The problem of entanglement produced by an arbitrary operator is formulated and a related measure of entanglement production is introduced. This measure of entanglement production satisfies all properties natural for such a characteristic. A particular case is the entanglement produced by a density operator or a density matrix. The suggested measure is valid for operations over pure states as well as over mixed states, for equilibrium as well as nonequilibrium processes. Systems of arbitrary nature can be treated, described either by field operators, spin operators, or any other kind of operators, which is realized by constructing generalized density matrices. The interplay between entanglement production and phase transitions in statistical systems is analysed by the examples of Bose-Einstein condensation, superconducting transition, and magnetic transitions. The relation between the measure of entanglement production and order indices is analysed.

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I. INTRODUCTION

Entanglement is the term used by Schrödinger with respect to the superposition principle applied to composite systems [1–3]. If two quantum particles have interacted, their state cannot be presented as a tensor product of single-particle states, but it is entangled, being a superposition of such products. The notion of entanglement is now at the heart of such interrelated intriguing problems as quantum measurement, quantum information processing, and quantum computing, which have been expounded in several books and reviews [4–12].

In the literature, one distinguishes between the entanglement of quantum states and the entanglement produced by quantum operators over unentangled wave functions. Both these types of entanglement, to be well defined, require the knowledge of quantifying characteristics. In the present paper, as follows from its title, we shall consider only the second type of entanglement produced by quantum operations. In quantifying the latter, one usually envokes the characteristics measuring the entanglement of states. Because of this, it is useful to briefly mention the problem of quantifying the entanglement of quantum states, which, at the same time, would make clearer the difference between these two types of entanglement.

Quantifying entanglement of quantum states, one usually deals with bipartite systems. Several ways of measuring entanglement in such systems have been suggested, the methods being based on the notions of either reduced or relative entropy. Thus, mutual information is a linear combination of the von Neumann reduced entropies [13,14]. The reduced entropies themselves define the entropy of entanglement, which serves as a measure for entanglement of formation [15,16]. Another measure of entanglement is defined by minimizing over disentangled states the Kullback-Leibler distance, yielding the relative entropy of entanglement [17–19]. A measure, not envolving the notion of entropy, could be introduced as the number of maximally entangled pairs that can be purified from a given state, which results in entanglement of distillation [15]. However, this measure depends on the particular process of purification, and it is not clear how to compute it in an efficient and unique way [17–19].

Entanglement of formation was also employed in the attempt of characterizing the entanglement for mixed states [20], but it was concluded that this way did not uniquely define mixed-state entanglement [20,21]. A suggestion for measuring covariance entanglement in bipartite systems by means of correlation functions squared was advanced in [22]. However, such a measure, as is accepted by the authors themselves, does not possess all necessary properties [8–12] to be considered as really a measure. One often connects the existence of entanglement with the violation of the Bell inequalities, which can be formulated for bipartite systems with both orthogonal and nonorthogonal states [23]. This point of view is based on the Gisin theorem [24], according to which any pure entangled state of two particles violates a Bell inequality for two-particle correlation functions. Nevertheless, there exist pure entangled $N > 2$ qubit states that do not violate any Bell inequality for $N$-particle correlation functions [25]. It seems that correlation functions do not provide the best tool for characterizing entanglement. At present, there is no such a
general definition of entanglement measure that would be valid for bipartite as well as multipartite systems, for pure as well as mixed states, for equilibrium as well as nonequilibrium processes.

Another problem is to quantify the entanglement produced by quantum operations on a given set of disentangled functions. The so produced entanglement is termed as entangling power, entanglement capacity, entanglement of evolution, entanglement generation, or entanglement production [26–30]. In what follows, we shall employ the term *entanglement production* as the most closely related to the meaning of this notion, being the entanglement produced by an operator. To quantify this type of entanglement, one usually resorts to the combination of measures defined for the entanglement of states. Since the latter are well defined only for pure bipartite systems, the entanglement production is also usually considered for such systems.

The aim of the present paper is to introduce a general *measure of entanglement production*, which could quantify the amount of entanglement produced by an arbitrary operator on a given disentangled set. The suggested approach can be applied to operators of any nature and to any physical systems, whether pure or mixed, bipartite or multipartite, equilibrium or not. Being justified for arbitrary operators, the approach can straightforwardly be applied to a particular kind of operators, as statistical or density operators.

II. DISENTANGLED AND ENTANGLED FUNCTIONS

First of all, it is necessary to give correct mathematical definitions for the notions of entangled or, conversely, disentangled functions and to specify the notations to be employed in what follows. From the very beginning, a multipartite composite system is kept in mind, consisting of an arbitrary number of subsystems enumerated by the index \( i = 1, 2, \ldots, p \), where \( p = 1, 2, \ldots \) can be any integer. Subsystems are treated as indivisible parts, because of which they could equivalently be called particles. These can be distinguishable or indistinguishable, which does not change the general mathematical structure, provided an appropriate labelling of partite states is employed, e.g., by means of collective mode labels or occupation numbers [31–33].

The *space of single-partite states* for each \( i \)-part is presented by the Hilbert space

\[
\mathcal{H}_i \equiv \mathbb{C}\{|n_i>\},
\]
being a closed linear envelope of a basis \( \{|n_i>\} \) composed of orthonormalized single-partite states \( |n_i> \). Hence, any vector \( \varphi_i \in \mathcal{H}_i \) can be expanded over the basis \( \{|n_i>\} \) as

\[
\varphi_i = \sum_{n_i} a_{n_i} |n_i>.
\]

The nature of the labels \( n_i \) here is of no importance. The vector norm

\[
||\varphi_i||_{\mathcal{H}_i} \equiv \sqrt{(\varphi_i, \varphi_i)}
\]
in \( \mathcal{H}_i \) is defined through the associated scalar product \((\varphi_i, \varphi_i)\).

It is worth emphasizing that \( |n> \) does not compulsorily mean a quantum state of a given physical particle. This case is not excluded if particles are distinguishable. But, if one deals with a system of indistinguishable particles, \( |n> \) should be understood as a single-particle mode.

The *space of composite-system states* is given by the \( p \)-fold tensor product

\[
\mathcal{H} \equiv \otimes_{i=1}^{p} \mathcal{H}_i,
\]
which is identified [34] with the closed linear envelope

\[
\mathcal{H} \equiv \mathbb{C}\{|n_1 \ldots n_p>\}
\]
over a normalized \( p \)-particle basis \( \{|n_1 \ldots n_p>\} \). The latter may be written as the tensor product

\[
|n_1 \ldots n_p> = \otimes_{i=1}^{p} |n_i>
\]
of the single-particle basis states. Any function \( \varphi \in \mathcal{H} \) can be presented as the sum

\[
\varphi = \sum_{\{n_i\}} c_{n_1 \ldots n_p} |n_1 \ldots n_p>
\]
over the multiparticle basis \(|n_1 \ldots n_p\rangle\), where \(|n_i\rangle = n_1, n_2, \ldots, n_p\). The vector norm \(||\varphi||_H = \sqrt{\langle \varphi, \varphi \rangle}\) in \(H\) is generated by the related scalar product \(\langle \varphi, \varphi \rangle\).

Note that it is not compulsory to deal with the total tensor-product space (3). In some cases, because of physical restrictions, additional selection rules may be superimposed on the admissible states of \(H\). This is, e.g., the case of systems composed of identical particles, whose quantum states are to be either symmetrized or antisymmetrized, according to whether the particles are bosons or fermions. Then the space of admissible states is reduced to a subspace of space (3). In what follows, we shall keep in mind the possibility of such additional restriction rules. For short, according to whether the particles are bosons or fermions. Then the space of admissible states is reduced to a subspace of systems composed of identical particles, whose quantum states are to be either symmetrized or antisymmetrized, restrictions, additional selection rules may be superimposed on the admissible states of \(H\).

