IDENTITIES ON MAXIMAL SUBGROUPS OF $GL_n(D)$

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Abstract

Let $D$ be a division ring with centre $F$. Assume that $M$ is a maximal subgroup of $GL_n(D)$, $n \geq 1$ such that $Z(M)$ is algebraic over $F$. Group identities on $M$ and polynomial identities on the $F$-linear hull $F[M]$ are investigated. It is shown that if $F[M]$ is a PI-algebra, then $[D : F] < \infty$. When $D$ is noncommutative and $F$ is infinite, it is also proved that if $M$ satisfies a group identity and $F[M]$ is algebraic over $F$, then we have either $M = K^*$, where $K$ is a field and $[D : F] < \infty$ or $M$ is absolutely irreducible. For a finite dimensional division algebra $D$, assume that $N$ is a subnormal subgroup of $GL_n(D)$ and $M$ is a maximal subgroup of $N$. If $M$ satisfies a group identity, it is shown that $M$ is abelian-by-finite.

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1 Introduction

Let $D$ be a division ring with centre $F$, and let $n$ be a positive integer. Denote by $A := M_n(D)$ the full $n \times n$ matrix ring over $D$ and by $A^* := GL_n(D)$, the units of $A$. Given a subgroup $M$ of $A^*$, we shall say that $M$ is maximal in $A^*$ if for any subgroup $L$ of $A^*$ with $M \subseteq L$, one concludes that $L = A^*$. The study of maximal subgroups of $A^*$ begins in \cite{1} and \cite{9} in relation with an investigation of the structure of finitely generated normal subgroups of $GL_n(D)$, where $D$ is of finite dimension over its centre $F$. In those papers we essentially show that maximal subgroups arise naturally in $A^*$, and finitely generated subnormal subgroups of $A^*$, are central. This result is used to prove that a maximal subgroup of $A^*$ cannot be finitely generated. The reader may consult \cite{7}, and the references therein for more recent results on multiplicative subgroups of $A^*$. The objective of this note is to investigate the algebraic structure of $D$ when the $F$-linear hull $F[M]$ satisfies a polynomial identity. We also study the structure of $M$ whenever $M$ satisfies a group identity. To be more precise, let $M$ be a maximal subgroup of $A^*$ such that the centre of $M$, $Z(M)$ is algebraic over $F$. It is shown that if $F[M]$ is a PI-algebra, then $[D : F] < \infty$. As a consequence of this, some results of \cite{1} that are proved there for the case $n = 1$ may be generalized for $n > 1$. For example, it is shown if $[A : F] = \infty$, then $A^*$ contains no abelian maximal subgroup which is algebraic over $F$. Also, if $M$ satisfies a multilinear polynomial identity, then $[D : F] < \infty$. In this direction, it is proved that if $D$ is noncommutative and $M/F^*$ is locally finite, then we have either $M$ is absolutely irreducible and $D$ is locally finite dimensional or $[D : F] < \infty$ and $M = K^*$, where $K$ is a subfield of $A$. We then turn to the case where $M$ satisfies a group identity. Let $D$ be a noncommutative division ring with infinite centre $F$ and $n \geq 1$. Assume that $M$ is a maximal subgroup of $A^*$ such that $F[M]$ is algebraic over $F$. It is shown that if $M$ satisfies a group identity, then we have either $M = K^*$, where $K$ is a field and $[D : F] < \infty$ or $M$ is absolutely irreducible. In particular, if $M$ is a noncommutative soluble maximal subgroup of $A^*$ such that $F[M]$ is algebraic over $F$, then $M$ is abelian-by-locally finite. For a noncommutative finite dimensional $F$-central division algebra $D$, assume that $N$ is a subnormal subgroup of $A^*$ and $M$ is a maximal subgroup of $N$. It is proved that if $M$ satisfies a group identity, then $M$ is abelian-by-finite.

