Conformal Boundary States in $\widehat{su}(2)_1/G$

Atsushi YAMAGUCHI$^*$

*Theory Division, High Energy Accelerator Research Organization (KEK), Tsukuba, 305-0801, Japan

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We construct boundary states in a particular $c = 1$ conformal field theory, the $\widehat{su}(2)_1/G$ orbifold with $G$ a binary finite subgroup of $SU(2)$. These states preserve the conformal symmetry, at least, but break rational symmetries of the $\widehat{su}(2)_1/G$ orbifold in general.

§1. Introduction and summary

In the study of conformal field theories with boundaries, it is important to construct boundary states generally breaking extended symmetries but preserving the (reduced) conformal symmetry. This problem has been clarified in $c = 1$ irrational theories, in particular, the circle theory$^{1-8}$ and its $Z_2$ orbifold.$^{8,9}$ In another $c = 1$ CFT realized by the $\widehat{su}(2)_1/G$ orbifold$^{10}$ the authors of Ref. 11) constructed the boundary states preserving the so-called automorphism-type rational symmetries. After that work, a set of boundary states, which generally break such rational symmetries but preserve the conformal symmetry, were constructed in Ref. 12) in the case that $G$ is the order-8 dihedral group. Our purpose here is to generalize this result to all binary finite subgroups of $SU(2)$.

In this note, we construct a set of conformal boundary states (in the $\widehat{su}(2)_1/G$ orbifold) characterized by $G$-orbits of the $SU(2)$ Lie group constituting a parameter space of prototypical conformal boundary states in the $\widehat{su}(2)_1$ WZW model, the parent theory of the orbifold. With respect to stabilizers in $G$, which fix these $G$-orbits, our boundary states are classified into two types, referred to as a bulk type and a fractional type.$^{9,13}$ The bulk type boundary states exist on the $G$-orbits with the trivial stabilizers including only the identity and the center. The fractional type boundary states exist on the $G$-orbits with non-trivial stabilizers and also correspond to irreducible representations of these stabilizers. An important property is that a bulk type boundary state extended to an arbitrary $G$-orbit with a nontrivial stabilizer is no longer fundamental but splits into a linear combination of the fractional type boundary states existing on this particular $G$-orbit. Among the boundary states mentioned above, some preserve rational symmetries on special $G$-orbits (although most break these symmetries). In particular, the Cardy states,$^{14}$ which preserve the symmetry of the full chiral orbifold algebra $\widehat{su}(2)_1/G$, correspond to the conjugacy classes of $G \subset SU(2)$.

This note is organized as follows. In the next section, we briefly review some aspects of boundary states in CFTs. There we explain Cardy’s conditions$^{14}$ and the

$^*$ E-mail: ayamagu@post.kek.jp
notion of fundamental boundary states. In \( \mathfrak{su}(2)_1 \) we construct boundary states in the \( \tilde{\mathfrak{su}}(2)_1/G \) orbifold. First, we introduce boundary states in the \( \tilde{\mathfrak{su}}(2)_1 \) WZW model. Modding out these boundary states using the symmetry \( G \), we next construct bulk type boundary states existing on the \( G \)-orbits with the trivial stabilizers. Then, we explain that bulk type boundary states extended to \( G \)-orbits with non-trivial stabilizers are not fundamental. Non-fundamental boundary states are generally decomposed into linear combinations of fundamental boundary states.\(^*\) We attempt to construct such additional fundamental boundary states, which we refer to as the “fractional type”, with the following ansatz. We start from consistent boundary states, for example, the Cardy states. Next, we decompose them into linear combinations of the Virasoro Ishibashi states.\(^{16}\) Then, we deform the coefficients of these linear combinations to obtain expressions of more generic boundary states. This deformation is carried out so that the resulting boundary states satisfy a set of strong constraints known as Cardy’s conditions. In this way, we can obtain other boundary states which are to be identified with fractional type boundary states. At the end of \( \mathfrak{su}(2)_1/G \) we mention which boundary states in this note preserve rational symmetries in the \( \tilde{\mathfrak{su}}(2)_1/G \) orbifold. In the Appendix, we give some technical details.

§2. Boundary states and Cardy’s conditions

Here, we give a short review of boundary states in a generic CFT with central charge \( c \).

