H} F := \frac{1}{2} f. \]  
(64)
This is the correct definition because $dz d\bar{z}$ is invariant under $U_q(e(2))$ transformations, hence the Hodge dual satisfies
\[ (*_{H} F) \triangleleft u = *_{H} (F \triangleleft u) \]  
(65)
\(^5\)This is not natural for $q \neq 1$, since then $dz, d\bar{z}$ do not commute with functions.
for all \( u \in U_q(e(2)) \). We can now write down the following action using one of the invariant integrals found in Section 3.11

\[
S := \int q F(*_H F) \bar{z}^{-1} z^{-1} = \int q \frac{1}{2} f^2 \bar{z}^{-1} z^{-1} \theta \bar{\theta}.
\]  

(66)

The factor \( \bar{z}^{-1} z^{-1} \) is required by gauge invariance under (57), using the property

\[
\int q g \bar{z}^{-1} z^{-1} = \int q g f \bar{z}^{-1} z^{-1}
\]

which follows from Lemma 3. In the classical limit we obtain

\[
S \xrightarrow{q \to 1} \int \frac{1}{2} (\partial_z A_\bar{z} - \partial_{\bar{z}} A_z)^2 \bar{z}^{-1} z^{-1} dzd\bar{z}.
\]

The “measure factor” \( \bar{z}^{-1} z^{-1} \) breaks the \( E_q(2) \)-invariance explicitly. Unfortunately, it is required by gauge invariance. In other words, the invariant integral seems incompatible with this kind of gauge invariance, and one is faced with the choice of giving up either gauge invariance or \( E_q(2) \)-invariance\(^6\). In this paper, we will insist on gauge invariance.

There are several possibilities how this problem might be avoided. One may try to modify the gauge transformation, e.g. by using some kind of \( q \)-deformed gauge invariance as in [17]. Unfortunately we were not able to find a satisfactory prescription here [18]. Alternatively, we will propose in the next section a mechanism using spontaneous symmetry breaking, which yields an \( E_q(2) \)-invariant action for low energies. In any case, the above action is certainly appealing because of its simplicity, and the gauge transformations (66) are very natural. This problem may also be a hint that the quantum group spacetime-symmetry has not been correctly implemented in the field theory, beyond a formal level. A proper treatment would presumably require a second quantization, such that the \( E_q(2) \)-symmetry acts on a many-particle Hilbert space and the quantum fields, as in [19].

Let us briefly discuss the critical points of the above action. The absolute minima are given by solutions of the zero curvature condition \( F = 0 \). In terms of the coordinates \( D = D_i \theta^i \) this leads to

\[
D_2 D_1 = q^2 D_1 D_2.
\]

This is the defining relation of the deformed Euclidean plane with opposite multiplication. One solution is of course \( D = \Theta \), and we get all possible solutions in terms of the automorphisms of \( \mathbb{R}^2 \).

5 Restoring \( E_q(2) \)-invariance through spontaneous symmetry breaking

The explicit “weight” factor \( \bar{z}^{-1} z^{-1} \) in (66) is rather unwelcome, because it explicitly breaks the \( E_q(2) \)-invariance of the action, which was the starting point for our consider-

\( ^6 \)In the classical limit, the measure function can be written as \( \bar{z}^{-1} z^{-1} dzd\bar{z} = \frac{1}{r^2} (rdrd\phi) = d(lnr)d\phi \), which is the volume form on a cylinder. Therefore this action could be interpreted as Yang-Mills action on a quantum cylinder. However this is not the aim of this paper.
erations. One could in principle interpret it as some kind of additional “metric” term in the action, which is required by gauge invariance. However, it is also possible to cancel it by the vacuum-expectation value (VEV) of a suitable scalar field: Consider the action

\[ S_1 := \int^q F(*_HF) e^\phi \overline{z}^{-1} z^{-1}. \quad (68) \]

This is gauge invariant if \( \phi \) transforms in the adjoint:

\[ \phi \rightarrow i[A, \phi]. \quad (69) \]

We can then add an action for \( \phi \), such as

\[ S_2 = \int^q V(\phi) \overline{z}^{-1} z^{-1} \]

where \( V(x) \) is an ordinary function, which is again gauge invariant. If we could find a potential \( V(\phi) \) which has \( e^\phi = z\overline{z} \) as solution, we would obtain the following “low-energy” action

\[ S_1[A, \langle \phi \rangle] = \int^q F(*_HF) \]

replacing \( \phi \) by its VEV \( \langle \phi \rangle \). This is formally invariant under \( E_2(2) \), while the gauge invariance is spontaneously broken rather than explicitly. To find such a potential \( V \), consider the equation of motion

\[ \delta S_2[\phi] = \int^q \delta \phi V'(\phi) \overline{z}^{-1} z^{-1} = 0 \]

using the cyclic property of the integral, where \( V' \) denotes the ordinary derivative of the power series \( V(x) \). We therefore need a potential \( V(x) \) such that \( V'(\ln(z\overline{z})) = 0 \). For a given irrep \( L_{r_0} \) labeled by \( r_0 \) as in [39], the eigenvalues of \( z\overline{z} \) are \( r_0^2 q^{2n} = e^{2n \ln(q) + 2 \ln(r_0)} \) for \( n \in \mathbb{Z} \). Therefore

\[ V'_{r_0}(2n \ln(q) + 2 \ln(r_0)) = 0, \quad n \in \mathbb{Z}. \quad (73) \]

This certainly holds for \( V'_{r_0}(x) \propto \sin(2\pi \frac{x - 2 \ln(r_0)}{2 \ln(q)}) \), thus

\[ V_{r_0}(x) = -V_0 \cos(2\pi \frac{x - 2 \ln(r_0)}{2 \ln(q)}) \]

is a possible potential. Hence we will use the representation \( L_{r_0} \), and the quantum trace \( \int^{q,(r_0)} \) on \( L_{r_0} \) as invariant integral for the action. Note furthermore that

\[ \delta_\phi S_1 = 0 \]

for \( F = 0 \), therefore \( e^\phi = z\overline{z}, F = 0 \) is indeed a possible “vacuum” of the combined action

\[ S = S_1 + S_2 = \int^{q,(r_0)} \left( F(*_HF) e^\phi + V(\phi) \right) \overline{z}^{-1} z^{-1}. \]

\[ \text{(76)} \]
Replacing \( \phi \rightarrow \langle \phi \rangle = \ln(z \overline{z}) \), it reduces to
\[
\int^{q,(r_0)}_{q} F(\star_H F) + \text{const}, \tag{77}
\]
as desired. The fluctuations in \( \phi \) are suppressed if \( V_0 \) is chosen large enough. Of course there are other solutions for \( \phi \), which would give a nontrivial “effective metric” \( e^{(\phi)} \overline{z}^{-1} z^{-1} \) in the action. This is somewhat reminiscent of the low-energy effective actions in string theory, where the dilaton enters in a similar way.