2 Notations and conventions

Let $D$ be a division ring with centre $F$ and $G$ be a subgroup of $A^* = GL_n(D)$. We denote by $F[G]$ the $F$-linear hull of $G$, i.e., the $F$-algebra generated by elements of $G$ over $F$. We also denote by $D^n$ the space of row $n$-vectors over $D$. Then $D^n$ is a $D - G$ bimodule in the obvious manner. $G$ is said to be an irreducible (reducible) subgroup of $GL_n(D)$ whenever $D^n$ is irreducible (reducible) as $D - G$ bimodule. Considering the elements of $D^n$ as column vectors, we may regard $D^n$ as a $G - D$ bimodule. It is easily shown that $D^n$ is irreducible (reducible) as
a \( G - D \) bimodule precisely when it has the property as \( D - G \) bimodule. We shall say that \( G \) is \textit{absolutely irreducible} if \( M_n(D) = F[G] \). For any group \( G \) we denote its centre by \( Z(G) \). Given a subgroup \( H \) of \( G \), \( N_G(H) \) means the \textit{normalizer} of \( H \) in \( G \), and \( <H,K> \) the group generated by \( H \) and \( K \), where \( K \) is a subgroup of \( G \). We shall say that \( H \) is \textit{abelian-by-finite} if there is an abelian normal subgroup \( K \) of \( H \) such that \( H/K \) is finite. Let \( S \) be a subset of \( M_n(D) \), then the \textit{centralizer} of \( S \) in \( M_n(D) \) is denoted by \( C_{M_n(D)}(S) \). We shall identify the centre \( FI \) of \( M_n(D) \) with \( F \). By a dilatation matrix \( D_{ii}(d), d \in D^* \) we understand a diagonal \( n \times n \) matrix whose diagonal entries are all 1 except the \((i,i)\)-th entry which is \( d \). Some notations and conventions for linear groups and skew linear groups from [11] and [12] are frequently used throughout.

3 Polynomial identities on \( F[M] \)

Given a maximal subgroup of \( A^* \), this section essentially deals with conditions on \( M \) that imply either the commutativity of \( M \) or \( [D:F] < \infty \). The main result is Theorem 5 which asserts that if \( F[M] \) is a PI-algebra and \( Z(M) \) is algebraic over \( F \), then \( [D:F] < \infty \). Using this, it is shown that if \( M \) satisfies a multilinear polynomial identity and \( Z(M) \) is algebraic over \( F \), then \( [D:F] < \infty \). Furthermore, it is proved that if either \( n = 1 \) and \( D \) is noncommutative or \( n > 1 \) and \( D \) is infinite, then there exists no maximal subgroup \( M \) of \( A^* \) containing \( F^* \) such that \( [M:F^*] < \infty \). For a noncommutative division ring with centre \( F \), it is also shown that if \( M/F^* \) is locally finite, then we have either \( M \) is absolutely irreducible and \( D \) is locally finite dimensional or \( [D:F] < \infty \) and \( M = K^* \), where \( K \) is a subfield of \( A \). We begin our material with

\textbf{Proposition 3.1.} Let \( D \) be a division ring with centre \( F \) and \( n \geq 1 \). Assume that \( M \) is a maximal subgroup of \( GL_n(D) \). Then \( M \) is either irreducible or it contains an isomorphic copy of \( D^* \).

\textbf{Proof.} The case \( n = 1 \) follows from Proposition 1 of [1]. So, we may assume that \( n \geq 2 \). Now, consider the \( F \)-algebra \( F[M] \). Since \( M \) is maximal we conclude that either \( GL_n(D) = F[M]^* \) or \( F[M]^* = M \). The first case implies that \( M_n(D) = F[M] \), i.e., \( M \) is absolutely irreducible and so it is irreducible. Thus, we may assume that \( F[M]^* = M \). If \( M \) is not irreducible, then \( D^n \) has a nontrivial submodules as \( D - F[M] \) bimodule. Thus, by 1.1.1 of [12], there exists an invertible \( n \times n \) matrix \( P \) over \( D \) such that \( PMP^{-1} \subset \Sigma \), where \( \Sigma = \{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in GL_n(D), C \in GL_{n-s}(D), B \in M_{s \times (n-s)}(D) \} \). It is clear that \( PMP^{-1} \) is also a maximal subgroup of \( GL_n(D) \) and we have \( PMP^{-1} \subseteq \Sigma \subset GL_n(D) \). Therefore, \( PMP^{-1} = \Sigma \) and since \( \Sigma \) contains a copy of \( D^* \) we obtain the result.

We shall need the following lemmas to prove our main theorem.

