Analytic structure of the gluon and quark propagators in Landau gauge QCD *

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In Landau gauge QCD the infrared behavior of the propagator of transverse gluons can be analytically determined to be a power law from Dyson–Schwinger equations. This propagator clearly shows positivity violation, indicating the absence of the transverse gluons from the physical spectrum, i.e. gluon confinement. A simple analytic structure for the gluon propagator is proposed capturing all important features. We provide arguments that the Landau gauge quark propagator possesses a singularity on the real timelike axis. For this propagator we find a positive definite Schwinger function.

The standard model of particle physics consists of gauge field theories. These have been postulated on the basis of symmetries and their elementary excitations do not reflect the observed particle spectrum. In the quantum formulation of these theories, especially in Poincaré-covariant gauges, an intricate problem is posed by the separation of physical and unphysical degrees of freedom.

In Quantum Electrodynamics (QED) in linear covariant gauges, the electromagnetic field can be decomposed into transverse, longitudinal and timelike photons, however, only transverse polarizations are observed. From a purely mathematical point of view, this can be understood from the representations of the Poincaré group for massless states: massless particles have only two possible polarizations [1]. The apparent contradiction is resolved by the fact that timelike and longitudinal photons cancel exactly in the S-matrix [2]. In this context we emphasize that the states of quantum gauge field theories in covariant gauges necessarily constitute an indefinite metric space. In covariant gauge QED one thus has to sacrifice the principle of positivity of the representation space.

In Quantum Chromodynamics (QCD) in linear covariant gauges, the cancellation of unphysical degrees of freedom in the S-matrix is substantially complicated by the self-interaction of the gauge fields and by the ghost fields that are necessarily present in the quantum formulation of these theories [3]. To order $\alpha_s^2$ in perturbation theory, one obtains amplitudes for the scattering of two transverse gluons into one transverse and one longitudinal gluon. However, at the same order, a ghost loop appears and cancels the various gluon loops, and scattering of transverse to longitudinal gluons does not occur. It is possible to prove this cancellation to all orders in perturbation theory on the basis of the BRS [4] symmetry of the covariantly gauge fixed theory. This symmetry can be represented by gauge transformations with the ghost field as a parameter. The ghost fields, being scalar, anti-commuting fields, are necessarily in the unphysical part of the representation space. In covariant gauge QCD, the physical (and thus positive definite) part of the state space is conjectured to be the set of BRS singlets [5].

Gauge fixed QCD is invariant under transformations related to the ghost number, and employing such transformations, one can show that BRS non-singlets occur in quartets [5]. Such a BRS-quartet consists of two parent and two daugh-

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ter states of respectively opposite ghost numbers. The latter states are BRS-exact and thus BRS-closed because the BRS transformation is nilpotent. The BRS daughters are orthogonal to all other states in the positive definite subspace and therefore do not contribute to physical S-matrix elements. The parent states belong to the indefinite metric part of the representation space and are expected to violate positivity.

The Kugo–Ojima confinement scenario describes a mechanism by which the positive semi-definite, physical state space contains only colorless states. Colored states are not BRS singlets and therefore do not appear in S-matrix elements: they are confined. Thus an investigation of (non-)positivity of transverse gluons and quarks allows us to understand confinement via the BRS quartet mechanism in more detail.

Here we present analytic properties of the gluon and quark propagators in Landau gauge QCD as they result from non-perturbative calculations (more details can be found in Ref. [17]). We confirm previous results on positivity violation for the gluon propagator. We also provide a parameterization of the gluon propagator that is analytic throughout the complex $p^2$ plane except on the real timelike axis and which decreases to zero in every direction of the complex $p^2$ plane. Such behavior satisfies the usual axioms of local quantum field theory (except positivity). For the quark propagator, we analyze several general constraints, lattice data, and solutions of the coupled quark-gluon-ghost Dyson–Schwinger equations (DSEs). None of these contradict positivity of the quark propagator.