For reducible representations of \( \mathbb{R}^2_q \) one could still find such potentials, but if we take continuous superpositions as in \( \text{(17)} \) in order to have the classical integral (as in the Seiberg-Witten approach below), this is no longer possible.

\section{Star Product Approach}

We now want to study gauge theory on \( \mathbb{R}^2_q \) using the star product approach, which was developed in \textit{[20, 16]}. We will denote classical variables on \( \mathbb{R}^2 \) by greek letters \( \zeta, \zeta \overline{\iota} \) in this section, in order to distinguish them from the generators \( z, \overline{z} \) of the algebra \( \mathbb{R}^2_q \).

A star product corresponding to \( \mathbb{R}^2_q \) is defined as the pull-back of the product in \( \mathbb{R}^2_q \) via an invertible map
\[
\rho : \mathbb{R}[[\zeta, \overline{\zeta}]] [[\hbar]] \rightarrow \mathbb{R}^2_q \tag{78}
\]
of vector spaces,
\[
f \star g := \rho^{-1}(\rho(f)\rho(g)), \tag{79}
\]
where
\[
q = e^h. \tag{80}
\]
For example, the star product corresponding to normal ordering in \( \mathbb{R}^2_q \) (i.e. commuting all \( z \) to the left and all \( \overline{z} \) to the right) reads \textit{[16]}
\[
f \star g = \mu \circ e^{-2h(\zeta \partial_{\zeta} \zeta \partial_{\zeta}) (f \otimes g)} . \tag{81}
\]
For our purpose the following star product will be more useful
\[
f \star_q g := \mu \circ e^{h(\zeta \partial_{\zeta} \zeta \partial_{\zeta}) + \overline{\zeta} \partial_{\overline{\zeta}} \overline{\zeta} \partial_{\overline{\zeta}}) (f \otimes g) = fg + h \zeta \overline{\zeta} (\partial_{\zeta} f \partial_{\overline{\zeta}} g - \partial_{\overline{\zeta}} f \partial_{\zeta} g) + \mathcal{O}(h^2), \tag{82}
\]
because it is hermitian, i.e. \( f \star_q g = g \star_q f \), and satisfies other nice properties as shown in Lemma \textit{[11]} (see equation \textit{(88)} below). The corresponding Poisson structure reads\textsuperscript{7}
\[
\theta^{ij} = -2i \zeta \overline{\zeta} \epsilon^{ij}. \tag{83}
\]
This star product is equivalent to the normal ordered one \textit{[81]} via the equivalence transformation
\[
T := e^{-h \zeta \partial_{\zeta} \zeta \partial_{\zeta}}.
\]
\textsuperscript{7}The Poisson structure is given by \([f \star_q g] = i\hbar \theta^{ij} \partial_i f \partial_j g + O(h^2)\).

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To see this, we first note that
\[ \zeta \partial_\zeta \zeta \partial_\zeta \circ \mu = \mu \circ (\zeta \partial_\zeta \zeta \partial_\zeta \otimes id + id \otimes \zeta \partial_\zeta \zeta \partial_\zeta + \zeta \partial_\zeta \otimes \zeta \partial_\zeta \otimes \zeta \partial_\zeta \otimes \zeta \partial_\zeta). \]

This leads to
\[
T(f *_q g) = e^{-h \zeta \partial_\zeta \zeta \partial_\zeta} \circ \mu \circ e^{h (\zeta \partial_\zeta \zeta \partial_\zeta \otimes \zeta \partial_\zeta \otimes \zeta \partial_\zeta)} (f \otimes g)
\]
\[
= \mu \circ e^{-h (\zeta \partial_\zeta \zeta \partial_\zeta \otimes id + id \otimes \zeta \partial_\zeta \zeta \partial_\zeta + \zeta \partial_\zeta \otimes \zeta \partial_\zeta \otimes \zeta \partial_\zeta)} \circ e^{h (\zeta \partial_\zeta \zeta \partial_\zeta \otimes \zeta \partial_\zeta \otimes \zeta \partial_\zeta \otimes \zeta \partial_\zeta)} (f \otimes g)
\]
\[
= \mu \circ e^{-2h (\zeta \partial_\zeta \otimes \zeta \partial_\zeta)} (e^{-h \zeta \partial_\zeta \zeta \partial_\zeta} f \otimes e^{-h \zeta \partial_\zeta \zeta \partial_\zeta} g)
\]
\[
= T(f) *_n T(g),
\]

hence \( T \) is indeed an equivalence transformation from \(*_n\) to \(*_q\). If we denote the normal ordering by \( \rho_n \), this new star product can be obtained by \( f *_q g := \rho_q^{-1}(\rho_q(f)\rho_q(g)) \) in terms of an “ordering prescription” \( \rho_q \) given by
\[
\rho_q := \rho_n \circ T.
\]

For illustration we give the image of \( \rho_q \) of some simple polynomials:
\[
\begin{align*}
\zeta^n & \mapsto z^n \\
\zeta & \mapsto z \\
(\zeta \bar{\zeta})^n & \mapsto q^{-n}(z \bar{z})^n.
\end{align*}
\]

Moreover, \(*_q\) is compatible with \( J \):
\[
(f *_q g) \circ J = (f \circ J) *_q g + f *_q (g \circ J)
\]
where the action of \( J \) on \( \mathbb{R}^2 \) is the obvious one.