\textbf{Lemma 3.2.} Let \( D \) be a division ring of infinite dimension over its centre \( F \) and \( n \geq 1 \).
Assume that $M$ is a maximal subgroup of $GL_n(D)$. If $F[M]$ is a PI-algebra, then $F[M]$ is a prime ring.

**Proof.** By Proposition 1, we know that either $M$ contains a copy of $D^*$ or $M$ is irreducible. If the first case happens, then $D$ is a PI-algebra. This implies, by Kaplansky’s Theorem (cf. [11]), that $[D : F] < \infty$ which is a contradiction. So we may assume that $M$ is irreducible. Now, using 1.1.14 of [12, p. 9], we conclude that $F[M]$ is a prime ring.

**Lemma 3.3.** Let $D$ be a division ring of infinite dimension over its centre $F$ and $n \geq 1$. Assume that $M$ is a maximal subgroup of $GL_n(D)$. If $F[M]$ is a PI-algebra, then $Z(F[M])$ is a field.

**Proof.** Set $A = F[M]$. Assume that $X \in Z(A)$. We consider two cases:

Case 1. If $X \in GL_n(D)$ and $X^{-1} \notin A$, then we have $< M, X > \subseteq C_{M_n(D)}(Z(A))$ and by maximality of $M$ we conclude that $C_{M_n(D)}(Z(A)) = M_n(D)$. This means that $Z(A) = F$ which is a contradiction. Thus $X^{-1} \in Z(A)$.

Case 2. Assume that $X \notin GL_n(D)$. Then there exists an $n \times n$ matrix $P$ in $GL_n(D)$ such that the first row of $PXP^{-1}$ is zero. We note that $PMP^{-1}$ is a maximal subgroup of $GL_n(D)$, $F[PMP^{-1}]$ is a PI-algebra, and $PXP^{-1} \in Z(F[PMP^{-1}]) = PZ(F[M])P^{-1}$. Set $J := XF[M]$. Then $J$ is an ideal in $F[M]$ and $PXP^{-1}(PP[M]P^{-1}) = PJP^{-1} = PXP^{-1}F[PMP^{-1}]$ is also an ideal in $F[PMP^{-1}]$. Put $J' = PJP^{-1}$ and $B := \{ Y \in M_n(D) | YJ' \subset J' \}$. It is clear that $B$ is a ring and $PAP^{-1} \subseteq B$. On the other hand $PXP^{-1}$ is a matrix whose first row is zero. Therefore, $D_{11}(d) \in B$ for all $d \in D$. It is clear that $PMP^{-1} \subseteq B^*$. Thus, $B^* = GL_n(D)$ or $B^* = PMP^{-1}$. In the first case $B = M_n(D)$ and so $J$ is a right ideal and clearly $J$ is a right ideal in $M_n(D)$.

In the second case $D_{11}(d) \in PMP^{-1}$ for all $d \in D$. Therefore, $F[PMP^{-1}]$ contains a copy of $D^*$ and so $D$ is a PI-algebra which implies $[D : F] < \infty$ that is a contradiction. Similarly, there exists a matrix $Q \in GL_n(D)$ such that $QXQ^{-1}$ is a matrix whose first column is zero. Set $C = \{ Y \in M_n(D) | (QJQ^{-1})Y \subset QJQ^{-1} \}$. As above, one may show that $J$ is a left ideal in $M_n(D)$. Consequently, $J$ is an ideal in $M_n(D)$. Since $J \neq 0$ we obtain $J = M_n(D)$. Therefore, $F[M] = M_n(D)$ and so $[D : F] < \infty$ which is a contradiction. Thus, $Z(A)$ is a field and the proof is complete.

**Lemma 3.4.** Let $D$ be a division ring of infinite dimension over its centre $F$ and $n \geq 1$. Assume that $M$ is a maximal subgroup of $GL_n(D)$. If $F[M]$ is a PI-algebra, then $F[M]$ is simple and we have $[F[M] : Z(F[M])] < \infty$.

**Proof.** By Lemmas 3.2-3, we conclude that $F[M]$ is a prime ring whose centre is a field. Since $F[M]$ is also a PI-algebra, by Theorem 7.5 of [3], we conclude that $F[M]$ is simple. Finally, the rest of the result follows from Kaplansky’s Theorem.