We shall continue denoting by \(D\) the vector norm \(|\varphi_i||_{H_i}\).

The compliment \(\mathcal{H} \setminus D\) to \(D\) forms the set of entangled states.

To illustrate in an explicit way the principal difference of disentangled states from entangled ones, let us consider a bipartite system, with \(p = 2\) and let the single-particle states be two-dimensional. Then, writing, for compactness, 1 and 2 instead of \(n_1\) and \(n_2\), for state (6), we have

\[
\varphi = c_{11}|11\rangle + c_{12}|12\rangle + c_{21}|21\rangle + c_{22}|22\rangle,
\]

while for the factor state (8), we get

\[
f = a_1b_1|11\rangle + a_1b_2|12\rangle + a_2b_1|21\rangle + a_2b_2|22\rangle.
\]

As is evident, the state \(\varphi \in \mathcal{H}\) is more general than \(f \in D\). The space \(\mathcal{H}\) contains the entangled states, such as \(c_{12}|12\rangle + c_{21}|21\rangle \) or \(c_{11}|11\rangle + c_{22}|22\rangle\) that in no way can be reduced to the factor states \(f \in D\). In general, no entangled state can be presented as a product of single-particle states.

### III. Measure of Entanglement Production

The multiparticle space (3) contains both entangled and factor states. For the time being, we follow the abstract terminology of Sec. II, implying no physical applications that will be treated later. An abstract mathematical level of consideration provides the best way for making transparent what actually is entanglement production and how to measure it.

The term entanglement production as such means an action that transforms disentangled states into entangled ones. A transformation can be described by the action of an operator. Thus, one may investigate entanglement produced by different operators. Let \(A\) be an arbitrary linear bounded operator acting on the tensor-product space \(\mathcal{H}\). A complete theory of linear bounded operators, defined on the tensor-product spaces, can be found in Refs. [35–39].

The norm of a linear operator \(A\) on \(\mathcal{H}\) can be given by

\[
||A||_H = \sup_{||\varphi||_{\mathcal{H}} = 1} ||A\varphi||_H = \sup_{||\varphi||_{\mathcal{H}} = ||\varphi'||_{\mathcal{H}}} |\langle \varphi, A\varphi' \rangle|.
\]
For a bounded operator, the norm is finite. If the operator \( A \) is Hermitian, then
\[
||A||_{\mathcal{H}} = \sup_{||\varphi||_{\mathcal{H}}=1} |(\varphi, A\varphi)| \quad (A^+ = A) .
\]

Let us introduce the projector \( \mathcal{P}_D \), which projects the total space (3) onto its subset given by the disentangled set (7), so that
\[
\mathcal{P}_D \mathcal{H} = \mathcal{D} ,
\]
with the standard properties of projecting operators
\[
\mathcal{P}_D^2 = \mathcal{P}_D , \quad \mathcal{P}_D^+ = \mathcal{P}_D , \quad ||\mathcal{P}_D||_{\mathcal{H}} = 1 .
\]
The projector in Eq. (12) is nonlinear, therefore the equality for its norm has to be understood as a definition. And let us define the norm of \( A \) on \( \mathcal{D} \) as
\[
||A||_{\mathcal{D}} = ||\mathcal{P}_D A \mathcal{P}_D||_{\mathcal{H}} .
\]
This, in view of the structure of the disentangled set (7), can be presented as
\[
||A||_{\mathcal{D}} = \sup_{||f||_{\mathcal{D}}=1, ||f'||_{\mathcal{D}}=1} |(f, A f')| .
\]

An operator \( A \), acting on \( f \in \mathcal{D} \), generally, transforms \( f \) to an \( \varphi \in \mathcal{H} \). That is, an operator \( A \), in general, entangles the factor states. The operator \( A \), having the property \( A \mathcal{D} \subset \mathcal{H} \setminus \mathcal{D} \) will be termed entangling operator.

Similarly to the existence of entangled and factor states, there exist entangling operators and nonentangling ones. The latter should, clearly, have the structure of a direct product \( \otimes_{i=1}^{p} A_i \) of single-particle operators \( A_i \) acting on \( \mathcal{H}_i \). Let the algebra of all linear bounded operators on \( \mathcal{H} \) be denoted by \( \mathcal{A} \equiv \{ A \} \). And let us separate out from this algebra a subset \( \mathcal{A}^\otimes \equiv \{ A^\otimes \} \subset \mathcal{A} \) of nonentangling, or product, operators having the structure of a product \( \otimes_{i=1}^{p} A_i \). Thus, by construction,
\[
\mathcal{A} \mathcal{D} \subset \mathcal{H} , \quad \mathcal{A}^\otimes \mathcal{D} \subset \mathcal{D} .
\]

Analogously to the projection of \( \mathcal{H} \) onto \( \mathcal{D} \) by means of the projector \( \mathcal{P}_D \) given in Eq. (12), we may denote the reduction of \( \mathcal{A} \) to \( \mathcal{A}^\otimes \) with the help of a projector \( \mathcal{P}_\otimes \), such that
\[
\mathcal{A}^\otimes = \mathcal{P}_\otimes (\mathcal{A}) ,
\]
which is a superoperator acting on the Hilbert space of linear operators, with the properties
\[
\mathcal{P}_\otimes^2 = \mathcal{P}_\otimes , \quad \mathcal{P}_\otimes^+ = \mathcal{P}_\otimes , \quad ||\mathcal{P}_\otimes||_{\mathcal{D}} = 1
\]
being valid. The equality for the norm has to be understood as a definition. An explicit construction of a product operator \( \mathcal{A}^\otimes \subset \mathcal{A}^\otimes \), associated with a given operator \( \mathcal{A} \subset \mathcal{A} \), can be done as follows. Let us define a reduced single-particle operator \( A_i \) on \( \mathcal{H}_i \) as
\[
A_i = C_i \text{Tr}_{(\mathcal{H}_{j \neq i})} \ A ,
\]
where the trace runs over all \( \mathcal{H}_j \) except the case \( j = i \). The set of constants \( C_i \) is chosen so that to satisfy the normalization condition
\[
\text{Tr}_{\mathcal{H}} A = \text{Tr}_{\mathcal{D}} A^\otimes .
\]
In this way, we obtain the product operator
\[
A^\otimes = \frac{\text{Tr}_{\mathcal{H}} A}{\text{Tr}_{\mathcal{D}} \otimes_{i=1}^{p} A_i} \otimes_{i=1}^{p} A_i ,
\]
where
\[
\text{Tr}_{\mathcal{D}} \otimes_{i=1}^{p} A_i = \prod_{i=1}^{p} \text{Tr}_{\mathcal{H}_i} A_i .
\]
Thus, from a given multiparticle space (3), it is possible to separate out the disentangled set (7) and, similarly, for an arbitrary operator $A$, one can put into correspondence the product operator (19). That is, there exist entangled and disentangled states as well as entangling and nonentangling operators.

Let us be interested in the entanglement produced by an operator $A$ on $\mathcal{H}$. How could we measure the resulting entanglement production? It would seem natural that we should somehow compare the actions of the given operator $A$ and its nonentangling counterpart $A^\otimes$. But what quantity should be defined for this purpose? Following the common ideology, one should construct a sort of entropy, comparing, say, $\text{Tr}_D A \ln A$ and $\text{Tr}_D A^\otimes \ln A^\otimes$. However, this is not the best way. The pivotal idea, we shall follow, is the observation that the norm of an operator characterizes a kind of order associated with this operator [40]. For instance, invoking the norm and trace of reduced density operators, or reduced density matrices, it is possible to define the density order indices [41]. Generalizing the latter, one can define the operator order indices [40] for arbitrary operators. These order indices provide a complete classification for different types of order, long-range and mid-range, off-diagonal and diagonal, because of which they are applicable for describing both phase transitions and crossover phenomena.