One can easily extend the star product formalism to include differential forms, which will be useful in Section 6.3. We simply use the invertible map
\[
\begin{align*}
\Omega^* & \rightarrow \Omega_q^* , \\
f = f(\zeta, \bar{\zeta}) & \mapsto \rho_q(f) \\
\zeta^{-1} \theta d\zeta & \mapsto \theta \\
\zeta \bar{\zeta}^{-1} d\bar{\zeta} & \mapsto \bar{\theta}
\end{align*}
\]
(extended in the obvious way) from the differential forms on \( \mathbb{R}^2 \) to the calculus \( \Omega^*_q \) defined in Section 2.3 and define the “star-wedge” \( \wedge_q \) on \( \Omega^* \) as the pull-back of \( \Omega^*_q \). Using the same notation \( \theta = \zeta^{-1} \theta d\zeta, \bar{\theta} = \zeta \bar{\zeta}^{-1} d\bar{\zeta} \) as in the noncommutative algebra, one has for example \( \theta \wedge_q \bar{\theta} = -q^2 \bar{\theta} \wedge \theta \) in \( \Omega^* \), as in \( \Omega^*_q \). Clearly \( \theta \wedge_q f = f \wedge_q \theta \) in self-explanatory notation, and we will omit the star in this case from now on.
6.1 $E_q(2)$-invariance of the Riemann integral

Since there exists an integral on the commutative space, it is natural to use the isomorphism $\rho$ corresponding to the star product, and define

$$\int^\rho f(z, \overline{z}) := \int \rho^{-1}(f)(\zeta, \overline{\zeta})d\zeta d\overline{\zeta}.$$  \hspace{1cm} (87)

In general, one should not expect that the integral defined in this way is invariant under $E_q(2)$. Nevertheless, for the star product $*_q$ defined by $\rho_q$, this integral is indeed invariant, i.e. \footnote{10} is satisfied. We want to explain this in detail. Consider

$$f(z, \overline{z}) = \sum_{n=-\infty}^{\infty} z^n f_n(z\overline{z}) \in \mathbb{R}_q^2.$$ 

Applying $\rho_q^{-1}$ gives

$$\rho_q^{-1}(f) = \sum_{n=-\infty}^{\infty} \zeta^n *_q \rho_q^{-1}(f_n(r^2)).$$

On the other hand, we can write the function $\rho_q^{-1}(f)$ in polar coordinates, and expand it in a Fourier series with $r$-dependent coefficients

$$\rho_q^{-1}(f) = \sum_{n=-\infty}^{\infty} e^{i n \phi} a_n(r).$$

Then

$$a_0(r) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \rho_q^{-1}(f)(\phi, r).$$

Since $\zeta = re^{i\phi}$ and $\rho_q^{-1}(f_n(r^2)) = f_n(qr^2)$ is a function of $r^2$ by \footnote{10} and using the fact \footnote{10} that $*_q$ is compatible with $J$, it follows that

$$a_0(r) = \rho_q^{-1}(f_0)(r^2) = f_0(qr^2).$$

Therefore

$$\int \rho_q^{-1}(f)(\zeta, \overline{\zeta})d\zeta d\overline{\zeta} = 2\pi \int dr f_0(qr^2).$$

This agrees essentially with \footnote{17}, which is indeed invariant under $U_q(e(2))$ transformations as was shown there.

From now on, we will use the Riemann integral \footnote{17} in this context, and omit the superscript $\rho = \rho_q$ for brevity.

6.2 Trace property and measure

The Riemann integral does not possess the trace property, i.e. star multiplication is not commutative under the integral. However the trace property is necessary to obtain a gauge invariant action. We therefore look for a measure $\mu(\zeta, \overline{\zeta})$ such that

$$\int f(\zeta, \overline{\zeta}) *_q g(\zeta, \overline{\zeta}) \mu(\zeta, \overline{\zeta})d\zeta d\overline{\zeta} = \int g(\zeta, \overline{\zeta}) *_q f(\zeta, \overline{\zeta}) \mu(\zeta, \overline{\zeta})d\zeta d\overline{\zeta}.$$
Such a measure function can indeed be found.

**Lemma 4.** Let \( f, g \) be two arbitrary functions which vanish sufficiently fast at infinity. Then

\[
\int f(\zeta, \bar{\zeta}) \ast_q g(\zeta, \bar{\zeta}) \frac{1}{\zeta \bar{\zeta}} d\zeta d\bar{\zeta} = \int g(\zeta, \bar{\zeta}) \ast_q f(\zeta, \bar{\zeta}) \frac{1}{\zeta \bar{\zeta}} d\zeta d\bar{\zeta} = \int f(\zeta, \bar{\zeta}) g(\zeta, \bar{\zeta}) \frac{1}{\zeta \bar{\zeta}} d\zeta d\bar{\zeta}. \tag{88}
\]

**Proof.** See Appendix A.4 \hfill \Box

Equation (88) has also an analog on the canonical quantum plane \( \mathbb{R}^2_q \), see e.g. [1].

A small puzzle arises here: since the Riemannian integral is invariant under \( E_q(2) \) as we argued above, we also have the following cyclic property

\[
\int f(\zeta, \bar{\zeta}) \ast_q g(\zeta, \bar{\zeta}) \ast_q \bar{\zeta}^{-1} \ast_q \zeta^{-1} d\zeta d\bar{\zeta} = \int g(\zeta, \bar{\zeta}) \ast_q f(\zeta, \bar{\zeta}) \ast_q \bar{\zeta}^{-1} \ast_q \zeta^{-1} d\zeta d\bar{\zeta} \tag{89}
\]

because of Lemma 3. These two cyclic properties are in fact equivalent, because

\[
\int G(\zeta, \bar{\zeta}) \ast_q (\bar{\zeta}^{-1} \ast_q \zeta^{-1}) d\zeta d\bar{\zeta} = q^{-1} \int G(\zeta, \bar{\zeta}) \frac{1}{\zeta \bar{\zeta}} d\zeta d\bar{\zeta}. \tag{90}
\]

To see this, note that the second equality in (88) implies

\[
\int G(\zeta, \bar{\zeta}) \ast_q \bar{\zeta}^{-1} \ast_q \zeta^{-1} d\zeta d\bar{\zeta} = \int ((G(\zeta, \bar{\zeta}) \ast_q \bar{\zeta}^{-1} \ast_q \zeta^{-1}) \ast_q \zeta \bar{\zeta}) \frac{1}{\zeta \bar{\zeta}} d\zeta d\bar{\zeta}. \tag{91}
\]

With \( \zeta \bar{\zeta} = q^{-1} \zeta \ast_q \bar{\zeta} \) which is easy to verify, it follows that

\[
\int G(\zeta, \bar{\zeta}) \ast_q \bar{\zeta}^{-1} \ast_q \zeta^{-1} d\zeta d\bar{\zeta} = q^{-1} \int G(\zeta, \bar{\zeta}) \frac{1}{\zeta \bar{\zeta}} d\zeta d\bar{\zeta} \tag{92}
\]

using the associativity of the star product. This shows the equivalence of the cyclic properties (88) and (89).