Let us consider boundary states \( |B\rangle \) that belong to a set denoted by \( \mathcal{B} \) (i.e. \( |B\rangle \in \mathcal{B} \)) and satisfy the conformally invariant “gluing” condition,

\[
(L_n - \tilde{L}_{-n})|B\rangle = 0 .
\]  

(1)

These boundary states are decomposed into linear combinations of the Ishibashi states\(^{16}\) \( |j\rangle^{\text{Vir}} \), with \( j \) being the spin-less irreducible representations of the left and right Virasoro algebras. We can normalize the Virasoro Ishibashi states \( |j\rangle^{\text{Vir}} \) to satisfy the relations

\[
\langle j|q^H|j'\rangle^{\text{Vir}} = \delta_{jj'}\chi_j(\tau) ,
\]

where \( q = e^{2\pi i\tau} , \ H = 1/2(L_0 + L_0 - c/12) \), and \( \chi_j \) denotes the (left or right) Virasoro characters of \( j \). Now, let us decompose the boundary states \( |B\rangle \in \mathcal{B} \) as follows:

\[
|B\rangle = \sum_j C^j_B |j\rangle^{\text{Vir}} .
\]

The coefficients \( C^j_B \) must satisfy a set of consistency conditions referred to as the sewing constraints (see Refs. 17 and 18 for details). Among them, particularly important is the set referred to as Cardy’s conditions,\(^{14}\) which require a certain modular covariance of the annulus amplitudes determined from boundary states in

\(^*\) Non-fundamental states are also relevant in statistical physics context (see, e.g., Ref. 15) and references therein).
the set $\mathcal{B}$. These conditions are explained as follows. Let us introduce arbitrary boundary states $|B\rangle, |B'\rangle \in \mathcal{B}$. From these states, we obtain a cylinder amplitude,
\[
\langle B| q^{H_c} | B'\rangle = \sum_{k} \mathcal{N}_{kB}^{B_k} \chi_{kB}(1/\tau),
\]
where $\mathcal{N}_{kB}^{B_k} = \sum_{j} C_{kB}^{j} S_{jk}(C_{B}^{j})^{*}$. Here we have assumed that the characters $\chi_{j}$ are mapped to themselves under the modular S-transformation in the following manner:
\[
\chi_{j}(\tau) = \sum_{k} S_{jk} \chi_{k}(1/\tau).
\]
From generic properties expected in boundary CFTs, Cardy placed consistent conditions on the coefficients $\mathcal{N}_{kB}^{B_k}$. These are called Cardy’s conditions and are expressed as
\[
\mathcal{N}_{kB}^{B_k} \in \mathbb{Z}_{\geq 0}, \mathcal{N}_{0B}^{B_k} = \delta_{BB_k}, \forall |B\rangle, |B'\rangle \in \mathcal{B},
\]
where $0$ in $\mathcal{N}_{0B}^{B_k}$ denotes the Virasoro identity representation. Among these conditions, the particular ones $\mathcal{N}_{0B}^{B_k} = 1$ are necessary for the boundary states $|B\rangle$ to be fundamental. In other words, boundary states cannot be fundamental if there are degeneracies of the Virasoro identity character in their self-overlaps. At this point, we should mention whether Cardy’s conditions are satisfied for the set of boundary states constructed below. These boundary states will be explicitly expressed as linear combinations of the Virasoro Ishibashi states in the $\tilde{su}(2)_{1}/G$ orbifold. With the help of these expressions, it is not difficult to see that our set of boundary states indeed does satisfy Cardy’s conditions described above. (This is implicitly explained in the Appendix.)

§3. Boundary states in $\tilde{su}(2)_{1}/G$

In this section, we construct bulk type and fractional type boundary states in the $\tilde{su}(2)_{1}/G$ orbifold.

Bulk type boundary states

We begin with the $\tilde{su}(2)_{1}$ WZW model, in which there are a set of fundamental conformal boundary states given (see, e.g., Ref. 3)) by
\[
|g\rangle_{\tilde{su}(2)_{1}} = 2^{-1/4} \sum_{j \in \frac{1}{2} \mathbb{Z}_{\geq 0}} \sum_{m,n=-j} D_{mn}^{j}(g^{-1}) |j; m, -n\rangle_{\text{Vir}}, \quad g \in SU(2),
\]
where $D_{mn}^{j}(g)$ denotes the matrix elements of the spin-$j$ representations of $SU(2)$, and $|j; m, n\rangle_{\text{Vir}}$ denotes the Virasoro Ishibashi states of the conformal weights $j^2$, which satisfy
\[
\text{Vir} \langle j; m, n| q^{H_c} | j', m', n'\rangle_{\text{Vir}} = \delta_{j,j'} \delta_{m,m'} \delta_{n,n'} \frac{1}{\eta(\tau)} (q^{j^2} - q^{(j+1)^2}),
\]
where $\eta(\tau)$ denotes Dedekind’s eta function. These boundary states are mapped as $|g\rangle^{\tilde{su}(2)_{1}} \rightarrow |g L g R^{-1}\rangle^{\tilde{su}(2)_{1}}$ under $SU(2)_{L} \times SU(2)_{R}$; the symmetry of the left-mover and right-mover modes of the $\tilde{su}(2)_{1}$ WZW model. This symmetry group includes an orbifolding group $G$ as a diagonal subgroup. Modding out the Hilbert space by the symmetry $G$, we obtain the untwisted sectors of the $\tilde{su}(2)_{1}/G$ orbifold with the charge conjugation modular invariant.\footnote{For details of rational orbifold CFTs with the charge conjugate modular invariant, see Ref 19.} Thus, the $G$-invariant combinations,

$$
|g\rangle^{G} = \frac{1}{\sqrt{2|G|}} \sum_{h \in G} |h g h^{-1}\rangle^{\tilde{su}(2)_{1}}, \quad g \in SU(2)
$$