Within the framework of a Euclidean quantum field theory (used in the following), positivity is formulated in terms of the Osterwalder–Schrader axiom of reflection positivity. In the special case of a two point correlation function, $\Delta(x − y)$, the condition of reflection positivity can be written as

$$\int d^4x \ d^4y \ f(\vec{x}, \vec{y}) \Delta(x − y) \ f(\vec{y}, \vec{x}) \geq 0 \ , \ (1)$$

where $f(\vec{x}, \vec{y})$ is a complex valued test function with support in $\{(\vec{x}, \vec{y}) : x_0 > 0\}$. After a three-dimensional Fourier transformation, this condition can be given in terms of the Schwinger function

$$\Delta(t) = \frac{1}{\pi} \int dp \ cosp(tp) \sigma(p^2) \geq 0 \ , \ (2)$$

where $\sigma(p^2)$ is a scalar function extracted from the corresponding two-point function.

The elementary two-point functions of QCD are the ghost, gluon, and quark propagators. In Landau gauge these renormalized momentum-space propagators $D_G(p)$, $D_{\mu\nu}(p)$, and $S(p)$ can be generically written as

$$D_G(p) = \frac{G(p^2)}{p^2} \ , \ (3)$$

$$D_{\mu\nu}(p) = \left( \delta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right) \frac{Z(p^2)}{p^2} \ , \ (4)$$

$$S(p) = \frac{1}{-ip \ A(p^2) + B(p^2)} = \frac{i p \sigma_s(p^2) + \sigma_s(p^2)}{.} \ (5)$$

Note that the quark propagator $S(p)$ is decomposed into scalar and vector parts, $\sigma_s(p^2)$ and $\sigma_v(p^2)$. Violation of reflection positivity can be studied by calculating Eq. (2) with $\sigma_s(p^2) = Z(p^2)/p^2$ for the transverse gluons and $\sigma_v(p^2)$ for the quarks. (The ghost propagator violates reflection positivity by the way ghosts are introduced in Faddeev–Popov quantization.)

The coupled set of DSEs (for recent reviews see e.g. [14]) for the ghost and gluon propagators can be solved analytically for $p^2 → 0^+$ [15]. One finds simple power laws for the gluon and ghost dressing functions

$$Z(p^2) \sim (p^2)^{2\kappa} \ , \ G(p^2) \sim (p^2)^{-\kappa} \ . \ (6)$$

such that the product $Z(k^2)G^2(k^2) \sim \alpha_S(k^2)$ goes to a constant in the infrared: there is an infrared fixed point for the running coupling. The value of the exponent $\kappa$ is in the range $0.5 < \kappa < 0.7$, depending on details of the truncation of the set of DSEs [12,14,16,18]. Here we use a self-consistent truncation scheme which neglects the effects of the four-gluon interaction and employs ansätze for the ghost-gluon, quark-gluon, and three-gluon vertices such that two important constraints are fulfilled: the running coupling, $\alpha_S(p^2)$, is independent of the renormalization point, and the anomalous dimensions of
the propagators are reproduced at one-loop level for large momenta. In this particular truncation \( \kappa = (93 - \sqrt{1201})/98 \approx 0.595 \) is an irrational number, and \( \alpha_S(0) \approx 2.972 \). Infrared dominance of the gauge fixing part of the QCD action \([12]\) implies infrared dominance of ghosts in the DSEs which, in turn, can be used to show \([17]\) that \( \alpha_S(0) \) depends only weakly on the dressing of the ghost-gluon vertex and not at all on other vertex functions. Recently, the same infrared behavior of the propagators has been found using the method of Exact Renormalization Group Equations \([20]\).