### 6.3 Seiberg-Witten map

The map \( \rho_q \) defines a one-to-one correspondence between noncommutative and commutative functions, and we can identify \( f \) with \( \rho_q^{-1}(f) \). We construct a Seiberg-Witten map for the noncommutative fields expressing them by their commutative counterparts [4]:

\[
\Lambda = \Lambda_\alpha[a_i]
\]

\[
A_i = A_i[a_i]
\]

\[
\Psi = \Psi[\psi, a_i]
\]

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Here $a_i$ is the classical gauge field, $\alpha$ the classical gauge parameter and $\psi$ a classical matter field. The noncommutative gauge transformations are defined as in Section 4 and will be spelled out below. We assume that it is possible to expand in orders of $h$

$$\Lambda_\alpha[a_i] = \alpha + h\Lambda_\alpha^1[a_i] + h^2\Lambda_\alpha^2[a_i] + \ldots$$

$$A_i[a_i] = A_i^0 + hA_i^1[a_i] + h^2A_i^2[a_i] + \ldots \quad (93)$$

$$\Psi[\psi, a_i] = \psi + h\Psi^1[\psi, a_i] + h^2\Psi^2[\psi, a_i] + \ldots .$$

The explicit dependence on the commutative fields can be obtained by requiring the following consistency condition [16]

$$(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\Psi = \delta_{-i[\alpha, \beta]}\Psi$$

$$\Leftrightarrow i\delta_\alpha \Lambda_\beta - i\delta_\beta \Lambda_\alpha + [\Lambda_\alpha, ^* \Lambda_\beta] = i\Lambda_{-i[\alpha, \beta]}, \quad (94)$$

which amounts to requiring that the noncommutative gauge transformations are induced by the commutative gauge transformations of the commutative fields:

$$A_i[a_i] + \delta_\Lambda A_i[a_i] = A_i[a_i + \delta_\alpha a_i]$$

$$\Psi[\psi, a_i] + \delta_\Lambda \Psi[\psi, a_i] = \Psi[\psi + \delta_\alpha \psi, a_i + \delta_\alpha a_i].$$

The consistency condition has the well-known solution [21]

$$\Lambda_\alpha[a_i] = \alpha + h\frac{1}{2}\theta^{ij}\partial_i\alpha a_j + \mathcal{O}(h^2). \quad (96)$$

This solution is hermitian for real gauge parameters $\alpha$ and for gauge fields $a_i$ corresponding to the hermitian connection form $a = a^c d\zeta + a_\zeta d\tilde{\zeta}$. As usual, this solution is not unique. Solutions to the homogeneous part of the corresponding Seiberg-Witten equation may be added leading to field redefinitions [22].

The crucial point of our approach is that we will essentially work with 1-forms and their components $A_i$ w.r.t the frame $\theta^i = (\theta, \bar{\theta})$,

$$A = A_i \theta^i = \theta^i A_i = \tilde{A}_i dz^i, \quad (97)$$

and that we are gauging the one-form $\Theta$ as in Section 4. In this way we naturally obtain a noncommutative gauge field and field strength, with the correct classical limit. This is not the case if one introduces covariant coordinates to define gauge fields and field strengths [21] [18], because $\theta^{ij}$ is not constant here. Using $[\Theta, f] = df = [\lambda_i, f] \theta^i$, this led to the gauge transformation law in the noncommutative algebra

$$\delta_\Lambda A_i = [\lambda_i, \Lambda] + i[\Lambda, A_i] \quad (98)$$

where

$$\lambda_z = \frac{1}{1 - q^{-2} z^{-1}} \quad \text{and} \quad \lambda_{\bar{z}} = \frac{-1}{1 - q^{-2} \bar{z}^{-1}}.$$

Since the commutator with $\lambda_z$ satisfies the usual Leibniz rule we do not have to introduce a "vielbein" field that transforms under gauge transformations as in [3].
In order to translate the above gauge transformation law to the star product approach, we simply have to apply $\rho_q^{-1}$. This leads to

$$\delta_\Lambda A_i = [\lambda_i \ast_q \Lambda] + i[\Lambda \ast_q A_i] ,$$  \hspace{1cm} (99)

where we note that

$$\rho_q^{-1}(z_i^{-1}) = \zeta_i^{-1} .$$

Furthermore, we remark that

$$\frac{1}{1 - q^{-2}} = \frac{1}{2\hbar}(1 + h + \mathcal{O}(h^2))$$

such that to zeroth order we have for the gauge field

$$\delta_\alpha A^0_1 = \zeta\bar{\zeta}^{-1}\partial_\alpha \alpha .$$  \hspace{1cm} (100)

An analogous calculation for $A^0_2$ leads to the solution

$$A^0_i = c_i a_i ,$$  \hspace{1cm} (101)

where

$$c_\alpha = \zeta\bar{\zeta}^{-1} \hspace{1cm} \text{and} \hspace{1cm} c_\bar{\alpha} = \zeta^{-1}\bar{\zeta} .$$  \hspace{1cm} (102)

This is the solution for the gauge field, written in the basis $(\theta, \bar{\theta}) = (c_\alpha^{-1}d\zeta, c_\bar{\alpha}^{-1}d\bar{\zeta})$ of one-forms (cp. \textit{[27]}). To obtain the components in the more familiar basis $(d\zeta, d\bar{\zeta})$ we have to multiply the above solution by $c_i^{-1}$, and we indeed obtain the classical gauge field $a_i$ in zeroth order:

$$\tilde{A}_i^0 = a_i .$$  \hspace{1cm} (103)