**Theorem 3.5.** Let $D$ be a division ring with centre $Z(D) = F$ and $n \geq 1$. Assume that $M$ is a maximal subgroup of $GL_n(D)$ such that $Z(M)$ is algebraic over $F$. If $F[M]$ is a PI-algebra,
then $[D : F] < \infty$.

**Proof.** We have $M \subseteq F[M]^*$. By maximality of $M$ we conclude that either $F[M]^* = GL_n(D)$ or $F[M]^* = M$. The first case gives us $F[M] = M_n(D)$. Now, use Kaplansky’s Theorem to obtain $[D : F] < \infty$. Therefore, we may assume that $F[M]^* = M$. To complete the proof, we show that the assumption $[D : F] = \infty$ leads to a contradiction. Thus, suppose $[D : F] = \infty$. Then, by Lemma 4 and Artin-Wedderburn’s Theorem, we have $F[M] \cong M_n(D_1)$ for some positive integer $n_1$ and division ring $D_1$, and so $M \cong GL_{n_1}(D_1)$. We claim that $Z(M) = F^*$. For otherwise, since $Z(M)$ is algebraic over $F$ there exists $a \in Z(M) \setminus F^*$ such that $[F(a) : F] < \infty$. Now, we have $F[M] \subseteq C_{M_n(D)}(F(a)) := A$. If $F[M] \neq A$, then, by the Centralizer Theorem, we conclude that $A$ is a simple Artinian ring. Therefore, there exists a positive integer $n_2$ and a division ring $D_2$ such that $A \cong M_{n_2}(D_2)$ and so $A^* \cong GL_{n_2}(D_2)$. We know that $M \subseteq A^*$. If $M = A^*$, then we clearly have $F[M] = A$ which is a contradiction to our assumption. Thus, by maximality of $M$, one concludes that $A^* = GL_n(D)$, i.e., $C_{M_n(D)}(F(a)) = A = M_n(D)$ which contradicts the fact that $a \in Z(M) \setminus F$. Therefore, we must have $A = F[M]$ and so $Z(F[M]) = Z(A) = F(a)$ by the Centralizer Theorem. Thus, by Lemma 4, this means that $[F[M] : F] < \infty$. Now, apply the Centralizer Theorem again to obtain $M_n(D) \otimes F(a)^{0p} \cong M_s(F) \otimes F[M]$ for some positive integer $s$. The last isomorphism implies that $[D : F] < \infty$ which contradicts our assumption. Therefore, we must have $Z(M) = F^*$ and the claim is established. Now, since $F[M]$ is simple and $F[M]^* = M$ we obtain $Z(F[M]) = F$. Finally, consider the simple Artinian ring $B := C_{M_n(D)}(F[M])$. There exists a positive integer $n_3$ and a division ring $D_3$ such that $B \cong M_{n_3}(D_3)$. If $F \neq C_{M_n(D)}(F[M])$, then $F^* \subseteq B^* \cong GL_{n_3}(D_3)$. Therefore, there is an $X \in B^* \setminus F^*$ such that $<M, X> \subseteq C_{M_n(D)}(F(X))$ and so $M_n(D) = C_{M_n(D)}(F(X))$ which is a contradiction to the fact that $X \notin F^*$. Thus, $F = C_{M_n(D)}(F[M])$. Now, using the Centralizer Theorem as above, one concludes that $[D : F] < \infty$ which is a contradiction and this completes the proof.

The next result generalizes Theorem 4.1 of [1].

**Corollary 3.6.** Let $D$ be a division ring with centre $Z(D) = F$ and $n \geq 1$. Assume that $M$ is a maximal subgroup of $GL_n(D)$ such that $Z(M)$ is algebraic over $F$. If $M$ satisfies a multi-linear polynomial identity, then $[D : F] < \infty$.

The following result is also a generalization of Corollary 4.2 of [1].

**Corollary 3.7.** Let $D$ be a division ring of infinite dimension over its centre $Z(D) = F$ and $n \geq 1$. Then $GL_n(D)$ contains no abelian maximal subgroup which is algebraic over $F$.

**Corollary 3.8.** Let $D$ be a division ring with centre $F$. If either $n = 1$ and $D$ is non-commutative or $n > 1$ and $D$ is infinite, then there exists no maximal subgroup $M$ of $GL_n(D)$, $n \geq 1$, containing $F^*$ such that $[M : F^*] < \infty$.