(7)

belong to these untwisted sectors. Here note that the combinations $|g\rangle^{G}$ depend only on the $G$-orbits of $g$. We can regard these combinations as fundamental conformal boundary states if the associated stabilizers,

$$
N(g) = \left\{ h \in G \mid h g h^{-1} = g \right\},
$$

(8)

are trivial, i.e., $N(g) = \{1, -1\}$. In this case, we call them bulk type boundary states. However, if the stabilizers $N(g)$ are non-trivial, the expressions (7) cannot represent fundamental boundary states. This is because we find (by explicit calculation) $|N(g)|/2$-fold degeneracies of the Virasoro identity character in their self-overlaps. In this case, however, we can resolve these degeneracies by decomposing the boundary states $|g\rangle^{G}$ into linear combinations of other fundamental boundary states. Before demonstrating such a resolution, let us ask which $G$-orbits in $SU(2)$ have non-trivial stabilizers.

**Fixed orbits**

In the following, $G$-orbits in $SU(2)$ with non-trivial stabilizers are simply called fixed-orbits. We denote by $\mathcal{F}$ the set of fixed orbits and denote by $\mathcal{F}^{0}$ the supplement of $\mathcal{F}$ in the set of all distinct $G$-orbits in $SU(2)$. Prototypical fixed orbits are the conjugacy classes $C(G)$ (the $G$-orbits of the $G$-elements). These classes are introduced in the following manner. First, we denote by $g_{a}(a = 1, 2, \cdots)$ and $g_{b}(b = 1, 2, \cdots)$ representatives of the distinct mutually non-conjugate $G$-elements and assume that the relations $g_{a}^{-1} \in [g_{a}]$, $g_{b}^{-1} \notin [g_{b}]$ hold. Here, we denote the $G$-orbits of $g \in SU(2)$ by $[g] = \{ h g h^{-1} \mid h \in G \}$. We can always find the smallest positive integers $n_{a}$ and $n_{b}$ satisfying $g_{a}^{n_{a}} = g_{b}^{n_{b}} = 1$. (Here, $a$ and $b$ are regarded as collections of indices $1, 2, \cdots$.) Then, the $G$-elements $g_{a}$ and $g_{b}$ represent some classes in $C(G)$. All such classes are expressed as follows:

$$
[\pm 1]; \quad [g_{a}^{s}], \quad s = 1, \cdots, n_{a} - 1; \quad [\pm g_{b}^{s}], \quad s = 1, \cdots, n_{b} - 1.
$$

(9)

Here we note that the stabilizers of $[\pm 1], [g_{a}^{s}]$ and $[\pm g_{b}^{s}]$ are given, respectively, by

$$
G, \quad N_{a} = \{ g_{a}^{s} \mid s = 1, \cdots, 2n_{a} \} \quad \text{and} \quad N_{b} = \{ g_{b}^{s} \mid s = 1, \cdots, 2n_{b} \}.
$$

(10)

In addition to the above fixed orbits identified with $C(G)$, we can obtain the other fixed orbits by defining $2 \times 2$ hermitian matrices $\sigma_{a}$ and $\sigma_{b}$ from the relations $g_{a} = e^{i \pi/n_{a} \sigma_{a}}$ and $g_{b} = e^{i \pi/n_{b} \sigma_{b}}$. In this way, we can identify all the fixed orbits as in TableII.
The $\hat{su}(2)_1/G$ Ishibashi states

We propose a resolution of $|g\rangle^G$ with $[g] \in \mathcal{F}$, by expressing $|g\rangle^G$ as linear combinations of other fundamental boundary states. Although it is difficult to construct the latter states directly, we can guess their expressions by studying other consistent boundary states, for example, the Cardy states,\cite{14} in the $\hat{su}(2)_1/G$ orbifold. In other words, we use the Cardy states to probe the space of generic boundary states. To obtain these Cardy states, we need to identify the $\hat{su}(2)_1/G$ modular S-matrix and the $\hat{su}(2)_1/G$ Ishibashi states. The former was obtained in Refs. 11) and 19). The latter are in one-to-one correspondence with the chiral primary fields of $\hat{su}(2)_1/G$.\cite{16} We next review the $\hat{su}(2)_1/G$-primary-field content\cite{11,19} before identifying the $\hat{su}(2)_1/G$ Ishibashi states.