Defining $c_i =: \frac{1}{1 - q^{-2}}l_i$, i.e. $l_\zeta := \zeta^{-1}$ and $l_{\bar{\zeta}} := -\zeta^{-1}$, we obtain to first order the equation

$$\delta_\alpha A^1_i = \frac{1}{2}\theta^{kl}\partial_k l_i \Lambda_\alpha^1 - \theta^{kl}\partial_k \alpha \partial_l(c_i a_i) + \frac{1}{2}\theta^{kl}\partial_k l_i \partial_l \alpha ,$$  \hspace{1cm} (104)

which admits the solution

$$A^1_i = c_i\left(\frac{-1}{2}\theta^{kl}a_k(\partial_i a_i + F^0_{ij})\right) - \frac{1}{2}\theta^{kl}a_k \partial_l(c_i a_i) + c_i a_i ,$$  \hspace{1cm} (105)

where

$$F^0_{ij} := \partial_i a_j - \partial_j a_i$$

is the usual, commutative field strength. This solution satisfies $\tilde{A}_i^1 = A^1_2, A^2_2 = A^1_1$.

We now define the noncommutative field strength as in Section \textit{[4]}

$$F = (\lambda_i \ast_q A_j + A_i \ast_q \lambda_j - iA_i \ast_q A_j)\theta^i \wedge_q \theta^j = f \theta \wedge_q \bar{\theta}$$  \hspace{1cm} (106)

using the “star-calculus” defined by \textit{[80]}, because it satisfies the correct transformation law

$$\delta f = i[\Lambda \ast_q f] .$$  \hspace{1cm} (107)
The above solution then leads to
\[
f = F^0_{12} + h \left\{ F^0_{12} + \Theta^{12}(F^0_{12} F^0_{12} - a_\zeta \partial_\zeta F^0_{12} + a_\zeta \partial_\zeta a_\zeta + a_\bar{\zeta} \partial_{\bar{\zeta}} a_{\bar{\zeta}} + 2 a_\zeta \partial_\zeta a_{\bar{\zeta}}) + \Theta^{12}(a_\zeta \partial_\zeta a_\zeta + a_\zeta \partial_\zeta a_\zeta + a_\bar{\zeta} \partial_{\bar{\zeta}} a_{\bar{\zeta}} + 2 a_\zeta \partial_\zeta a_{\bar{\zeta}}) \right\} + \mathcal{O}(h^2).
\]  

We can now write down the following action using the classical integral:
\[
S := \frac{1}{2} \int f \star_q f \frac{1}{\zeta d\zeta d\bar{\zeta}}.
\]  

Recall that the measure function \( \mu(\zeta, \bar{\zeta}) = \frac{1}{\zeta d\zeta d\bar{\zeta}} \) is necessary to ensure gauge invariance of the action, using the trace property of the integral by Lemma 1. This action can be written in terms of commutative fields using the above result:
\[
S = \int d\zeta d\bar{\zeta} \frac{1}{\zeta d\zeta d\bar{\zeta}} \left\{ \frac{1}{2} F^0_{12} F^0_{12} + h \left(F^0_{12} F^0_{12} + \Theta^{12}(F^0_{12} F^0_{12} - a_\zeta \partial_\zeta F^0_{12} + a_\zeta \partial_\zeta a_\zeta + a_\bar{\zeta} \partial_{\bar{\zeta}} a_{\bar{\zeta}} + a_\zeta \partial_\zeta a_{\bar{\zeta}} + a_\bar{\zeta} \partial_{\bar{\zeta}} a_{\zeta}) + \Theta^{12}(a_\zeta \partial_\zeta a_\zeta + a_\zeta \partial_\zeta a_\zeta + a_\bar{\zeta} \partial_{\bar{\zeta}} a_{\bar{\zeta}} + 2 a_\zeta \partial_\zeta a_{\bar{\zeta}}) \right) \right\} + \mathcal{O}(h^2).
\]

Observe that this action is also the Seiberg-Witten form of (60), because
\[
S = \frac{1}{2} \int f \star_q f \frac{1}{\zeta d\zeta d\bar{\zeta}} = \frac{q}{2} \int f \star_q f \star_q (\bar{\zeta}^{-1} \star_q \zeta^{-1}) d\zeta d\bar{\zeta}
\]

using (90). We see that as in the algebraic approach of Section 4, gauge invariance requires a measure function \( \mu(\zeta, \bar{\zeta}) = \frac{1}{\zeta d\zeta d\bar{\zeta}} \) which breaks translation invariance. However, one should realize that even without this measure function, this “classical” action would not be invariant under \( E(2) \), because the star product is not compatible with the symmetry (only for rotations (85) holds). This would only be the case if one could find a star product on \( \mathbb{R}^q \) which is compatible with the coproduct of \( E_q(2) \), c.p. 23, 13.

### 6.4 The classical limit and the measure function

The measure function \( \mu(\zeta, \bar{\zeta}) = \frac{1}{\zeta d\zeta d\bar{\zeta}} \) survives in the classical limit \( q \to 1 \). If we want a deformation of the classical theory, this should not be the case. We therefore would like to get rid of this measure function in the classical limit. This can be achieved by multiplying the action with a gauge-covariant expression, which in the classical limit exactly cancels the measure function \( \mu \). For this purpose we introduce covariant coordinates (10):
\[
Z_i := \zeta_i + A_i.
\]

Here \( A_i \) should not be confused with \( A_i \). The one-form \( A_i d^i \) is a noncommutative analog of the classical gauge field, because its gauge transformation law (99) is the noncommutative generalization of the classical gauge transformation law. Indeed, we