For quenched QCD, the gluon propagator, as it results from numerical solution of the coupled DSEs for the gluon and ghost propagators, agrees very well with recent lattice data \([21]\), see Fig. \( \text{H} \). The unquenched DSE gluon propagator is significantly suppressed in the intermediate momentum region, where screening effects of \( q\bar{q} \) pairs becomes important. For both \( N_f = 0 \) and \( N_f = 3 \) the infrared behavior is given by Eq. \( \text{B} \).

The corresponding running coupling can be accurately represented by \([12]\)

\[
\alpha_{\text{fit}}(p^2) = \frac{1}{1 + (p^2/\Lambda_{\text{QCD}}^2)} \left[ \frac{\alpha_S(0)}{4\pi} + \frac{1}{\ln(p^2/\Lambda_{\text{QCD}}^2)} - \frac{1}{p^2/\Lambda_{\text{QCD}}^2 - 1} \right], \tag{7}
\]

with \( \beta_0 = (11 N_c - 2N_f)/3 \). Note that the Landau pole at spacelike \( p^2 = \Lambda_{\text{QCD}}^2 \) is canceled, c.f. Ref. \([22]\). The expression \( \text{7} \) is analytic in the complex \( p^2 \) plane except on the real timelike axis \( (p^2 < 0) \) where the logarithm produces a cut.

Since the infrared exponent, \( \kappa \), is an irrational number, one knows that the corresponding gluon propagator possesses a cut on the negative real axis as well. It is possible to fit the DSE solution for the gluon propagator without introducing further singularities. The fit to the gluon renormalization function with the form

\[
Z_{\text{fit}}(p^2) = w \left( \frac{p^2}{\Lambda_{\text{QCD}}^2 + p^2} \right)^{2\kappa} \left( \alpha_{\text{fit}}(p^2) \right)^{-\gamma}, \tag{8}
\]

with \( w = 2.5 \) and \( \Lambda_{\text{QCD}} = 510 \) MeV, is in good agreement with the DSE solution. Here \( w \) is a normalization parameter, and \( \gamma = (-13N_c + 4N_f)/(22N_c - 4N_f) \) is the one-loop value for the anomalous dimension of the gluon propagator. The discontinuity across the cut in \( Z_{\text{fit}}(p^2) \) vanishes for \( p^2 \to 0^- \), diverges to \( +\infty \) at \( p^2 = -\Lambda_{\text{QCD}}^2 \) on both sides and drops to zero for \( p^2 \to -\infty \). Additional parameterizations have been explored in Ref. \([7]\).

The corresponding Schwinger function, \( \Delta_g(t) \), based on the fit, Eq. \( \text{8} \), is compared to the DSE solution in Fig. \( \text{I} \). To enable a logarithmic scale, the absolute value is displayed. \( \Delta_g(t) \) has a zero for \( t \approx 5 \text{GeV}^{-1} \approx 1 \text{fm} \) and is negative for larger Euclidean times. I.e. we clearly observe positivity violations in the DSE gluon propagator and the agreement of the numerical Schwinger function with the Fourier transformed fit is also excellent. The crucial property for positivity violation in the gluon propagator is that it vanishes for \( p^2 \to 0^+ \). This can be seen from the relation \( 0 = \sigma_g(p^2 = 0) = \int d^4x \ D(x) \) which implies that the gluon propagator in coordinate space, \( D(x) \), is trivially zero or necessarily contains positive as well as negative contributions.
In expression (8), the overall magnitude, \( w \), is arbitrary because of renormalization properties (it is determined via the choice of the renormalization scale). The infrared exponent, \( \kappa \), is determined from the infrared analysis of the DSEs, and the one-loop value is used for the gluon anomalous dimension, \( \gamma \). Thus, we have found a parameterization of the gluon propagator which effectively only one parameter, the scale \( \Lambda_{\text{QCD}} \). This and the relatively simple analytic structure gives us confidence that we have succeeded in uncovering the important features of the Landau gauge gluon propagator.