Entanglement, in some sense, is also a characteristic of order (or disorder). Hence it should be well characterized by an operator norm. To this end, we introduce the measure of entanglement production

$$\varepsilon(A) \equiv \log \frac{||A||_D}{||A^\otimes||_D}$$

(20)

generated by the given operator $A$. Here the logarithm can be taken with respect to any base that would be convenient, e.g., to the base 2.

As follows from definition (20), the entanglement-production measure quantifies the amount of entanglement produced by an operator $A$ over a set $D$, because of which this measure should, in general, be denoted as $\varepsilon(A, D)$. When working with a fixed set $D$, one may, for simplicity, shorten the notation writing $\varepsilon(A) = \varepsilon(A, D)$. Quantity (20) satisfies all natural properties that are compulsory for being really a measure:

1. **Measure is semipositive.**
   This is evident by construction, since
   $$||A^\otimes||_D = ||\mathcal{P}_\otimes(A)||_D \leq ||A||_D ,$$
   because of which
   $$\varepsilon(A) \geq 0 .$$
   (21)

2. **Measure is continuous.**
   This implies the following. Assume that for any operator $A$ of $\mathcal{H}$ there exists a family $\{A(t)\}$ of operators $A(t)$ parameterized with $t \in \mathbb{R}$, so that $A(t) \to A$ as $t \to 0$ in the sense of the norm convergence
   $$||A(t)||_D \to ||A||_D \quad (t \to 0) .$$
   If so, then for the measure (20) one has
   $$\varepsilon(A(t)) \to \varepsilon(A) \quad (t \to 0) .$$
   (22)
   The nature of the parameter $t$ can be arbitrary. In physical applications, this may be time, temperature, density, interaction parameters, and so on.

3. **Measure is zero for a nonentangling operator.**
   A nonentangling operator has the form of $A^\otimes$. As is obvious, for $A = A^\otimes$,
   $$\varepsilon(A^\otimes) = 0 .$$
   (23)
   In particular, there is no self-entanglement of a single-particle system, when $A = A_1 = A^\otimes$.

4. **Measure is additive.**
   Let now $\{A_\nu\}$ be a set of copies of $A$, with $\nu = 1, 2, \ldots$, so that $A = \otimes_{\nu} A_\nu$. Such a case may happen, e.g., in treating heterophase systems [42]. Then $A^\otimes = \otimes_{\nu} A_\nu^\otimes$ and one gets
   $$||A||_D = \prod_{\nu} ||A_\nu||_D , \quad ||A^\otimes||_D = \prod_{\nu} ||A_\nu^\otimes||_D .$$
From here, it follows that
\[ \varepsilon (\otimes_{\nu} A_{\nu}) = \sum_{\nu} \varepsilon (A_{\nu}) . \] (24)

5. Measure is invariant under local unitary operations.

Such operations are described by a set \( \{U_i\} \) of unitary operators \( U_i \) on \( \mathcal{H}_i \), with \( U_i U_i^+ = 1 \) and \( i = 1, 2, \ldots, p \). Using the properties of the operator norm and trace, it is easy to show that
\[ \varepsilon \left( \otimes_{i=1}^p U_i^+ A \otimes_{i=1}^p U_i \right) = \varepsilon (A) . \] (25)

Thus, the entanglement-production measure (20) can be introduced for any operator. The requirement that the latter be bounded can be relaxed in the following way. Assume that \( A \) defined on \( \mathcal{H} \) is unbounded. Introduce a restricted space \( \mathcal{R}_N \) such that the norm \( \|A\|_{\mathcal{R}_N} \) is finite and \( \mathcal{H} \) can be treated as the inductive limit \( \mathcal{R}_N \to \mathcal{H} \) as \( N \to \infty \). Here \( N \) is not necessarily the number of particles, but may be any labelling number. Following the procedure described above, one can separate out from the space \( \mathcal{R}_N \) the disentangled set \( \mathcal{D}_N \), whose inductive limit is \( \mathcal{D} = \lim_{N \to \infty} \mathcal{D}_N \). Then the entanglement measure, instead of Eq. (20), can be defined as
\[ \varepsilon (A) = \lim_{N \to \infty} \log \frac{\|A\|_{\mathcal{D}_N}}{\|A^\otimes\|_{\mathcal{D}_N}} . \]

In particular, this can correspond to the thermodynamic limit.

IV. PURE STATE ENTANGLEMENT

Entanglement production, as is explained above, can be generated by any operator. In physical applications, one often implies that this is due to the action of a von Neumann operator. There are two types of density operators, pure-state and mixed-state. A density operator \( \hat{\rho}_N \) for an ensemble of \( N \) particles is termed a pure-state operator when it is an idempotent operator, such that \( \hat{\rho}_N^2 = \hat{\rho}_N \). Below, we shall consider several examples of calculating the entanglement-production measure (20) generated by pure-state density operators.

A. Einstein-Podolsky-Rosen states

These states provide a classical example of entangled states. They correspond to a bipartite system composed of two-dimensional parts, so that such a state can be presented as
\[ |EPR > \equiv \frac{1}{\sqrt{2}} (|12 > \pm |21 >) . \] (26)

The related density operator is
\[ \hat{\rho}_{EPR} \equiv |EPR><EPR| \quad (\text{Tr}_N \hat{\rho}_{EPR} = 1) , \] (27)
with \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). Reduced single-particle density operators, according to definition (17), have the form
\[ \hat{\rho}_i^1 = \frac{1}{2} (|1><1| + |2><2|) , \]
being given on \( \mathcal{H}_i \), with \( i = 1, 2 \). The product operator (19) is \( \hat{\rho}_{EPR}^\otimes = \hat{\rho}_1^1 \otimes \hat{\rho}_2^1 \). Calculating the norm (13), we have for the density operator (27)
\[ \|\hat{\rho}_{EPR}\|_{\mathcal{D}} = \|\hat{\rho}_i^1\|_{\mathcal{H}_i} = \frac{1}{2} , \]
while for the product operator,
\[ \|\hat{\rho}_{EPR}^\otimes\|_{\mathcal{D}} = \|\hat{\rho}_i^1\|^2_{\mathcal{H}_i} = \frac{1}{4} . \]

Thence, the entanglement-production measure (20) is
\[ \varepsilon (\hat{\rho}_{EPR}) = \log 2 . \] (28)
Assuming the logarithm to the base 2, one has \( \varepsilon (\hat{\rho}_{EPR}) = 1 \).
B. Bell states

Similarly to the previous case, it is easy to find the entanglement-production measure for the density operator formed by the Bell states

\[ |B> \equiv \frac{1}{\sqrt{2}} (|11> \pm |22>) , \]  

so that the corresponding density operator is

\[ \hat{\rho}_B = |B><B| \quad (\text{Tr}_H \hat{\rho}_B = 1) . \]  

For the entanglement-production measure (20), this gives

\[ \varepsilon(\hat{\rho}_B) = \log_2 . \]  

Keeping in mind the logarithm to the base 2 yields \( \varepsilon(\hat{\rho}_B) = 1 \).

Note that the Bell states can be treated as a two-particle generalization of the single-particle Schrödinger cat state.