**Proof.** Assume that there is a maximal subgroup $M$ such that $[M : F^*] < \infty$. Then,
we have $[F[M] : F] < \infty$, i.e., $F[M]$ is a $PI$-algebra and $M$ is algebraic over $F$. Thus, by Theorem 5, we obtain $[D : F] < \infty$. Let $x_1, \ldots, x_t$ be the representatives for cosets of $F^*$ in $M$, i.e., $M = F^*x_1 \cup \cdots \cup F^*x_t$. Then, we have $M = \langle x_1, \ldots, x_t \rangle > F^*$, where $\langle x_1, \ldots, x_t \rangle$ is the group generated by $x_1, \ldots, x_t$. Take $x \in GL_n(D) \setminus M$. By maximality of $M$, we obtain $GL_n(D) = \langle x_1, \ldots, x_t, x \rangle > F^*$. Put $H = \langle x_1, \ldots, x_t, x \rangle$. Thus, $GL_n(D) = HF^*$ and consequently we have $SL_n(D) = H' < H$, i.e., $H$ is normal in $GL_n(D)$. Now, by Theorem 5 of [2], we conclude that $H \subset F^*$, i.e., $GL_n(D) = F^*$ which means that $n = 1$ and $D = F$ that is a contradiction and so the result follows.

**Corollary 3.9.** Let $R$ be a semisimple Artinian $F$-algebra with $[Z(R) : F] < \infty$. Assume that $M$ is a maximal subgroup of $R^*$ such that $Z(M)$ is algebraic over $F$. If $F[M]$ satisfies a polynomial identity, then we have $[R : F] < \infty$.

Now, assume that $M$ is a maximal subgroup of $GL_n(D)$ containing $F^*$ such that $M/F^*$ is locally finite. The next result gives some information about the algebraic structure of $M$.

**Theorem 3.10.** Let $D$ be a noncommutative division ring with centre $F$ and $n \geq 1$. Assume that $M$ is a maximal subgroup of $GL_n(D)$ containing $F^*$ such that $M/F^*$ is locally finite. Then we have either $M$ is absolutely irreducible and $D$ is locally finite dimensional or $[D : F] < \infty$ and $M = K^*$, where $K$ is a subfield of $M_n(D)$.