We now turn to the analytic structure of the quark propagator. In the DSE studies of Ref. [12] it has been assumed that the non-Abelian part of the quark-gluon vertex can be factored out from the tensor structure. This structure is then given by the Curtis–Pennington vertex [23] which, by construction, is multiplicatively renormalizable and satisfies the Abelian Ward–Takahashi identity of QED. In this particular case one has a term

\[
\Delta B_\nu := i \frac{B(p^2) - B(q^2)}{p^2 - q^2} (p + q)_\nu ,
\]  

(9)

in the vertex. For the present discussion it is important to note that the exact quark-gluon vertex (which satisfies the more complicated Slavnov–Taylor identity of QCD), and future improved approximations to it, will almost certainly also exhibit such a quark-gluon coupling proportional to the sum of quark momenta. This coupling, being effectively scalar, is not invariant under chiral transformations in contrast to the leading \( \gamma_\nu \) part of the vertex. However, the term in Eq. (9) only appears once chiral symmetry is broken dynamically and is thus consistent with the chiral Ward identities. Its existence provides a significant self-consistent enhancement of dynamical chiral symmetry breaking in the quark DSE, which is necessary to produce an acceptable value of the chiral condensate [12]. Quite independently of the form of the gluon propagator, the resulting quark propagator respects positivity if the term in Eq. (9) is included in the quark-gluon vertex [24], as evidenced by the Schwinger functions in Fig. 3 (solid and dashed curves). From this figure we also see that positivity is violated if only the bare quark-gluon vertex is used (dotted curve), in agreement with previous work employing this approximation [24].
A single real pole on the negative momentum axis results in a pure exponential decay of the corresponding Schwinger function. This is not what is observed in $\Delta_s(t)$ for small times where there is some curvature (see Fig. 3). To give an acceptable reproduction of $\Delta_s(t)$, analytic parameterizations of the quark propagator must contain additional sub-dominant structure. By Fourier transforming combinations of functions with cuts and poles at timelike momenta we have determined three simple sources for such curvature. The dominant singularity may be accompanied by additional real singularities at larger mass scales, or by complex conjugate singularities with a larger real part of the mass, or it may be the starting point of a branch cut on the negative real momentum axis $[7]$. Of course such a list is by no means exhaustive.

In Ref. $[7]$, we have used various general requirements, lattice data $[11]$, and the DSE solutions $[12]$ to constrain these various parametric forms. Such analysis leads to an important conclusion: in all parameterizations, the dominant singularity must occur on (or very close to) the real, timelike half-axis at a scale $m \sim 350$ to 500 MeV, the lower estimate coming when lattice data are used for the fit, the upper estimate for fits to the DSE solutions. This scale may relate to a constituent quark mass.

Whilst this result is robust, our current methods are not able to accurately determine the nature of the dominant singularity or reliably constrain the additional sub-dominant structures. In particular, the regular infrared behavior of quark propagator found in the DSE and lattice calculations makes determining its detailed analytic properties very difficult without explicitly probing the timelike momentum half-plane. A more conclusive analysis of the structure of the quark propagator would result from direct solution of the DSEs over an appropriate region of the complex momentum plane.

To summarize: We have proposed the relatively simple function, Eq. (8), to describe the full (non-perturbative) Landau gauge gluon renormalization function for all complex values of momentum. The corresponding gluon propagator agrees well with DSE solutions and lattice data for spacelike momenta, it is positivity violating, and it is analytic everywhere except for a cut on the negative real $p^2$ half-axis. Thus it implies that in Landau gauge QCD, the confinement of transverse gluons is related to the violation of Osterwalder–Schrader reflection positivity. We have also provided evidence that the Schwinger functions related to the quark propagator are positive definite, and consequently quark confinement is not manifest at the level of the propagator. A number of relatively simple parameterizations have been suggested for this propagator in terms of real and complex conjugate poles and branch cuts. In all cases, the dominant singularity is real and occurs at a scale 350 to 500 MeV.

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REFERENCES