C. Greenberger-Horne-Zeilinger states

Such states, having the form

\[ |GHZ> \equiv \frac{1}{\sqrt{2}} (|11\ldots1> \pm |22\ldots2>) , \]  

can be considered as an \( N \)-particle generalization of the single-particle Schrödinger cat state [43,44]. The associated density operator is

\[ \hat{\rho}_{GHZ} \equiv |GHZ><GHZ| \quad (\text{Tr}_H \hat{\rho}_{GHZ} = 1) . \]  

Following the procedure of Sec. III, we find

\[ ||\hat{\rho}_{GHZ}||_D = ||\hat{\rho}_1^{\dagger}||_{\mathcal{H}} = \frac{1}{2} , \]

while for the corresponding product operator,

\[ ||\hat{\rho}_{GHZ}^\otimes||_D = \frac{1}{2^N} . \]

From here,

\[ \varepsilon(\hat{\rho}_{GHZ}) = (N-1)\log_2 . \]  

Hence, for the logarithm to the base 2, one gets \( \varepsilon(\hat{\rho}_{GHZ}) = N-1 \). As it should be, there is no entanglement production for a single particle, when \( N = 1 \), and the case of two particles, when \( N = 2 \), reduces to measures (28) or (31).

D. Multicat states

The multiparticle Schrödinger cat states, or, for short, multicat states, are sometimes also called the generalized GHZ states. They have the form

\[ |MC> \equiv c_1|11\ldots1> + c_2|22\ldots2> , \]  

where \( N \) two-dimensional parts are assumed and the coefficients are arbitrary complex numbers satisfying the normalization \( |c_1|^2 + |c_2|^2 = 1 \). The related density operator is

\[ \hat{\rho}_{MC} \equiv |MC><MC| \quad (\text{Tr}_H \hat{\rho}_{MC} = 1) . \]
For the norm (13), we find
\[ \|\hat{\rho}_{MC}\|_D = \|\hat{\rho}_1\|_H = \sup\{|c_1|^2, |c_2|^2\}. \]
The entanglement-production measure (20) becomes
\[ \varepsilon(\hat{\rho}_{MC}) = (1 - N) \log \sup\{|c_1|^2, |c_2|^2\}. \tag{37} \]
Its value lies in the interval
\[ 0 \leq \varepsilon(\hat{\rho}_{MC}) \leq (N - 1) \log 2. \tag{38} \]
The maximal entanglement production occurs when \(|c_1|^2 = |c_2|^2 = 1/2\), which goes back to the GHZ states. And the entanglement production disappears if \(|c_i| = 1\) for any \(i = 1, 2\).

Multicat states can be realized for systems of particles with two internal single-particle states, such as trapped ions subject to the action of resonant laser beams [45,46] or Bose-condensed neutral atoms with an effective interaction due to coherent Raman scattering [47]. Instead of internal single-particle states, one can create collective nonlinear states by invoking the resonant excitation of topological coherent modes in trapped Bose-Einstein condensates [48–50]. Such two-level or two-mode states are usually accompanied by atomic squeezing [50,51].

**E. Multimode states**

A natural generalization of the multicat states are the multimode states describing a system of \(N\) parts, each of which can be in one of \(m\) different modes. Such a state reads
\[ |MM\rangle \equiv \sum_n c_n |nn...n\rangle, \tag{39} \]
where
\[ \sum_n |c_n|^2 = 1, \quad \sum_n 1 = m. \]
The corresponding density operator is
\[ \hat{\rho}_{MM} \equiv |MM><MM| (\text{Tr}_H \hat{\rho}_{MM} = 1). \tag{40} \]
For this case, the entanglement-production measure writes
\[ \varepsilon(\hat{\rho}_{MM}) = (1 - N) \log \sup_n |c_n|^2. \tag{41} \]
It varies in the range
\[ 0 \leq \varepsilon(\hat{\rho}_{MM}) \leq (N - 1) \log m. \tag{42} \]
The maximal entanglement production is reached when \(|c_n| = 1/m\), while entanglement is absent if any of \(|c_n| = 1\). For the two-mode case, when \(m = 2\), one returns to the multicat states. In general, states (39) are related to coherent states with a fixed number of particles [40,52,53].

**F. Hartree-Fock states**

These states, typical of indistinguishable particles, have the structure
\[ |HF\rangle \equiv \frac{1}{\sqrt{N!}} \sum_{sym} |12...N\rangle, \tag{43} \]
where a symmetrized or antisymmetrized sum is assumed according to either bosons or fermions are considered. Note that here it is implied that \(N\) particles are in \(N\) different single-particle states. The density operator is
Calculating
\[ \|\hat{\rho}_{HF}\|_{\mathcal{D}} = \frac{1}{\sqrt{N!}}, \quad \|\hat{\rho}_{i}\|_{\mathcal{H}_{i}} = \frac{1}{\sqrt{N}}, \]
we find the entanglement-production measure
\[ \varepsilon(\hat{\rho}_{HF}) = \log \frac{N^N}{N!}. \] (45)
For \( N = 2 \), this reduces to
\[ \varepsilon(\hat{\rho}_{HF}) = \log 2 \quad (N = 2), \]
as it should be as far as state (43) reduces to the EPR state (26). And for large \( N \), one gets
\[ \varepsilon(\hat{\rho}_{HF}) \simeq N \log e \quad (N \to \infty). \]
It is worth noting that in all examples considered above the limit
\[ \lim_{N \to \infty} \frac{1}{N} \varepsilon(\hat{\rho}_{N}) < \infty \]
exists. The existence of such a limit is sometimes required as a prerequisite property of any entanglement measure. In our case, the existence of this limit is guaranteed by the property of additivity (24).

V. MIXED STATE ENTANGLEMENT

Mixed states of physical systems are characterized by density operators that are not idempotent, so that \( \hat{\rho}_{N}^2 \neq \hat{\rho}_{N} \). Since the entanglement-production measure (20) is defined for arbitrary operators, there is no principal problem of applying this definition to any density operators, including the mixed-state density operators. When a statistical operator is entangling, the related measure of produced entanglement must be nonzero.

To concretize the aforesaid, let us consider the operator
\[ \hat{\rho} \equiv \frac{1}{2} (|11><11| + |22><22|). \]
This form corresponds to what one terms a separable statistical operator. However, despite its simple form, the operator \( \hat{\rho} \) produces entanglement, i.e., it is entangling. This is easy to demonstrate by taking, for instance, a product function
\[ f = \sqrt{2} (|1><2|) \otimes (|1><2|) \]
from the disentangled set \( \mathcal{D} \). The action of \( \hat{\rho} \) on this function gives
\[ \hat{\rho} f = \frac{1}{\sqrt{2}} (|11><22|), \]
which is a Bell state, that is, a maximally entangled state. Here \( \hat{\rho} \) really does produce entanglement. Therefore, the related measure of produced entanglement (20) has to be nonzero. The latter can be easily verified following the general scheme. Thus, we have
\[ \hat{\rho}_i = \text{Tr}_{\mathcal{H}_{\bar{x}}} \hat{\rho}, \quad \hat{\rho} = \frac{1}{2} (|1><1| + |2><2|), \]
so that \( \hat{\rho}^\otimes = \hat{\rho}_1 \otimes \hat{\rho}_2 \). Calculating the norms
\[ ||\hat{\rho}||_{\mathcal{D}} = \frac{1}{\sqrt{2}}, \quad ||\hat{\rho}_i||_{\mathcal{H}_i} = \frac{1}{\sqrt{2}}, \quad ||\hat{\rho}^\otimes||_{\mathcal{D}} = \frac{1}{4}, \]
we find \( \varepsilon(\hat{\rho}) = \log 2 \), as it should be for an entangling operator \( \hat{\rho} \).
**A. Quantum mechanics**

Let us consider mixed-state density operators obtained by taking partial traces of pure-state quantum density operators. For instance, let us consider the Hartree-Fock pure state density operator (44). Its partial traces produce the mixed-state reduced density operators

\[ \hat{\rho}_{HF}^p \equiv \text{Tr}_{H_{p+1}} \cdots \text{Tr}_{H_N} \hat{\rho}_{HF} , \]

with \( p = 1, 2, \ldots N - 1 \). It is easy to find the norm

\[ ||\hat{\rho}_{HF}^p||_D = \frac{(N - p)!}{N!} , \]

from where the entanglement-production measure (20) becomes

\[ \varepsilon(\hat{\rho}_{HF}^p) = \log \left( \frac{(N - p)!N^p}{N!} \right) . \]

(47)

For a large number of particles and finite \( p \) this gives

\[ \varepsilon(\hat{\rho}_{HF}^p) \approx \frac{p(p - 1)}{2N} \log e \quad (N \to \infty) . \]

(48)

Measure (47) describes the amount of entanglement produced by the reduced density operator (46) defined for \( p \) particles from a quantum ensemble of \( N \) particles.