The distinct twisted sectors in the $\hat{su}(2)_1/G$ orbifold are in one-to-one correspondence with $C(G)$. Here note that the untwisted sectors correspond to the classes $\{\pm 1\}$. Let us consider a class $[g] \in C(G)$ and the corresponding $g$-twisted sector. The $g$-twisted sector is projected onto irreducible representations of the stabilizer $N(g)$. The set of these representations, denoted by $\mathcal{R}(g)$, is chosen\cite{19} as a suitable set of irreducible representations of $N(g)$. To express $\mathcal{R}(g)$, we need to explain some notation. Let $\text{Irr}(N(g))$ denote the set of all irreducible representations of $N(g)$. Let us introduce two subsets of $\text{Irr}(N(g))$ by

$$\text{Irr}^\pm(N(g)) = \{ r \in \text{Irr}(N(g)) \mid r(-1) = \pm r(1) \} .$$

(11)

Then, $\mathcal{R}(g)$ is identified with either of $\text{Irr}^\pm(N(g))$, as shown explicitly in Table I. We assign the indices $(g, r)$ with $r \in \mathcal{R}(g)$, to the primary fields in the $g$-twisted sector. In this way, all the chiral primary fields of $\hat{su}(2)_1/G$ are represented by $(g, r)$, with $[g] \in C(G)$ and $r \in \mathcal{R}(g)$. As mentioned above, the primary fields $(g, r)$ correspond to the $\hat{su}(2)_1/G$ Ishibashi states. The corresponding states are denoted by $|r\rangle_{[g]}^{\hat{su}(2)_1/G}$ and are identified below.

We can decompose each of the $\hat{su}(2)_1/G$ Ishibashi states into a linear combination of the Virasoro Ishibashi states in the untwisted and twisted sectors. The Virasoro Ishibashi states in the untwisted sectors were already introduced as $|j; m, n\rangle^{\text{Vir}}$, which belong to the 1-twisted and $-1$-twisted sectors, respectively, if $j \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$. They are combined into the $\hat{su}(2)_1$ Ishibashi states of the spin-0 and spin-1/2 highest-weight representations of the $\hat{su}(2)_1$ WZW model (the parent theory of the orbifold) in the following manner:

$$|0\rangle^{\hat{su}(2)_1} = \sum_{j \in \mathbb{Z}_{\geq 0}} \sum_{n=-j}^j |j; n, -n\rangle^{\text{Vir}}, \quad |1/2\rangle^{\hat{su}(2)_1} = \sum_{j \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \sum_{n=-j}^j |j; n, -n\rangle^{\text{Vir}} .$$

(12)

The states $|0\rangle^{\hat{su}(2)_1}$ and $|1/2\rangle^{\hat{su}(2)_1}$ are projected onto the $\hat{su}(2)_1/G$ Ishibashi states of $(\pm 1, r)$ with $r \in \text{Irr}^\pm(G)$, via suitable projection operators.\cite{19} This projection can be expressed as follow:

$$|r\rangle_{[\pm 1]}^{\hat{su}(2)_1/G} = \frac{\sqrt{\text{ch}_r^G(1)}}{|G|} \sum_{h \in G} \text{ch}_r^G(h)^* \sum_{j \in \mathbb{Z}_{\geq 0}} \sum_{m, n=-j}^j D_{mn}^j(h) |j; m, -n\rangle^{\text{Vir}} .$$

(13)
Here we have denoted by $\text{ch}^G_r$ the irreducible characters of $G$, with $*$ representing the complex conjugate. Next, we consider the Virasoro Ishibashi states in the twisted sectors to decompose the $\widehat{s\mathfrak{u}(2)}_1/G$ Ishibashi states in these sectors. The $\pm g_2^s$-twisted sectors with $j = a, b$ and $s = 1, \ldots, n_j - 1$ are projected onto the representations $r \in \mathcal{R}(\pm g_2^s)$ of $(\pm g_2^s, r)$. From the Virasoro characters of $(\pm g_2^s, r)^{(11), (19)}$ we can determine which Virasoro Ishibashi states are contained in the $\widehat{s\mathfrak{u}(2)}_1/G$ Ishibashi states of $(\pm g_2^s, r)$. In this way, we find the Virasoro Ishibashi states in the twisted sectors as

$$|m\rangle^{\text{Vir}}_{[g_2^s]}, m \in \mathbb{Z}; \quad |m\rangle^{\text{Vir}}_{[\pm g_2^s]}, m \in \mathbb{Z}.$$

where the states $|m\rangle^{\text{Vir}}_{[\pm g_2^s]}$ are built up from the (non-degenerate) Virasoro highest-weight states of conformal weights $(m + s/2n_j)^2$ in the $\pm g_2^s$-twisted sectors and satisfy

$$\langle [g_2^s]|(q^{H_\eta}|m \rangle)^{\text{Vir}}_{[g_2^s]} = \delta_{mm} \delta_{g_2^s} \left( rac{1}{\eta(\tau)} \right) q^{(m + s/2n_j)^2},$$

where $\epsilon, \epsilon' = \pm 1$. The Virasoro Ishibashi states $|m\rangle^{\text{Vir}}_{[\pm g_2^s]}$ are combined into the $\widehat{s\mathfrak{u}(2)}_1/G$-Ishibashi states in the $\pm g_2^s$-twisted sectors as follows:

$$|r_a^+\rangle^{\widehat{s\mathfrak{u}(2)}_1/G}_{[g_2^s]} = \sum_{m \in \mathbb{Z}} \left| n_a m + r_a^+ \right\rangle^{\text{Vir}}_{[g_2^s]},$$

$$|r_b^+\rangle^{\widehat{s\mathfrak{u}(2)}_1/G}_{[g_2^s]} = \sum_{m \in \mathbb{Z}} \left| n_b m + r_b^+ \right\rangle^{\text{Vir}}_{[g_2^s]},$$

$$|r_b^-\rangle^{\widehat{s\mathfrak{u}(2)}_1/G}_{[-g_2^s]} = \sum_{m \in \mathbb{Z}} \left| n_b m + r_b^- + \frac{1}{2} \right\rangle^{\text{Vir}}_{[-g_2^s]}.$$

Here we have identified the representations $r_a^+ \in \text{Irr}^+(N_a)$ and $r_b^\pm \in \text{Irr}^+(N_b)$ with the positive integers $r_a^+ = 1, \ldots, n_a$ and $r_b^\pm = 1, \ldots, n_b$, so that the corresponding characters of $N_a$ and $N_b$ are expressed as

$$\text{ch}^{N_a}_{r_a^+}(g_a) = e^{2\pi i r_a^+}, \quad \text{ch}^{N_b}_{r_b^+}(g_b) = e^{2\pi i r_b^+}, \quad \text{ch}^{N_b}_{r_b^-}(g_b) = e^{2\pi i r_b^- + \frac{1}{4}}.$$  

It is not difficult to see that the expressions (13) and (16)–(18) reproduce the expected overlaps, (16)

$$\langle [g]|(q^{H_\eta}|r \rangle)^{\widehat{s\mathfrak{u}(2)}_1/G}_{[g]} = \delta_{(g,r),(g',r')} \chi(g,r)(\tau),$$

where $\chi(g,r)$ denotes the Virasoro characters of the chiral primary fields $(g, r)$, with $[g] \in C(G)$ and $r \in \mathcal{R}(g)$.

**Fractional type boundary states**

Defining the $\widehat{s\mathfrak{u}(2)}_1/G$-Ishibashi states as above leads to the unique set of the Cardy states (given later). From these Cardy states, we can guess the expressions of more generic (fundamental) boundary states, as mentioned in the Introduction.
Although such a prescription may not lead to a unique set of boundary states, we propose a possible set below. Each of the boundary states we propose corresponds to a fixed orbit. Conversely, an arbitrary fixed orbit \([g] \in \mathcal{F}\) is associated with a set of fundamental boundary states. These states are denoted by \([g;r]^G\) with \(r \in \mathcal{R}(g)\).

Here, the set \(\mathcal{R}(g)\) is identified with either \(\text{Irr}^+(N(g))\) or \(\text{Irr}^-(N(g))\), as shown in Table I.\(^*\)

This identification results in the following relations for irreducible characters of \(N(g)\):

\[\frac{|N(g)|}{2} = \sum_{r \in \mathcal{R}(g)} (\text{ch}_r^{N(g)}(1))^2.\]  \(21\)

We construct the states \([g;r]^G\) to satisfy

\[|g]^G = \sum_{r \in \mathcal{R}(g)} \text{ch}_r^{N(g)}(1)|g;r|^G.\] \(22\)

Clearly, the right-hand side and the character relation \(21\) explain the origin of the \(|N(g)|/2\)-fold degeneracy associated with the non-fundamental boundary state \([g]^G\), the left-hand side. The generic expressions of the states \([g;r]^G\), with \([g] \in \mathcal{F}\) and \(r \in \mathcal{R}(g)\), are given by

\[|g;r|^G = \frac{2\text{ch}_r^{N(g)}(1)}{|N(g)|} |g|^G + |g;r|^T.\] \(23\)

Here, \([g;r]^G_T\) denotes contributions to the states \([g;r]^G\) from the non-trivial twisted sectors in the \(SU(2)_1/G\) orbifold. To express these contributions, we define the following combinations of the Virasoro Ishibashi states in the \(\pm g_j^a\)-twisted sectors with \(j = a, b\) and \(s = 1, \ldots, n_j\):

\[|e^{i\theta a^j}|_{\pm g_j^a}^G = \frac{2^{-1/4}}{\sqrt{n_j}} \sum_m e^{-i\theta(2m+s/n_j)}|m\rangle^{\text{Vir}}_{\pm g_j^a}, \quad -\pi < \theta \leq \pi,\] \(24\)

where the indices \(m\) are summed over the ranges given in \(13\). For convenience, let us also introduce sets of mutually non-conjugate \(G\)-elements:

\[\tilde{N}_a = \{g_a^s \mid 1, \ldots, n_a - 1\}, \quad \tilde{N}_b = \{|\pm g_b^s| 1, \ldots, n_b - 1\}.\] \(25\)