**Proof.** Consider the $F$-algebra $F[M]$. If $F[M]^* = GL_n(D)$, then we have $F[M] = M_n(D)$. Since $M/F^*$ is locally finite we conclude that $D$ is locally finite dimensional and so the result follows. Thus, we may assume throughout that $F[M]^* \neq GL_n(D)$. By maximality of $M$ we conclude that $F[M]^* = M$. We observe that since $M/F^*$ is locally finite then $F[M]$ is locally finite dimensional. Therefore, for each finite set of elements $m_1 \cdots m_r \in M$ we have $[F[m_1 \cdots m_r] : F] < \infty$. We next claim that $F[M]$ is a $PI$-algebra satisfying $P(X, Y) = (XY - YX)^n$. To do this, let $x, y \in F[M]$. Then, there exist elements $m_i, n_j \in M$ with $1 \leq i \leq t$, $1 \leq j \leq s$ such that $x = m_1 \cdots m_t$ and $y = n_1 \cdots n_s$, and we have $[F[m_1 \cdots m_t, n_1 \cdots n_s] : F] < \infty$. Therefore, $A = F[m_1 \cdots m_t, n_1 \cdots n_s]$ is an Artinian $PI$-ring and so the Jacobson radical $J = J(A)$ of $A$ is nilpotent. Therefore, by 1.39 of [12], we conclude that $J^n = 0$. Now, by the Wedderburn-artin Theorem, there exist positive integers $n_1, \cdots, n_k$ such that $B := A/(J(A) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ as $F$-algebras, for some division rings $D_i$, where $1 \leq i \leq k$. Now, we have $B^* = \{m + J|m \in M \cap A\} \cong GL_{n_1}(D_1) \times \cdots GL_{n_k}(D_k)$. Since $B^*$ is torsion over $F$ we conclude that each $D_i$ is torsion over $F$ where $F \subset Z(D_i)$. Thus $D_i$ is torsion over $Z(D_i)$ and by a result of Kaplansky (cf. [5]), we obtain $D_i = Z(D_i) := F_i$. Therefore, we obtain $B \cong M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_k)$. If there exists $i$ such that $n_i \neq 1$, then the dilatation $n_i \times n_i$ matrix $D_{1i}(f)$, where $f \in F_i$, is torsion over $F$. In particular, this implies that $F^*$ is a torsion group. Now, $F^*$ and $M/F^*$ are locally finite. Therefore, by a well-known result of group theory, $M$ is locally finite. By Proposition 1, either $M$ contains an isomorphic copy of $D^*$ or $M$ is irreducible. The first case says that $D^*$ is locally finite. This implies, by a
result of Jacobson (cf. [5]), that $D = F$ which is a contradiction. Thus, we may assume that $M$ is irreducible and so it is completely reducible. Therefore, by 1.1.14 of [12], we conclude that $F[M]$ is semisimple Artinian. Thus, as in the above case, there exist positive integers $m_1, \ldots, m_k$ such that $F[M] \cong M_{m_1}(D_1) \times \cdots \times M_{m_k}(D_k)$, for some division rings $D_i$, where $1 \leq i \leq k$. Now, we have $F[M]^* \cong M \cong GL_{m_1}(D_1) \times \cdots \times GL_{m_k}(D_k)$. Since $M$ is locally finite, as above, we conclude that $D_i = F_i = Z(D_i)$. Therefore, $F[M] \cong M_{m_1}(F_1) \times \cdots \times M_{m_k}(F_k)$. This means that $F[M]$ satisfies a polynomial identity and so by Theorem 5 we conclude that $[D : F] < \infty$. Since $F^*$ is torsion we obtain $D = F$ which is a contradiction. Therefore, in the decomposition of $B$ for all $i$ we have $n_i = 1$, i.e., $B \cong F_1 \times \cdots \times F_k$. This implies that for each $x, y \in A$ we have $xy - yx \in J$ and therefore $(xy - yx)^n = 0$ since $J^n = 0$. Thus, $F[M]$ satisfies $P(X, Y) = (XY - YX)^n$. Therefore, by Theorem 5, we obtain $[D : F] < \infty$. Since $M$ is irreducible and $[D : F] < \infty$, by 1.1.12 of [12], we conclude that $F[M]$ is simple Artinian, i.e., $F[M] \cong M_t(D_1)$ for some $t$ and division ring $D_1$. Now, we may use a similar argument as above to deduce that $D_1 = F_1 = Z(D_1)$. If $t \neq 1$, we may conclude that $F$ is torsion and since $[D : F] < \infty$ we obtain $D = F$ which is a contradiction. Therefore, $t = 1$ and we have $F[M]^* = M \cong F_1^*$ which completes the proof.

Corollary 3.11. Let $D$ be a division ring with centre $F$ and $n \geq 1$. Assume that $M$ is a maximal subgroup of $GL_n(D)$. If $M$ is locally finite, then we have $D = F$.

Proof. If $F^* \not\subset M$, then $SL_n(D) \subset M$. Since $M$ is torsion, we conclude that $SL_n(D)$ is torsion. Thus, by Corollary 2 of [8], we conclude that $D = F$. So, we may assume that $F^* \subset M$. This implies that $F^*$ is torsion and so $M/F^*$ is torsion. Now, Theorem 10 implies that $D$ is algebraic over $F$. Therefore, $D$ is algebraic over its prime subfield and so by a result of Jacobson we conclude that $D = F$.

4 Group identities on $M$

Let $D$ be a noncommutative division ring with infinite centre $F$ and $n$ be a positive integer. Given a maximal subgroup $M$ of $A^*$ such that $F[M]$ is algebraic over $F$, the key result of this section is to show that if $M$ satisfies a group identity, then we have either $M = K^*$, where $K$ is a field and $[D : F] < \infty$ or $M$ is absolutely irreducible. Particularly, if $M$ is a noncommutative soluble maximal subgroup of $A^*$ such that $F[M]$ is algebraic over $F$, it is proved that $M$ is abelian-by-locally finite. When $D$ is of finite dimension over $F$, assume that $N$ is a subnormal subgroup of $A^*$ and $M$ is a maximal subgroup of $N$. It is proved that if $M$ satisfies a group identity, then $M$ is abelian-by-finite.