**B. Statistical mechanics**

Statistical properties of multiparticle systems are characterized by reduced density matrices [41]. Therefore, it is natural to consider entanglement realized by these matrices. The general scheme of measuring entanglement production can be as follows.

Let \( \mathcal{X} = \{x\} \) be a characteristic space of physical coordinates \( x \), whose concrete nature can be arbitrary. For example, \( x \) can be a set of spatial Cartesian coordinates, or it may be a set of momentum variables, or a set of quantum numbers or mode indices. Assume that there exist several such characteristic spaces \( \mathcal{X}_i = \{x_i\} \), with \( i = 1, 2, \ldots p \). And let each space \( \mathcal{X}_i \) be measurable, with a differential measure \( dx_i \) allowing for defining the Lesbegue integration over \( \mathcal{X}_i \). In the case of discrete variables, the differential measure is atomic, so that integration reduces to summation. The total characteristic space is the direct product

\[ \mathcal{X} \equiv \times_{i=1}^p \mathcal{X}_i \quad (p = 1, 2, \ldots N) . \]

The set \( \mathcal{X}^p \equiv \{x_1, x_2, \ldots, x_p\} \) is an element of \( \mathcal{X} = \{\mathcal{X}^p\} \). The space \( \mathcal{X} \) is measurable, with a differential measure \( dx_p \equiv \prod_{i=1}^p dx_i \).

Let the elements of the single-partite Hilbert space \( \mathcal{H}_i \) be the vectors \( \varphi_i = [\varphi_i(x_i)] \) treated as columns with respect to the variable \( x_i \), with the scalar product

\[ \varphi_i^\dagger \varphi'_i = (\varphi_i, \varphi'_i) = \int \varphi_i^*(x_i) \varphi'_i(x_i) \, dx_i . \]

A \( p \)-order reduced density matrix is defined as a matrix

\[ \rho_p = [\rho_p(x^p, y^p)] \]

with respect to \( x^p \) and \( y^p \), whose elements are

\[ \rho_p(x^p, y^p) = \text{Tr}_F \psi(x_1) \ldots \psi(x_p) \hat{\rho} \psi^\dagger(y_p) \ldots \psi^\dagger(y_1) , \]

(50)

where \( \psi(x) \) is a field operator, \( \hat{\rho} \) is a statistical operator, and the trace is over the Fock space \( F \). Definition (50) can equivalently be written as the statistical average

\[ \rho_p(x^p, y^p) = < \psi^\dagger(y_p) \ldots \psi^\dagger(y_1) \psi(x_1) \ldots \psi(x_p) > . \]
By this definition, the matrix $\rho_p$ is self-adjoint semipositive. An example is a first-order density matrix

$$\rho_i^1 = [\rho_i(x_i,y_i)] \quad (x_i,y_i \in X_i),$$

with

$$\rho_i(x,y) = <\psi^\dagger(y)\psi(x>).$$

A common name for $\rho_i^1$ is the single-particle density matrix, which, as was emphasized above, does not imply a concrete physical particle but rather tells what is the order of the reduced matrix.

The trace operations for the density matrices are given by the expressions

$$\text{Tr}_{H_i}\rho_i^1 = \sum_{n_i} <n_i|\rho_i^1|n_i> = \int \rho_i(x_i,x_i) \, dx_i = N,$$

and similarly

$$\text{Tr}_H\rho_p = \int \rho_p(x^p,x^p) \, dx^p = \frac{N!}{(N-p)!},$$

which yields the relation

$$\rho_i^1 = \frac{(N-p)!}{(N-1)!} \text{Tr}_{(H_{j\neq i})}\rho_p.$$

For the product operator (19), we now have

$$\rho_p^\otimes = \frac{N!}{(N-p)! N^p} \otimes_{i=1}^p \rho_i^1,$$

whose norm (14) reads

$$||\rho_p^\otimes||_D = \frac{N!}{(N-p)! N^p} \prod_{i=1}^p ||\rho_i^1||_{H_i}.$$

To effectively calculate the norms, entering the entanglement-production measure (20), we need to specify the single-partite spaces $H_i$. Recall that the latter can, generally, be chosen in an arbitrary way. However, for density matrices, there exists a natural choice related to the eigenvectors of $\rho_i^1$. These eigenvectors are termed natural orbitals [41]. Therefore, under $|n_i>$, we shall imply the eigenvectors of $\rho_i^1$. And a closed linear envelope of these natural orbitals gives the natural single-partite space $H_i$. Then one can write

$$\rho_i^1 = \sum_{n_i} D_{n_i,n_i}^1 |n_i><n_i|.$$

Keeping in mind the spectral norm, one has

$$||\rho_i^1||_{H_i} = \sup_{n_i} |D_{n_i,n_i}^1|.$$

A $p$-order density matrix can be presented as an expansion

$$\rho_p = \sum_{\{m,n_i\}} D_{\{m,n_i\}}^p |m_1 \ldots m_p><n_1 \ldots n_p|,$$

so that

$$||\rho_p||_D = \sup_{\{n_i\}} |D_{\{n_i\}}^p|.$$

In the case of a system of identical particles, when all $H_i$ are just copies of the same $H_1$, then

$$||\rho_i^1||_{H_i} = ||\rho_1||_{H_1}.$$

Therefore, the entanglement-production measure (20) becomes

$$\varepsilon(\rho_p) = \log \frac{(N-p)! N^p ||\rho_p||_D}{N! ||\rho_1||_{H_1}^p}.$$

Thus, the problem of quantifying the entanglement generated among any $p$ particles from a given ensemble of $N$ particles is reduced to calculating the norms of reduced density matrices.
The reduced density matrices (49), in general, depend on time,
\[ \rho_p(t) = [\rho_p(x^p, y^p, t)], \]
which enters definition (50) either through the field operators \( \psi(x, t) \) or through the statistical operator \( \hat{\rho}(t) \). Consequently, the entanglement-production measure (55) is, generally, also a function of time. Temporal dependence of entanglement is named evolutional entanglement [4].

As an example of entanglement generated in a nonequilibrium system, let us consider the case when the reduced density matrices have the structure of the mixed multimode state
\[ \rho_p(t) = \frac{N!}{(N-p)!} \sum_n w_n(t) |n...n\rangle \langle n...n|, \]
where \( w_n(t) \) are the fractional mode populations with the properties
\[ 0 \leq w_n(t) \leq 1, \quad \sum_n w_n(t) = 1. \]
Such multimode, or in the simplest case two-mode, states can be created in trapped Bose-Einstein condensates (see reviews [54–56]). This can be done, e.g., by separating a Bose condensate in a two-well, or a multiwell potential [55,56]. Another possibility is by generating topological coherent modes in a trapped condensate, by means of resonant alternating fields [48–50]. The mixed atom-molecule condensate can also be treated as a two-mode coherent system [57]. Coherent collisions between matter waves result in the formation of an effective multimode system [58]. A similar system can also be created in the process of superradiant scattering of an atomic Bose-condensed cloud [59–61]. A multimode Bose-Einstein condensate has many analogies with coherent optical systems and, in particular, with lasers [50,56,62].