\(^*\) Previously, we introduced the set \(\mathcal{R}(g)\) with \([g] \in C(G)\) to describe the primary-field content. Here we have not only extended the definition of \(\mathcal{R}(g)\) for the case \([g] \in C(G)\) to the case \([g] \in \mathcal{F}\), but also changed the role of \(\mathcal{R}(g)\) to describe the boundary-state content.
Using the quantities defined above, we can write the contributions \( |g; r \rangle^G_T \), with \( [g] \in \mathcal{F} \) and \( r \in \mathcal{R}(g) \) (also see Table I), as follows:

\[
| \pm 1; r \rangle^G_T = \sum_a \sum_{h \in \tilde{N}_a} c^G_r(h) | 1 \rangle^G_{[h]} + \sum_b \sum_{h \in \tilde{N}_b} c^G_r(h) | 1 \rangle^G_{[h]},
\]

(26)

\[
| e^{i \theta \sigma_a}; r \rangle^G_T = \sum_{h \in \tilde{N}_a} \left( c^G_r(h) | e^{i \theta \sigma_a} \rangle^G_{[h]} + c^G_r(h) | e^{-i \theta \sigma_a} \rangle^G_{[h]} \right),
\]

(27)

\[
| \pm e^{i \theta \sigma_b}; r \rangle^G_T = \sum_{h \in \tilde{N}_b} c^G_r(h) | e^{i \theta \sigma_b} \rangle^G_{[h]}.
\]

(28)

Clearly, the above contributions are made only from non-trivial twisted-sector states. This fact and the expressions (23) imply that the boundary states \( |g; r \rangle^G_T \) have fractional masses, \( 2c^G_r(1)/|N(g)| \), with respect to the bulk type boundary states (7). Therefore, we can regard the states \( |g; r \rangle^G_T \) as fractional type orbifold boundary states.

For clarity, we summarize the set of distinct boundary states constructed in this note (on the \( \hat{su}(2)_1/G \) orbifold). This set is given by

\[
\mathcal{B} = \{ |g\rangle^G \mid [g] \in \mathcal{F} \} \cup \{ |g; r \rangle^G \mid [g] \in \mathcal{F}, r \in \mathcal{R}(g) \},
\]

(29)

where the first (second) subset corresponds to the bulk type (fractional type) boundary states. By construction, the Cardy states belong to the above set. The set of Cardy states is given by

\[
\{ |g; r \rangle^G \mid [g] \in C(G), r \in \mathcal{R}(g) \} \subset \mathcal{B},
\]

(30)

where the states \( |g; r \rangle^G \) correspond to the Cardy states of the chiral primary fields \((g, r)\). This implies that the boundary states in the set (30) preserve the full \( \hat{su}(2)_1/G \) symmetry. There are other classes of boundary states that preserve some rational-symmetries in the \( \hat{su}(2)_1/G \) orbifold. In Ref. 11), the boundary states that preserve only the symmetry of the automorphism-type subalgebras of \( \hat{su}(2)_1/G \) were constructed from the Cardy states in the \( \hat{su}(2)_1/G \) orbifold, where \( G \) denotes the groups extended from \( G \) via the corresponding automorphism groups. Repeating the construction given there, we can show that these symmetry-breaking states also belong to the set \( \mathcal{B} \) and correspond to the conjugacy classes of \( \tilde{G} \).

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Appendix A

Non-Vanishing Overlaps and Some Useful Identities

Here we present several non-vanishing overlaps that contribute to the cylinder amplitudes determined from the boundary states given in this note. We also give
useful identities for checking Cardy’s conditions in the set of these boundary states. For these purposes, we first introduce some quantities. Let us introduce generalized characters by

$$\chi^{(n)}(\alpha|\tau) = \frac{1}{\eta(\tau)} \sum_{m \in \mathbb{Z}} q^{(nm+\alpha)^2}.$$  \hspace{2cm} \text{(A.1)}$$

We denote the representations in $\text{Irr}(N_a)$ and $\text{Irr}(N_b)$ by the positive integers $r_a = 0, 1, \cdots, 2n_a - 1$ and $r_b = 0, 1, \cdots, 2n_b - 1$, so that the corresponding characters have the following values at $g_a$ and $g_b$:

$$\chi_{r_a}^N(g_a) = e^{i\pi r_a/n_a}, \quad \chi_{r_b}^N(g_b) = e^{i\pi r_b/n_b}. \hspace{2cm} \text{(A.2)}$$

Let us set $\tilde{\tau} = -1/\tau$. Let us define a function of $g \in SU(2)$ by

$$\Delta(g) = \cos^{-1}\left(\frac{1}{2} \text{Tr}(g)\right). \hspace{2cm} \text{(A.3)}$$

Then, basic overlaps are given as follows:

- **Overlap 1.**

  $$\tilde{\sigma}_u^{(2)}(g|q^{H_c}|g')\tilde{\sigma}_u^{(2)} = \chi^{(1)}(\Delta_1|\tilde{\tau}), \quad g, g' \in SU(2), \quad \Delta_1 = \Delta(gg'^{-1}). \hspace{2cm} \text{(A.4)}$$