Theorem 4.1. Let $D$ be a noncommutative division ring with infinite centre $F$ and $n \geq 1$. Assume that $M$ is a maximal subgroup of $GL_n(D)$ such that $F[M]$ is algebraic over $F$. If $M$ satisfies a group identity, then we have either $M = K^*$, where $K$ is a field and $[D : F] < \infty$ or
$$M$$ is absolutely irreducible.

**Proof.** If $$F^* \not\subseteq M$$, then $$M$$ is normal in $$GL_n(D)$$. This means that $$F[M]$$ is normal in $$M_n(D)$$. Thus, by a result of [10], we conclude that either $$F[M] \subset F$$ or $$F[M] = M_n(D)$$. If the first case occurs, then we have $$M \subseteq F^*$$ which is a contradiction to the fact that $$M$$ is a maximal subgroup of $$GL_n(D)$$. The second case says that $$M$$ is absolutely irreducible. So, let $$F^* \subset M$$, we have two cases to consider. If $$F[M]^* = GL_n(D)$$, then $$M$$ is absolutely irreducible. So, assume that $$F[M]^* = M$$. By Proposition 3.1, we know that either $$M$$ contains an isomorphic copy of $$D^*$$ or $$M$$ is irreducible. If the first case occurs, then $$D^*$$ satisfies a group identity. But since $$F$$ is infinite this is impossible by a result of Amitsur (cf. [11]). Therefore, $$M$$ must be irreducible. Thus, by 1.1.14 of [12, p. 9], we conclude that $$F[M]$$ is a prime ring. Now, by Theorem 5.5 of [6], we have $$F[M] \cong M_r(K)$$ for some positive integer $$r$$, where $$K$$ is an extension field of $$F$$. This shows that $$F[M]$$ satisfies a polynomial identity. Now, by Theorem 3.5, we conclude that $$[D : F] < \infty$$. Finally, we have $$F[M]^* = M \cong GL_r(K)$$. If $$r \neq 1$$, then $$M$$ contains a copy of $$SL_r(K)$$, i.e., $$M$$ contains a free subgroup (cf. [12]). This contradicts the fact that $$M$$ satisfies a group identity. Therefore, we have $$r = 1$$ and this completes the proof.

**Corollary 4.2.** Let $$D$$ be a division ring with infinite centre $$F$$ and $$n \geq 1$$. Assume that $$M$$ is a noncommutative maximal subgroup of $$GL_n(D)$$ such that $$F[M]$$ is algebraic over $$F$$. If $$M$$ satisfies a group identity, then $$M_n(D)$$ is algebraic over $$F$$.

**Corollary 4.3.** Let $$D$$ be a division ring with infinite centre $$F$$ and $$n \geq 1$$. Assume that $$M$$ is a noncommutative nilpotent maximal subgroup of $$GL_n(D)$$ such that $$F[M]$$ is algebraic over $$F$$. Then $$M$$ is centre-by-locally finite. Therefore, $$M/F^*$$ is locally finite and so $$D$$ is locally finite dimensional.

**Proof.** $$M$$ satisfies a group identity since $$M$$ is nilpotent. Thus, by Theorem 1, we conclude that $$M$$ is absolutely irreducible. Now, by a result of [12, p. 213], this implies that $$M$$ is centre-by-locally finite and so the proof is complete.

**Corollary 4.4.** Let $$D$$ be a division ring with infinite centre $$F$$ and $$n \geq 1$$. Assume that $$M$$ is a noncommutative soluble maximal subgroup of $$GL_n(D)$$ such that $$F[M]$$ is algebraic over $$F$$. Then $$M$$ is abelian-by-locally finite.

**Proof.** $$M$$ satisfies a group identity since $$M$$ is soluble. By Theorem 1, we conclude that $$M$$ is absolutely irreducible. Now, by a result of [13], this implies that $$M$$ is abelian-by-locally finite and so the proof is complete.

To prove our final result, we need the following

**Lemma 4.5.** Let $$D$$ be a division algebra of finite dimension over its centre $$F$$. Assume that $$G$$ is a subgroup of $$D^*$$. If $$G$$ satisfies a group identity, then $$G$$ is abelian-by-finite.