For the density matrix (57), we get the norm
\[ \|\rho_p(t)\|_D = \frac{N!}{(N-p)!} \sup_n w_n(t), \]
while for the first-order matrix (52), we have
\[ \|\rho_1(t)\|_{\mathcal{H}} = N \sup_n w_n(t). \]
Thus, for the product operator (53), we find
\[ \|\rho^\otimes_p(t)\|_D = \frac{N!}{(N-p)!} \sup_n w_n^p(t). \]
As a result, the entanglement-production measure is
\[ \varepsilon(\rho_p) = (1-p) \log \sup_n w_n(t). \]

The temporal behaviour of the fractional mode populations \( w_n(t) \) are defined by the evolution equations describing the corresponding process. For instance, in the case of Bose-Einstein condensates, the evolution equations for the mode populations follow from the time-dependent Gross-Pitaevskii equation [48–50,62]. The fractional mode populations \( w_n(t) \), depending on the physical situation considered, vary in time between 0 and 1. The maximal entanglement production happens at that time when all populations coincide, so that \( w_n(t) = 1/m \), where \( m \equiv \sum_n 1 \). That is, the entanglement-production measure (60) can vary in the interval
\[ 0 \leq \varepsilon(\rho_p) \leq (p-1) \log m. \]

Thus, the entanglement-production measure (60) depends on time through the mode populations. The evolution of the latter can be regulated. For example, when the condensate mode structure is due to the resonant generation of topological coherent modes, the temporal behaviour of mode populations is governed by the applied alternating fields [48–50]. In this way, the time evolution of the entanglement-production measure (60) can be controlled, which opens wide possibility for information processing.

12
D. Spin entanglement

Entanglement production, being defined for arbitrary operators, can be considered for systems of any nature. An important class of systems is that corresponding to spin ensembles. Spin entanglement can be studied in the same way as entanglement of particles, by invoking a kind of density matrices.

Let spin operators $S_i = \{S_\alpha^i\}, \text{with } \alpha = x, y, z$, be associated with a lattice $\mathbb{Z}_N = \{n_i\}$ whose lattice sites are enumerated by an index $i = 1, 2, \ldots, N$. Similarly to the reduced density matrices (49), composed of field operators, we may introduce [40] spin density matrices

$$R_p = \left[R^\alpha_\beta_{ij}\right],$$

(61)

with the elements

$$R^\alpha_\beta_{ij} \equiv <S^\beta_j S^\alpha_i>$$

(62)

composed of statistical averages of spin operators. Form (61) is treated as a matrix with respect to all indices $\{ij\}$ as well as $\{\alpha\beta\}$, although in principle, one could also consider the matrices $R^\alpha_\beta$ and $R_{ij}$ defined as matrices with respect to only the indices $\{ij\}$ or $\{\alpha\beta\}$, under fixed other indices. The first-order spin density matrix (61) is $R_1 = [R^\alpha_\beta_{ij}],$ whose elements are the correlation functions

$$R^\alpha_\beta_{ij} \equiv <S^\beta_j S^\alpha_i>.$$  

(63)

Matrices (61) are self-adjoint, since such are the spin operators. With the traces

$$\text{Tr}_{\mathcal{H}_1} R_1 \equiv \sum_i \sum_\alpha <S^\alpha_i S^\alpha_i> = NS(S+1), \quad \text{Tr}_{\mathcal{H}_1} R_p = (NS)^p (S+1)^p,$$

where $S$ is the maximal quantum number of each spin, the product operator (19) becomes

$$R_p^\otimes \equiv R_1 \otimes R_1 \otimes \ldots \otimes R_1.$$

(64)

To calculate the operator norms, the single-partite space $\mathcal{H}_1$ can be specified as the closed linear envelope of natural spin orbitals, which are the eigenfunctions of $R_1$. This is in analogy with the case of reduced density matrices $\rho_p$. Finally, the spin entanglement measure (20) takes the form

$$\varepsilon(R_p) = \log \frac{||R_p||_D}{||R_p^\otimes||_D}.$$  

(65)

Explicit calculation of this measure will be illustrated in the following sections.

VI. STATISTICAL THERMAL ENTANGLEMENT

Instead of considering entanglement realized by reduced density matrices, as in the previous section, one may study entanglement produced by a statistical operator $\hat{\rho}$. The latter for a statistical system in thermal equilibrium reads

$$\hat{\rho} = \frac{1}{Z} e^{-\beta H}, \quad Z \equiv \text{Tr} e^{-\beta H},$$

(66)

where $\beta \equiv (k_B T)^{-1}$, with $k_B$ being the Boltzmann constant and $T$, temperature. One can define, in complete analogy with the previous sections, the reduced operators by tracing out some of the states of the total space $\mathcal{H}$. For example,

$$\hat{\rho}_1^\otimes \equiv \text{Tr}_{\{\mathcal{H}_{ij}\}} \hat{\rho}.$$  

(67)

Entanglement, related to statistical thermal operators, is usually characterized by the concurrence between a pair of qubits [63–68]. Here we show that the entanglement-production measure (20) provides a natural characteristic of the thermal entanglement, which is produced by the statistical operator (66).

Let us consider the Ising model
\[ H = -\frac{1}{2} \sum_{i \neq j}^{N} J_{ij} S_i^z S_j^z - B \sum_{i=1}^{N} S_i^z , \]  
(68)

with an exchange interaction \( J_{ij} = J_{ji} \) and an external magnetic field \( B \geq 0 \). Positive interaction \( J_{ij} > 0 \) is called ferromagnetic, while negative \( J_{ij} < 0 \), antiferromagnetic. It is natural to define the basis states \( |n_i> \) as the eigenvectors of \( S_i^z \). Then the single-partite space \( \mathcal{H}_i \) is the span of the basis \{\( |n_i> \)\}. For concreteness, let us take a two-site case, denoting \( J = J_{12} \), and let us consider spin-one-half operators \( S_i^z \), with \( S = 1/2 \). Introduce the dimensionless coupling \( g \) and magnetic field \( b \) by the expressions

\[ g \equiv \beta J S^2 , \quad b \equiv \beta B . \]  
(69)

Then we have

\[ \hat{\rho}_2 = \frac{1}{Z} \exp \{ 4g S_i^z S_j^z + b(S_i^z + S_j^z) \} , \]  
(70)

where

\[ Z = 2 \left( e^g \cosh b + e^{-g} \right) . \]

For the reduced operator

\[ \hat{\rho}_1 = \text{Tr}_{\mathcal{H}_{\neq i}} \hat{\rho}_2 , \]  
(71)

we find

\[ \hat{\rho}_1 = \frac{1}{Z} \exp \left\{ 2g S_i^z + b \left( S_i^z + \frac{1}{2} \right) \right\} + \frac{1}{Z} \exp \left\{ -2g S_i^z + b \left( S_i^z - \frac{1}{2} \right) \right\} . \]  
(72)

The disentangled set \( \mathcal{D} \) consists of the states \( |\uparrow \uparrow>, |\uparrow \downarrow>, |\downarrow \uparrow>, \text{ and } |\downarrow \downarrow> \). For the norms of \( \hat{\rho}_2 \) and \( \hat{\rho}_1 \), we get

\[ ||\hat{\rho}_2||_{\mathcal{D}} = \frac{1}{Z} \sup \{ e^{g+b}, e^{-g} \} \]  
(73)

and, respectively,

\[ ||\hat{\rho}_1||_{\mathcal{D}} = \frac{1}{Z} \left( e^{g+b} + e^{-g} \right) . \]  
(74)

This, taking into account that \( \hat{\rho}_2^0 = \hat{\rho}_1^0 \otimes \hat{\rho}_2^0 \), results in the entanglement-production measure

\[ \varepsilon(\hat{\rho}_2) = \log \left[ \frac{2(1 + e^{2g \cosh b})}{(1 + e^{b+2g})^2} \sup \{ 1, e^{b+2g} \} \right] , \]  
(75)

describing the pairwise spin entanglement production.