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8This was suggested by Peter Schupp.
recovered the classical gauge field $a_i$ with respect to the basis $d\zeta, d\bar{\zeta}$ in zeroth order of $h$. In contrast, the covariant coordinates are used here just as a quantity which transforms covariantly and reduces to the usual coordinates in the classical limit, in order to cancel the measure function. We will see that $A_i$ does not reduce to the classical gauge field for $q \to 1$. Requiring the covariant transformation rule $\delta Z_i = i[A \ast^q Z_i]$ leads to the following gauge transformation rule for $A_i$

$$\delta A_i = i[\zeta_i \ast^q \Lambda] + i[\Lambda \ast^q A_i]. \tag{113}$$

As before we can express $A_i$ in terms of commutative fields by solving the corresponding Seiberg-Witten equations. This gives \textcite{21}

$$A^i = h \theta^{ij} a_j + h^2 \frac{1}{2} \theta^{kl} a_i (\partial_k (\theta^{ij} a_j) - \theta^{ij} F^0_{jk}) + O(h^3). \tag{114}$$

In principle, covariant coordinates may be used to define noncommutative gauge fields and covariant expressions such as field strength \textcite{16,18}. However, the above equation shows that gauge fields and field strengths defined in that way do not lead to the classical gauge field $a_i$ and field strength $F^0_{ij}$ in the limit $h \to 0$ whenever the Poisson-structure is not constant and not invertible, as is the case here\textsuperscript{9}. Nevertheless they are a convenient tool for our purpose, because they satisfy

$$Z \ast_q \bar{Z} \to \zeta \bar{\zeta} \tag{115}$$

for $q \to 1$, and

$$\delta (Z \ast_q \bar{Z}) = i[\Lambda \ast_q Z \ast_q \bar{Z}]. \tag{116}$$

Now we can define a gauge-invariant action with the correct classical limit:

$$S' := \frac{1}{2} \int f \ast_q f \ast_q Z \ast_q \bar{Z} \frac{1}{\zeta \bar{\zeta}} d\zeta d\bar{\zeta}. \tag{117}$$

Expanded up to first order of $h$ we obtain

$$S' = \int d\zeta d\bar{\zeta} \frac{1}{2} F^0_{12} F^0_{12} + h (F^0_{12} F^0_{12} + \theta^{12} (F^0_{12} F^0_{12} F^0_{12} F^0_{12} a_\zeta F^0_{12} - \zeta F^0_{12} \partial_\zeta F^0_{12} + a_\bar{\zeta} F^0_{12} \partial_{\bar{\zeta}} F^0_{12}))$$

$$+ \partial_{\zeta} \theta^{12} F^0_{12} (2 a_\zeta \partial_{\zeta} a_\zeta + a_\zeta \partial_{\bar{\zeta}} a_{\bar{\zeta}} + a_{\bar{\zeta}} \partial_{\zeta} a_\zeta)$$

$$+ \partial_{\bar{\zeta}} \theta^{12} F^0_{12} (2 a_{\bar{\zeta}} \partial_{\bar{\zeta}} a_{\bar{\zeta}} + a_\zeta \partial_{\zeta} a_{\bar{\zeta}} + a_{\bar{\zeta}} \partial_{\bar{\zeta}} a_\zeta)$$

$$+ \frac{1}{\zeta \bar{\zeta}} \theta^{12} F^0_{12} (\zeta a_{\bar{\zeta}} - \zeta a_\zeta) - F^0_{12} F^0_{12} + \zeta \partial_\zeta (F^0_{12} F^0_{12} - \zeta \partial_\zeta (F^0_{12} F^0_{12})) + O(h^2). \tag{118}$$

This reduces indeed to a Yang-Mills theory in the classical limit. However, choosing $Z \ast_q \bar{Z}$ is only one possibility to cancel $\frac{1}{\zeta \bar{\zeta}}$. There are other expressions which are gauge-covariant, and lead to the same classical limit. Our choice is motivated by simplicity.

\textsuperscript{9}To obtain in the classical limit the classical gauge field $a_i$, we have to invert $\theta^{ij}$ and write $\frac{1}{h} \theta_{ij}^{-1} A^i$. This is only defined if $\theta$ is invertible, and even then it spoils the covariant transformation property whenever $\theta$ is not constant. To maintain covariance one has to "invert $\theta$ covariantly" as done in \textcite{18}, leading to complicated expressions. The approach that we propose in Section \textcite{23} does not have these problems. Gauging the one-form $\Theta$ instead of the coordinates leads very naturally to a noncommutative gauge-field \textcite{39} and field strength \textcite{109}. Compare also with \textcite{3}, where a different approach using a "vielbein" is discussed.
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A Mathematical Appendix

A.1 Coaction and action

Definition 1. A left coaction of a Hopf algebra \( \mathcal{H} \) on an algebra \( \mathcal{A} \) is a linear mapping

\[
\rho : \mathcal{A} \rightarrow \mathcal{H} \otimes \mathcal{A}
\]

which satisfies

\[
(id \otimes \rho) \circ \rho = (\Delta \otimes \text{id}) \circ \rho, \quad (\varepsilon \otimes \text{id}) \circ \rho = \text{id}
\]

\[
\rho(ab) = \rho(a) \rho(b), \quad \rho(1) = 1 \otimes 1.
\]

In Sweedler notation, one writes

\[
\rho(a) =: a_{-1} \otimes a_0.
\]

\( \mathcal{A} \) is then called a left \( \mathcal{H} \)-comodule algebra.

Definition 2. A Hopf algebra \( \mathcal{H} \) is acting on an algebra \( \mathcal{A} \) from the right if \( \mathcal{A} \) if there is an action \( \triangleleft : \mathcal{A} \otimes \mathcal{H} \rightarrow \mathcal{A} \) which satisfies

\[
ab \triangleleft h = (a \otimes b) \triangleleft \Delta(h) = (a \triangleleft h_{(1)})(b \triangleleft h_{(2)}) \quad \text{and} \quad 1 \triangleleft h = \varepsilon(h)1
\]

for any \( h \in \mathcal{H} \) and \( a, b \in \mathcal{A} \). \( \mathcal{A} \) is then called a right \( \mathcal{H} \)-module algebra.

By (111), these two notions are dual to each other. There are obvious analogs replacing left with right everywhere.