- **Overlap 2.**

  Let us extend the left-hand side of (27) to $0 \leq \theta \leq \pi$ and $r_a \in \text{Irr}(N_a)$ to define the corresponding combinations $|e^{i\theta\sigma_a}; r_a\rangle^G_T$. Then, we have the overlaps,

  $$\begin{align*}
  G_T(e^{i\theta\sigma_a}; r_a|q^{H_c}|e^{i\theta'\sigma_a}; r_a')^G &= \chi^{(n_a)}(\Delta_2^+|\tilde{\tau}) + \chi^{(n_a)}(\Delta_2^+|\tilde{\tau}) \\
  &- \frac{1}{n_a} \chi^{(1)}(\Delta_2^+|\tilde{\tau}) - \frac{1}{n_a} \chi^{(1)}(\Delta_2^+|\tilde{\tau}),
  \end{align*} \hspace{2cm} \text{(A.5)}$$

  where $0 \leq \theta, \theta' \leq \pi$, $r_a, r_a' \in \text{Irr}(N_a)$ and $\Delta_2^\pm = (\theta + \pi r_a) \pm (\theta' + \pi r_a')$.

- **Overlap 3.**

  Similarly to the above case, we extend the definition of $| \pm e^{i\theta\sigma_b}; r_b \rangle^G_T$ to cover the range $0 \leq \theta \leq \pi$ and the representations $r_b \in \text{Irr}(N_b)$. Then, we have the overlaps,

  $$\begin{align*}
  G_T(e^{i\theta\sigma_b}; r_b|q^{H_c}|e^{i\theta'\sigma_b}; r_b')^G &= \chi^{(n_b)}(\Delta_3|\tilde{\tau}) - \frac{1}{n_b} \chi^{(1)}(\Delta_3|\tilde{\tau}),
  \end{align*} \hspace{2cm} \text{(A.6)}$$

  where $-\pi \leq \theta, \theta' \leq \pi$, $r_b, r_b' \in \text{Irr}(N_b)$ and $\Delta_3 = (\theta + \pi r_b) - (\theta' + \pi r_b')$.

By combining the above overlaps, in principle, we can calculate all the cylinder amplitudes obtained in our set of boundary states. For such a calculation, however, it is useful to recognize the following facts (relations):

1. Let us introduce $|u\rangle = \frac{2}{\sqrt{|N(g)|}}|g\rangle^G$ and $|v\rangle = \frac{2}{\sqrt{|N(g')|}}|g'\rangle^G$ for $g, g' \in SU(2)$. Then, it follows that

  $$\langle u|q^{H_c}|v\rangle = \frac{2}{|N(g)|} \sum_{h \in [g']} \tilde{\sigma}_u^{(2)}(g|q^{H_c}|h)\tilde{\sigma}_u^{(2)}(g') \hspace{2cm} \text{(A.7)}$$

  $$= \frac{2}{|N(g')|} \sum_{h \in [g]} \tilde{\sigma}_u^{(2)}(h|q^{H_c}|g')\tilde{\sigma}_u^{(2)}(g') \hspace{2cm} \text{(A.8)}$$
2. There exist $2 \times 2$ matrices $\sigma_{am}$ with $m = 1, 2, \cdots$, and $\sigma_{bm}$ with $m = 1, 2, \cdots$, such that the $G$-orbits $[\sigma_a]$ and $[\sigma_b]$ are decomposed into $N_a$- and $N_b$-orbits as follows:

$$[\sigma_a] \rightarrow [\sigma_a]_{N_a} + [-\sigma_a]_{N_a} + \sum_m [\sigma_{am}]_{N_a}, \tag{A.9}$$

$$[\sigma_b] \rightarrow [\sigma_b]_{N_b} + \sum_m [\sigma_{bm}]_{N_a}. \tag{A.10}$$

Here note that the order of the orbits on the right-hand sides are given by $|[\pm \sigma_a]_{N_a}| = |[\sigma_b]_{N_b}| = 1$ and $|[\sigma_{am}]_{N_a}| = n_a$, $|[\sigma_{bm}]_{N_b}| = n_b$. After some calculations, we can conclude that

$$\frac{1}{(n_a)^2} |G_e^{iθ_a\sigma_a}| H e^{iθ_a\sigma_a} |G = \frac{1}{n_a} \chi(1) (\Delta_a^+ |\bar{\tau}) + \frac{1}{n_a} \chi(1) (\Delta_a^- |\bar{\tau}) + \sum_m \chi(1) (\Delta_{am} |\bar{\tau}), \tag{A.11}$$

$$\frac{1}{(n_b)^2} |G_e^{iθ_b\sigma_b}| H e^{iθ_b\sigma_b} |G = \frac{1}{n_b} \chi(1) (\Delta_b |\bar{\tau}) + \sum_m \chi(1) (\Delta_{bm} |\bar{\tau}), \tag{A.12}$$

where

$$0 \leq θ_a, θ_a' \leq π, \quad Δ_a^+ = θ_a \pm θ_a' \quad \text{and} \quad Δ_{am} = Δ(e^{iθ_a\sigma_a} e^{-iθ_a'\sigma_{am}}); \tag{A.13}$$

$$-π < θ_b, θ_b' \leq π, \quad Δ_b = θ_b - θ_b' \quad \text{and} \quad Δ_{bm} = Δ(e^{iθ_b\sigma_b} e^{-iθ_b'\sigma_{bm}}). \tag{A.14}$$