Analyzing the properties of measure (75), it is interesting to compare its behaviour with that of the average magnetization per spin

\[ M = -\frac{1}{N} \frac{\partial F}{\partial B} , \quad F = -\frac{1}{\beta} \ln Z , \]

for which one has

\[ M = \frac{e^{2g \sinh b}}{2(1 + e^{2g \cosh b})} . \]  
(76)

Expressions (75) and (76) are functions of two variables, coupling \( g \in (-\infty, +\infty) \) and magnetic field \( b \in [0, \infty) \).

In the case of zero magnetic field, when \( B \to 0 \) and \( b \to 0 \), one has \( M \to 0 \) and

\[ \lim_{b \to 0} \varepsilon(\hat{\rho}_2) = \log \left( \frac{e^g}{\cosh g} \right) . \]  
(77)
For low temperature, when $T \to 0$, hence $\beta \to \infty$ and $g \to \pm \infty$, we get
\[
\lim_{g \to \pm \infty} \lim_{b \to 0} \varepsilon(\hat{\rho}_2) = \log 2,
\]
that is, the maximal pairwise entanglement production. In the opposite case of high temperatures, when $T \to \infty$, so that $\beta \to 0$ and $g \to 0$, we find
\[
\lim_{g \to 0} \lim_{b \to 0} \varepsilon(\hat{\rho}_2) = 0,
\]
which means that high-temperature fluctuations destroy entanglement production. Note that the limits $b \to 0$ and $g \to 0$ commute with each other.

If the external magnetic field increases, with $B \to \infty$ and $b \to \infty$, then
\[
\lim_{b \to \infty} \varepsilon(\hat{\rho}_2) = 0, \quad \lim_{b \to \infty} M = \frac{1}{2}.
\]
This shows that there is no entanglement production between perfectly aligned spins.

When temperature diminishes, such that $T \to 0$ and $\beta \to \infty$, but the magnetic field is nonzero, $B \neq 0$, then the resulting expressions depend on the relation between $b \to \infty$ and $|g| \to \infty$. Thus, we obtain
\[
\lim_{T \to 0} \varepsilon(\hat{\rho}_2) = \begin{cases} 
\log 2 (b + 2g \to -\infty) \\
\log(3/4) (b + 2g \to 0) \\
0 (b + 2g \to +\infty)
\end{cases}
\]
(81)

And for magnetization (76), we find
\[
\lim_{T \to 0} M = \begin{cases} 
0 (b + 2g \to -\infty) \\
1/6 (b + 2g \to 0) \\
1/2 (b + 2g \to +\infty)
\end{cases}
\]
(82)

Magnetization plays the role of an order parameter. The above analysis shows that the larger is the magnetization, the smaller is the entanglement-production measure. In this case, entanglement production and order are complimentary to each other. To illustrate this by another example, let us consider the Hamiltonian (68) with ferromagnetic interactions $J_{ij} > 0$ of long-range type, when $J_{ij}$ depends on $N$ so that
\[
\lim_{N \to \infty} J_{ij} = 0, \quad \lim_{N \to \infty} \frac{1}{N} \sum_{i \neq j} J_{ij} < \infty.
\]
(83)

In that case, as is known [69], the Hamiltonian (68) is asymptotically, as $N \to \infty$, equivalent to the mean-field form $\sum_i H_i$, where
\[
H_i = -\left( \sum_j J_{ij} < S_j^z > + B \right) S_i^z + \frac{1}{2} \sum_j J_{ij} < S_i^z > < S_j^z >.
\]
This implies that the statistical operator (66) is asymptotically, as $N \to \infty$, equivalent to
\[
\hat{\rho}_N \to \otimes_{i=1}^N \hat{\rho}_i, \quad \hat{\rho}_i = \frac{1}{Z^1/N} e^{-\beta H_i}.
\]
(84)

But then $\hat{\rho}_N \to \hat{\rho}_N^\otimes$ and we come to the limit
\[
\lim_{N \to \infty} \varepsilon(\hat{\rho}_N) = 0.
\]
(85)

Long-range ferromagnetic interactions, satisfying condition (83), organize long-range magnetic order in the spin system with a finite magnetization $M$. At the same time, this yields the absence of entanglement production between ferromagnetically aligned spins.
The analysis of the previous section hints that there should be a relation between the entanglement-production measure and an order parameter. The latter, in turn, experiences dramatic changes at phase transitions. Hence, entanglement production may also exhibit essential changes under phase transformations, which is illustrated below.

### A. Bose-Einstein condensation

Let us study the entanglement realized by the reduced density matrices (49). The properties of the latter are described in detail in book [41]. At high temperature, much larger than the condensation temperature \( T_c \), one has

\[
\|\rho_p\|_D \simeq \|\rho_1\|_{\mathcal{H}_1} \quad (T \gg T_c) .
\]

Therefore, the entanglement-production measure (55) is

\[
\varepsilon(\rho_p) \simeq \log \left( \frac{(N-p)! N^p}{N!} \right) \quad (T \gg T_c) ,
\]

which is typical of the Hartree-Fock form.

At low temperature \( T \ll T_c \), we find [41] that

\[
\|\rho_p\|_D \simeq \frac{N!}{(N-p)!} , \quad \|\rho_1\|_{\mathcal{H}_1} \simeq N .
\]

Therefore, measure (55) becomes

\[
\varepsilon(\rho_p) \simeq 0 \quad (T \ll T_c) .
\]

This means that entanglement production diminishes when the Bose-Einstein condensation occurs.

### B. Superconducting transition

At temperatures much higher than the critical temperature \( T_c \), density matrices are of the Hartree-Fock type, which, for large \( N \gg 1 \), yields

\[
\varepsilon(\rho_p) \simeq \frac{p(p-1)}{2N} \log e \quad (T \gg T_c) .
\]

At temperatures below \( T_c \), the structure of the reduced density matrices essentially changes, as is thoroughly described in Ref. [41]. Then one has

\[
\|\rho_p\|_D \simeq c_p \times \left\{ \frac{N^{(p-1)/2}}{N^{p/2}} \begin{array}{c} \text{(p odd)} \\ \text{(p even)} \end{array} \right. ,
\]

where \( c_p \) is a constant of order one. From here,

\[
\|\rho_1\|_{\mathcal{H}_1} \simeq c_1 , \quad \|\rho_p^\otimes\|_D \simeq \frac{N! c^p_1}{(N-p)! N^p} .
\]

Thus, for \( T < T_c \), finite \( p \), and large \( N \gg 1 \), we obtain

\[
\varepsilon(\rho_p) \simeq \left\{ \begin{array}{ll} \frac{c^p_1}{2} \log N & \text{(p odd)} \\ \frac{1}{2} \log N & \text{(p even)} \end{array} \right. .
\]

Comparing Eqs. (90) and (92), we see that the entanglement-production measure \( \varepsilon(\rho_p) \) increases under arising superconductivity. This is contrary to what happens under Bose-Einstein condensation. Such a difference should not be surprising and it can be easily understood as follows. Considering here entanglement production, the single-partite space \( \mathcal{H}_1 \) has been treated as the space of single-particle quantum states. In the case of the Bose-Einstein condensation, there appears ordering of particles, which leads to the decrease of their entanglement production. But under superconducting transition, the arising order has to do with pairs of particles, that is, Cooper pairs, and not with separate particles. If we define the single-partite space as the space of the Cooper pairs quantum states, then the overall situation would become similar to that occurring at Bose-Einstein condensation. Then the order appearing between Cooper pairs under superconducting transition would result in the diminishing entanglement production of these pairs.
C. Magnetic transition

To study the interplay between magnetic ordering and entanglement production, we shall invoke the spin density matrices (61). As a particular case, let us consider these spin density matrices composed of the $z$-components $S^z_i$ of spin operators, with $i = 1, 2, \ldots, N$ enumerating lattice sites. Thus, the spin density matrix $R_p = [R_{ij}]$ possesses the elements

$$R_{ij} = \langle S^z_j S^z_i \rangle > .$$

(93)

Clearly, this is a Hermitian semipositive matrix.