For the action of \( J, T \) and \( \overline{T} \) on the generators \( z, \overline{z} \), we obtain

\[
\begin{align*}
    z \triangleleft T &= 1, & z \triangleleft \overline{T} &= 0, & z \triangleleft J &= i z \\
    \overline{z} \triangleleft T &= 0, & \overline{z} \triangleleft \overline{T} &= -q^2, & \overline{z} \triangleleft J &= -i \overline{z}.
\end{align*}
\]

The action on arbitrary functions is calculated in the following subsection.
A.2 The right action of $U_q(e(2))$ on $\mathbb{R}_q^2$

Knowing the structure maps for $J, T, \overline{T} \in U_q(e(2))$ and their action on $z, \overline{z}$ given above, we can determine the action of $J, T, \overline{T}$ on arbitrary functions using $(xy) \triangleleft U = (x \triangleleft U(y) \triangleleft U(x_2))$ for arbitrary $x, y \in \mathbb{R}_q^2, U \in U_q(e(2))$. Since an arbitrary function $f(z, \overline{z}) \in \mathbb{R}_q^2$ can be written as $f(z, \overline{z}) = \sum_{k \in \mathbb{Z}} z^k f_k(z, \overline{z})$, it is sufficient to know the action on the terms $z^k f(z, \overline{z})$,

where $f$ is a formal power series in $z, \overline{z}$. We will derive the formulas even for negative powers of $z, \overline{z}$, i.e. $f(z, \overline{z}) = \sum_{l \in \mathbb{Z}} a_l(z, \overline{z})^l$. We start with the action on $z^k$:

**Claim 1.** For $k \in \mathbb{Z}$ we have

$$z^k \triangleleft T = \frac{1 - q^{-2k}}{1 - q^{-2}} z^{k-1}$$

$$z^k \triangleleft \overline{T} = 0$$

$$z^k \triangleleft J = i^k z^k.$$  \hspace{1cm} (122)

**Proof.** The first equation can be shown by induction, using $z \triangleleft T = 1$ and $z^{-1} \triangleleft T = -q^2 z^{-2}$, which follows from

$$0 = 1 \triangleleft T = (z^{-1} z) \triangleleft T = (z^{-1} \triangleleft T)(z \triangleleft q^{2iJ}) + z^{-1}(z \triangleleft T) = (z^{-1} \triangleleft T)q^{-2} z + z^{-1}.$$  

The last two equations finally follow immediately with $z \triangleleft \overline{T} = 0$, $z \triangleleft J = iz$ and $\Delta(T) = T \otimes q^{2iJ} + 1 \otimes \overline{T}$.

The action on $f(z, \overline{z}) = \sum_{l \in \mathbb{Z}} a_l(z, \overline{z})^l$ follows from

**Claim 2.** For $l \in \mathbb{Z}$ we have

$$(z, \overline{z})^l \triangleleft T = q^2 \frac{1 - q^{-2l}}{1 - q^{-2}} (z, \overline{z})^{l-1} \overline{z}$$

$$(z, \overline{z})^l \triangleleft \overline{T} = -q^2 \frac{1 - q^{2l}}{1 - q^{2}} (z, \overline{z})^{l-1} z$$

$$(z, \overline{z})^l \triangleleft J = (z, \overline{z})^l.$$  \hspace{1cm} (123)

**Proof.** The last equation follows immediately with $z \triangleleft J = iz$, $\overline{z} \triangleleft J = -i\overline{z}$. The first equation follows again by induction, starting with $(z, \overline{z}) \triangleleft T = (z \triangleleft T)(\overline{z} \triangleleft q^{2iJ}) + z(\overline{z} \triangleleft T) = q^2 \overline{z}$, and concluding inductively

$$(z, \overline{z})^{l+1} \triangleleft T = ((z, \overline{z})^{l} \triangleleft T)((z, \overline{z}) \triangleleft q^{2iJ}) + (z, \overline{z})^{l}((z, \overline{z}) \triangleleft T) = q^2 \frac{1 - q^{-2l}}{1 - q^{-2}} (z, \overline{z})^{l-1} \overline{z}(z, \overline{z}) + (z, \overline{z})^{l}q^2 \overline{z}$$

$$= q^2 \frac{1 - q^{-2l-2}}{1 - q^{-2}} (z, \overline{z})^{l} \overline{z}$$

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for \( l > 0 \). If \( l = 0 \), then \( 1 \circ T = 0 \), which is consistent with the claim. To derive the action of \( T \) on \( (z\bar{z})^{-1} \) we calculate

\[
0 = ((z\bar{z})^{-1}(z\bar{z})) \circ T = ((z\bar{z})^{-1} \circ T)z\bar{z} + (z\bar{z})^{-1}((z\bar{z}) \circ T) = ((z\bar{z})^{-1} \circ T)z\bar{z} + (z\bar{z})^{-1}q^2z
\]

hence

\[
(z\bar{z})^{-1} \circ T = -(z\bar{z})^{-2}z,
\]

consistent with \([\ref{equation:theta}](\text{123})\). For \( l < 0 \) the claim follows similarly by induction, and the second equation follows also inductively.

Putting these results together and using \( f(z\bar{z}) = \sum_{i \in \mathbb{Z}} a_i(z\bar{z})^i \) we obtain

\[
z^k f(z\bar{z}) \circ T = (z^k \circ T)(f(z\bar{z}) \circ q^{2i}) + z^k(f(z\bar{z}) \circ T) = \frac{1 - q^{-2k}}{1 - q^{-2}}z^{k-1}f(z\bar{z}) + z^{k-1} \sum_{i \in \mathbb{Z}} a_i q^{1 - q^{-2}} q^{2(i-1)}(z\bar{z})^i
\]

\[
= \frac{z^{k-1}}{1 - q^{-2}}(f(q^2z\bar{z}) - q^{-2k}f(z\bar{z}))
\]

A similar calculation finally leads to \([\ref{equation:theta}](\text{14})\).