3. The contributions (26) can be written as sums of the contributions (27) and (28), if the latter contributions are extended to $θ = 0, π$. This is because the irreducible characters of $G$ are expressed in terms of irreducible characters of $N_a$ and $N_b$ for the $G$-elements in $\tilde{N}_a$ and $\tilde{N}_b$ [defined in (25)]. More concretely, for the representations $r \in \text{Irr}(G)$, there exist representations $r_a \in \text{Irr}(N_a)$ and $r_b \in \text{Irr}(N_b)$, and integers $c^a_r$ (or $c^{a'}_r$) and $c^b_r$, such that the following relations hold:

$$\text{ch}^G_r(h) = \begin{cases} 
 c^a_r \text{ch}^{N_a}_{r_a}(h) + c^a_r \text{ch}^{N_a}_{r_a}(h)^* \quad \text{if} \quad r_a \neq 0, n_a \\
 c^a_r \text{ch}^{N_a}_{r_a}(h) \quad \text{or} \quad c^{a'}_r \text{ch}^{N_a}_{r_a}(h) \quad \text{if} \quad r_a = 0, n_a \quad \text{for} \quad h \in \tilde{N}_a \\
 c^b_r \text{ch}^{N_b}_{r_b}(h) \quad \text{for} \quad h \in \tilde{N}_b \end{cases} \quad \tag{A.15}$$

$$c^a_r = \begin{cases} 
 1 \quad \text{ch}^G_r(1) = 2 \mod n_a \\
 0 \quad \text{ch}^G_r(1) = 0 \mod n_a \\
 -1 \quad \text{ch}^G_r(1) = -2 \mod n_a \end{cases} \quad \tag{A.16}$$

$$c^{a'}_r = \begin{cases} 
 1 \quad \text{ch}^G_r(1) = 1 \mod n_a \\
 0 \quad \text{ch}^G_r(1) = 0 \mod n_a \\
 -1 \quad \text{ch}^G_r(1) = -1 \mod n_a \end{cases} \quad \tag{A.17}$$
\[ \ell_r^b = \begin{cases} 1 & \text{ch}_r^G(1) = 1 \mod n_b, \\ 0 & \text{ch}_r^G(1) = 0 \mod n_b, \\ -1 & \text{ch}_r^G(1) = -1 \mod n_b. \end{cases} \] (A.18)

4. The characters \( \chi^{(n)} \) are related with the characters \( \chi^{(1)} \) in the following way:

\[ \chi^{(1)}(\alpha|\tau) = \sum_{k=0}^{n-1} \chi^{(n)}(\alpha + k|\tau). \] (A.19)

5. Let us assign indices \( \epsilon_r \) to the representations \( r \in \text{Irr}(G) \) as \( \epsilon_r = \pm 1 \) for \( r \in \text{Irr}^{\pm}(G) \). Then, we can show the relations,

\[ G\langle \epsilon_r \cdot 1; r|q^H|\epsilon_r \cdot 1; r'\rangle^G = \sum_{r'' \in \text{Irr}(G)} N_{r^{r''}} \chi_{(\epsilon_r, 1, r'')}(\tilde{\tau}), \] (A.20)

where

\[ N_{r^{r''}} = \frac{1}{|G|} \sum_{h \in G} \text{ch}_r^G(h)^* \text{ch}_{r'}^G(h) \text{ch}_{r''}^G(h). \] (A.21)

Note that the quantities \( N_{r^{r''}} \) are the non-negative integers that appear in the decompositions of tensor-products of \( r \in \text{Irr}(G) \) and \( r' \in \text{Irr}(G) \):

\[ (r) \otimes (r') = \sum_{r'' \in \text{Irr}(G)} N_{r^{r''}} (r''). \] (A.22)

It may be useful to present \( \chi_{(\epsilon_r, 1, r)} \), the Virasoro characters of the \( \widehat{su}(2)_1/G \) primary fields \( (\epsilon_r \cdot 1, r) \). These are explicitly expressed as follows:

\[ \chi_{(\epsilon_r, 1, r)}(\tilde{\tau}) = \frac{1}{|G|} q(\tau) \sum_{h \in G} \text{ch}_r^G(h)^* \sum_{m \in \mathbb{Z}} q^{m^2/4} e^{im\Delta(h)}. \] (A.23)

References