**Proof.** Assume that $$G$$ satisfies a group identity. Since $$[D : F] < \infty$$ we conclude that $$G$$ is a
linear group. By a theorem of Platonov (cf. [14, p. 149]), $G$ is soluble-by-finite, i.e., there exists a soluble normal subgroup $N$ of $G$ such that $G/N$ is finite. Since $[D : F] < \infty$ we conclude that $F[N]$ is a division ring, and therefore $F[N]$ is semisimple. Thus, $N$ as a linear group over $F$ is completely reducible. Therefore, $N$ is a completely reducible soluble linear group. So, by a theorem of [4, p. 111], $N$ is abelian-by-finite and consequently, $G$ is abelian-by-finite which completes the proof.

**Theorem 4.6.** Let $D$ be a noncommutative division algebra of finite dimension over its centre $F$ and $n \geq 1$. Assume that $N$ is a subnormal subgroup of $GL_n(D)$ and $M$ is a maximal subgroup of $N$. If $M$ satisfies a group identity, then $M$ is abelian-by-finite.

**Proof.** The case $n = 1$ follows from Lemma 4. So, we may assume that $n \geq 2$ and $N$ is a subnormal subgroup of $GL_n(D)$. By Theorem 11 of [8], we have either $N \subset F^*$ or $SL_n(D) \subset N$, i.e., $N$ is normal in $GL_n(D)$. We now claim that $M$ is irreducible. For otherwise assume that $M$ is reducible. Therefore, $D^n$ has a nontrivial submodule as $D - F[M]$ bimodule. Thus, there exists an invertible $n \times n$ matrix $P$ over $D$ such that $PMP^{-1} \subset \Sigma$, where $\Sigma = \{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} | A \in GL_a(D), C \in GL_{n-a}(D), B \in M_{a \times (n-a)}(D) \} \cap N$. It is clear that $PMP^{-1}$ is also a maximal subgroup of $N$. Set

$$T = \begin{pmatrix} SL_a(D) \\ o \\ SL_{n-a}(D) \end{pmatrix}.$$ 

Then, we have $T \subset SL_n(D)$ and so $T \subset \Sigma$. Now, we clearly have $PMP^{-1} \subset \Sigma \subset N$. By maximality of $PMP^{-1}$ we conclude that either $PMP^{-1} = \Sigma$ or $N = \Sigma$. If the second case occurs, then $I + e_{n1} \in SL_n(D)$ whereas $I + e_{n1} \notin \Sigma = N$ and this contradicts the fact that $SL_n(D) \subset N$. Therefore, $PMP^{-1} = \Sigma$ which implies that $\Sigma$ satisfies a group identity, and so $T \subset \Sigma$ satisfies a group identity. Now, by 4.5.1 of [12], we conclude that $D = F$ which is a contradiction. Thus, $M$ is irreducible as claimed. Therefore, by 1.1.12 of [12], $F[M]$ is simple Artinian. So, there exists a positive integer $t$ and a division ring $D_1$ such that $F[M] \cong M_t(D_1)$ as $F$-algebras. Thus, $F[M]^* \cong GL_t(D_1)$ and $(F[M]^*)' \cong SL_t(D_1)$. Since $N$ is normal in $GL_n(D)$ we have $(F[M]^*)' \subset N$. If $F[M]^* \cap N = M$, then we have $SL_t(D_1) \cong (F[M]^*)' \subset F[M]^* \cap N = M$. Since $M$ satisfies a group identity, by 4.5.1 of [12] again, we conclude that either $D_1$ is a locally finite field or $t = 1$ and $D_1 = Z(D_1)$. In the first case $F$ is also a locally finite field and since $[D : F] < \infty$ we conclude that $D = F$ which is a contradiction. The second case implies that $M$ is abelian and the result follows. Finally, if $N \cap F[M]^* = N$, then $N \subset F[M]^*$. Since $N$ is normal in $GL_n(D)$ we conclude that $F[M] = M_n(D)$, i.e., $M$ is absolutely irreducible. By a result of [14, p. 149], $M$ is soluble-by-finite, i.e., there is a soluble normal subgroup $N_1$ such that $M/N_1$ is finite. Now, by 1.2.12 of [12], $F[N_1]$ is semisimple Artinian. Therefore, $N_1$ is a soluble completely reducible linear group. By a theorem of [4, p. 111], $N_1$ is abelian-by-finite and therefore $M$ is abelian-by-finite which completes the proof.
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References