### A.3 Proof of Lemma \([\ref{lemma:commutation}]\)

**Proof.** Since the \( \theta^i \) commute with all functions, we have

\[
[\Theta, f] = \theta[\frac{1}{1 - q^{-2}z^{-1}}, f] - \bar{\theta}[\frac{1}{1 - q^{-2}z^{-1}}, f].
\]

Plugging in the explicit expressions \([\ref{equation:theta}](\text{27})\) for \( \theta^i \) we find

\[
[\Theta, f] = dz\bar{z}^{-1}[\frac{1}{1 - q^{-2}z^{-1}}, f] - dzz\bar{z}^{-1}[\frac{1}{1 - q^{-2}z^{-1}}, f],
\]

using the commutation relations \([\ref{equation:commutation}](\text{16})\). Taking \( f \equiv z \) and \( f \equiv \bar{z} \) we get

\[
z^{-1}[\frac{1}{1 - q^{-2}z^{-1}}, z] = \frac{1}{1 - q^{-2}} - \frac{q^{-2}}{1 - q^{-2}} = 1
\]

(124)

and

\[
z\bar{z}^{-1}[\frac{1}{1 - q^{-2}z^{-1}}, z] = 0.
\]

(125)

Thus \([\Theta, z] = dz\), and similarly \([\Theta, \bar{z}] = d\bar{z}\). Hence the claim is true on the generators of the algebra of functions, and since \([\Theta, \cdot] \) is a derivation we can conclude that

\[
df = [\Theta, f]
\]

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for all functions $f$.

To show $d\Theta = \Theta^2 = 0$, consider

\[
((1 - q^{-2})\Theta)^2 = (\theta \bar{z}^{-1} - \bar{\theta} z^{-1})^2 = (q^{-2}z^{-1}dz - \bar{z}^{-1}d\bar{z})^2
\]

\[
= -q^{-2}z^{-1}dzz^{-1}d\bar{z} - \bar{z}^{-1}d\bar{z}q^{-2}z^{-1}dz
\]

\[
= -q^{-4}z^{-1}\bar{z}^{-1}dzd\bar{z} - \bar{z}^{-1}z^{-1}d\bar{z}dz = 0,
\]

using the commutation relations (10), (16) and (18). Furthermore,

\[
(1 - q^{-2})d\Theta = d(q^{-2}z^{-1}dz - \bar{z}^{-1}d\bar{z}) = -q^{-4}z^{-2}dzd\bar{z} + q^2\bar{z}^{-2}d\bar{z}dz = 0
\]

where we used $d(z^{-1}) = -q^{-2}z^{-2}dz$ and $d(\bar{z}^{-1}) = -q^2\bar{z}^{-2}d\bar{z}$, which follows from the $q$–Leibniz rule applied to $0 = d1 = d(zz^{-1}) = d(zz^{-1})$.

\[
\square
\]

### A.4 Proof of Lemma 4

Proof. We have

\[
\int d\zeta d\bar{\zeta} f_q \star g = \int d\zeta d\bar{\zeta} \frac{\zeta \zeta}{\zeta} f + \int d\zeta d\bar{\zeta} \mu \circ \sum_{n=1}^{\infty} \frac{h^n}{n!} \left( \sum_{i_1,j_1=1}^{2} \epsilon_{i_1 j_1} \frac{\partial}{\partial \zeta_{i_1}} \otimes \frac{\partial}{\partial \zeta_{j_1}} \right)
\]

\[
\left( \sum_{i_2,j_2=1}^{2} \epsilon_{i_2 j_2} \frac{\partial}{\partial \zeta_{i_2}} \otimes \frac{\partial}{\partial \zeta_{j_2}} \right) \cdots \left( \sum_{i_n,j_n=1}^{2} \epsilon_{i_n j_n} \frac{\partial}{\partial \zeta_{i_n}} \otimes \frac{\partial}{\partial \zeta_{j_n}} \right) (f \otimes g).
\]

Consider the $n$-th term of the sum on the right hand side:

\[
\int d\zeta d\bar{\zeta} \frac{h^n}{n!} \mu \circ \left( \sum_{i_1,j_1=1}^{2} \epsilon_{i_1 j_1} \frac{\partial}{\partial \zeta_{i_1}} \otimes \frac{\partial}{\partial \zeta_{j_1}} \right) \left( \sum_{i_2,j_2=1}^{2} \epsilon_{i_2 j_2} \frac{\partial}{\partial \zeta_{i_2}} \otimes \frac{\partial}{\partial \zeta_{j_2}} \right) \cdots \left( \sum_{i_n,j_n=1}^{2} \epsilon_{i_n j_n} \frac{\partial}{\partial \zeta_{i_n}} \otimes \frac{\partial}{\partial \zeta_{j_n}} \right) (f \otimes g).
\]

Introducing the short hand notation

\[
f' \otimes g' := \left( \sum_{i_2,j_2=1}^{2} \epsilon_{i_2 j_2} \frac{\partial}{\partial \zeta_{i_2}} \otimes \frac{\partial}{\partial \zeta_{j_2}} \right) \cdots \left( \sum_{i_n,j_n=1}^{2} \epsilon_{i_n j_n} \frac{\partial}{\partial \zeta_{i_n}} \otimes \frac{\partial}{\partial \zeta_{j_n}} \right) (f \otimes g),
\]

the $n$–th term of the sum can be written as

\[
\int d\zeta d\bar{\zeta} \frac{h^n}{n!} \mu \circ \left( \sum_{i_1,j_1=1}^{2} \epsilon_{i_1 j_1} \frac{\partial}{\partial \zeta_{i_1}} \otimes \frac{\partial}{\partial \zeta_{j_1}} \right) (f' \otimes g').
\]
\[ \frac{h^n}{n!} \int d\zeta d\overline{\zeta} \sum_{i_1, j_1=1}^{2} \varepsilon_{i_1 j_1} \frac{\partial}{\partial \zeta_{i_1}} (f') \frac{\partial}{\partial \overline{\zeta}_{j_1}} (g'). \]

For \( n > 0 \), this leads after partial integration (assuming that the functions vanish at infinity) to

\[ -\frac{h^n}{n!} \int d\zeta d\overline{\zeta} \sum_{i_1, j_1=1}^{2} \varepsilon_{i_1 j_1} f' \frac{\partial}{\partial \zeta_{i_1}} \frac{\partial}{\partial \overline{\zeta}_{j_1}} (g') = 0. \]

This is valid for any summand corresponding to \( n > 0 \), so that only the zeroth order term does not vanish. Hence we find indeed

\[ \int \frac{d\zeta d\overline{\zeta}}{\zeta \overline{\zeta}} f \star_q g = \int \frac{d\zeta d\overline{\zeta}}{\zeta \overline{\zeta}} fg. \]

\[ \square \]

References