The eigenfunctions of $R_1 = [\langle S^z_j S^z_i \rangle]$ are the vectors

$$|k> = |\varphi_k(a_i)>, \quad \varphi_k(a) \equiv \frac{1}{\sqrt{N}} e^{ik \cdot a},$$

(94)
treated as $N$-order columns with respect to the lattice vectors $a_i \in \mathbb{Z}_N$. Functions (94) form a complete orthonormal basis $\{ |k> \}$. A single-partite space is defined as the closed linear envelope $H_1 \equiv \mathcal{L}(|k>)$. The norm of $R_p$ over $\mathcal{D}$ can be calculated as

$$||R_p||_\mathcal{D} = \sup_{\{k_i\}} <k_1 \ldots k_p | R_p | k_1 \ldots k_p > .$$

(95)

The elements of $R_1$ are the correlation function $\langle S^z_i S^z_j \rangle$ having the property [70]

$$\lim_{a_{ij} \to \infty} \langle S^z_i S^z_j \rangle = M^2 ,$$

(96)

where $a_{ij} \equiv |a_i - a_j|$ and $M$ is the magnetization

$$M = \frac{1}{N} \sum_{i=1}^{N} <S^z_i > .$$

Here we keep in mind ferromagnetic phase transition. Calculating norms (95), we use, for simplicity, the mean-field approximation and assume large $N \to \infty$. Above the critical temperature $T_c$, where $M = 0$, we find

$$||R_p||_\mathcal{D} = (2p - 1)!! S^{2p} \quad (T \geq T_c) ,$$

(97)

with

$$(2p - 1)!! \equiv 1 \times 3 \times 5 \times \ldots \times (2p - 1) = \frac{(2p)!}{2^p p!} .$$

And below $T_c$, where $M \neq 0$, we have

$$||R_p||_\mathcal{D} \simeq N^p M^{2p} \quad (T < T_c) .$$

(98)

In particular,

$$||R_1||_{H_1} = S^2 + NM^2$$

at all temperatures. Therefore, for the entanglement-production measure (65), we obtain

$$\epsilon(R_p) = \log \frac{(2p)!}{2^p p!} \quad (T \geq T_c)$$

(99)

above the transition temperature. For instance,

$$\epsilon(R_2) = \log 3 , \quad \epsilon(R_3) = \log 15 .$$

And below $T_c$, where there appears magnetic order, we get

$$\epsilon(R_p) \simeq 0 \quad (T < T_c)$$

(100)

for any finite $p$.

As is seen, the arising magnetic order also leads to the decrease of entanglement production, similarly to what happens at Bose-Einstein condensation. The same qualitative relation between entanglement production and ordering for different phase transitions can be understood if to keep in mind that a ferromagnetic phase transition is accompanied by the condensation of magnons [70].

17
D. Order indices

The intimate relation between entanglement production and ordering, illustrated above by several examples, can be explained by the most general arguments. For this purpose, we need to resort to the notion of operator order indices that can be introduced for arbitrary operators [40], and, in particular, for reduced density matrices [41]. The \textit{operator order index} of an operator $A$ is defined [40] as

$$\omega(A) \equiv \frac{\log \|A\|}{\log |\text{Tr} A|}.$$  

In describing the ordering in physical systems, the role of operators $A$ is played by reduced density matrices. The latter can be the standard density matrices constructed of field operators [41], as well as spin density matrices, lattice density matrices, or matrices composed of other operators [40]. The larger is the order index $\omega(A)$, the higher is the level of ordering corresponding to the operator $A$. Such order indices characterize both long-range as well as various types of mid-range order. They are suitable for describing off-diagonal and diagonal orders and can be applied for any physical system, equilibrium or nonequilibrium, infinite or finite. Also, they do not involve the notion of broken symmetry and can be employed when the order parameters are not defined.

If the operator $A$ represents a density matrix, it is semipositive. Then $\|A\| \leq \text{Tr} A$, because of which $\omega(A) \leq 1$. Assume that a semipositive operator $A$ is associated with a kind of ordering in a physical system. The order is absent when $\|A\| \ll \text{Tr} A$, and then $\omega(A) \ll 1$.

There are two types of long-range order that may develop in physical systems, total and even orders [40,41]. Under the arising \textit{total long-range order}, the norms of the related density matrices increase, so that $\|\rho_p\| \sim \text{Tr} \rho_p$, hence $\omega_p(\rho) \to 1$. The order is total, which implies that the property $\|\rho_1\| \sim \text{Tr} \rho_1$, from here, $\|\rho_p^\circ\| \sim \text{Tr} \rho_p^\circ$. According to the normalization condition (18), we have $\text{Tr} \rho_p = \text{Tr} \rho_p^\circ$. Therefore, $\|\rho_p\| \sim \|\rho_p^\circ\|$, which results in $\varepsilon(\rho_p) \to 0$. As is shown above, this situation takes place at Bose-Einstein condensation and ferromagnetic transition. Thus, under the appearing total long-range order, the order indices increase but the entanglement-production measure decreases,

$$\omega(\rho_p) \to 1, \quad \varepsilon(\rho_p) \to 0 \quad \text{(total order)}.$$  

Here increasing total order implies diminishing entanglement production.

The situation is different for \textit{even long-range order}. Then the norms of the density matrices also increase, but so that [40,41]

$$\|\rho_p\| \sim \begin{cases} \sqrt{\text{Tr} \rho_p / N} & (p \text{ odd}) \\ \sqrt{\text{Tr} \rho_p} & (p \text{ even}) \end{cases}.$$  

For instance, $\|\rho_1\| \sim \text{const}$ but $\|\rho_2\| \sim N$. Because of this, the order indices are different for $p$ odd and $p$ even,

$$\omega(\rho_p) \to \begin{cases} (p - 1)/2p & (p \text{ odd}) \\ 1/2 & (p \text{ even}) \end{cases}.$$  

There is no order in the single-particle matrix $\rho_1$, as far as $\omega(\rho_1) = 0$, but $\omega(\rho_2) = 1/2$. For all $p = 1, 2, \ldots$, we have $\omega_{2p-1} < \omega_{2p}$, and they coincide only in the limit $p \to \infty$. Such a behaviour is typical of superconducting transition. Entanglement production is also different for odd and even numbers of particles,

$$\varepsilon(\rho_p) \to \begin{cases} \log N & (p \text{ odd}) \\ \frac{1}{2} \log N & (p \text{ even}) \end{cases}.$$  

The entanglement-production measure increases, since the inequality $\|\rho_1\| \ll \|\rho_p\|$, with $p > 1$, yields $\|\rho_p^\circ\| \ll \|\rho_p\|$. Now both the order indices for $p > 1$ and entanglement-production measure increase together. Hence, increasing even order results as well in increasing entanglement production.

In conclusion, a general definition of entanglement-production measure for arbitrary operators is introduced. The concept is valid for systems of any nature. The interplay between entanglement production and phase transitions is elucidated.