Fractal Structure of 4D Euclidean Simplicial Manifold

H. S. Egawa\textsuperscript{a}, S. Horata\textsuperscript{b}\textsuperscript{}SOKENDAI\textsuperscript{[}Coordination Center for Research and Education, The Graduate University for Advanced Studies, Hayama, Kanagawa 240-0193 Japan\textsuperscript{\textcopyright} and T. Yukawa\textsuperscript{d}\textsuperscript{}SOKENDAI\textsuperscript{[}

\textsuperscript{a}Research Institute of Educational Development, Tokai University, 2-28-4 Tomigaya, Shibuya, Tokyo 151-8677 Japan
\textsuperscript{b}[Institute of Particle and Nuclear Studies, KEK, High Energy Accelerator Research Organization, 1-1 Oho, Tsukuba, Ibaraki 305-0801 Japan

The fractal properties of four-dimensional Euclidean simplicial manifold generated by the dynamical triangulation are analyzed on the geodesic distance $D$ between two vertices instead of the usual scale between two simplices. In order to make more unambiguous measurement of the fractal dimension, we employ a different approach from usual, by measuring the box-counting dimension which is computed by counting the number of spheres with the radius $D$ within the manifold. The numerical result is consistent to the result of the random walk model in the branched polymer region. We also measure the box-counting dimension of the manifold with additional matter fields. Numerical results suggest that the fractal dimension takes value of slightly more than 4 near the critical point. Furthermore, we analyze the correlation functions as functions of the geodesic distance. Numerically, it is suggested that the fractal structure of four-dimensional simplicial manifold can be properly analyzed in terms of the distance between two vertices. Moreover, we show that the behavior of the correlation length regards the phase structure of 4D simplicial manifold.

1. Introduction and model

Recently, Euclidean simplicial quantum gravity (SQG) with the additional matter fields has been investigated with using the Monte-Carlo simulation. From the previous numerical simulation\textsuperscript{[1,2]}, we expect the four-dimensional (4D) simplicial model realizes the realistic 4D quantum gravity. Especially, in Ref. [2], it is shown that the relation between SQG and the 4D conformal gravity\textsuperscript{[3]}, calculating with the grand-canonical method.

However, in order to discuss the relation between the discretized model and the continuum theory in detail, we need to analyze the correlations on the 4D simplicial manifold $\mathcal{M}_{DT}(4D)$. Namely, we need to discuss the geometry on $\mathcal{M}_{DT}(4D)$ instead of the statistical properties which have been discussed until now.

In order to discuss the geometry on $\mathcal{M}_{DT}(4D)$, we analyze the configurations of the 4D SQG coupled with $N_A$ U(1) gauge fields and $N_X$ massless scalar fields. The partition function $Z$ is given as the sum of the all of the configuration of 4-simplices $T$,

$$Z = \sum_T e^{-S_{EH}(T)} \prod_{N_A,N_X} \left( \int \prod_{l \in T} dA_l e^{-S_A(A_l)} \int \prod_{i \in T} dX_i e^{-S_X(X_i)} \right),$$

where $S_{EH}$, $S_X$ and $S_A$ is the action for the gravity, the scaler field $X_i$ on the vertex $i$ and the vector field $A_l$ on the link $l$ respectively. For the action of the gravity part, we use the discretized Einstein-Hilbert action,

$$S_{EH}[\kappa_2,\kappa_4] = \kappa_4 N_4 - \kappa_2 N_2$$

where $N_i$ denotes the number of the $i$-simplices, $\kappa_2$ is related to Newton’s constant and $\kappa_4$ is related to Cosmological constant. The action for
the scalar field $X_i$ on the vertex $i$ becomes

$$S_X = \sum_{ij} o(l_{ij}) (X_i - X_j)^2,$$  \hspace{1cm} (3)

where $o(l_{ij})$ is number of 4-simplices sharing link $l_{ij}$. Then, the action of vector field $A_{ij}$ on link $l_{ij}$

$$S_A = \sum_{ijk} o(t_{ijk}) (A(l_{ij}) + A(l_{jk}) + A(l_{ki}))^2,$$  \hspace{1cm} (4)

where $o(t_{ijk})$ is number of 4-simplices sharing triangle $t_{ijk}$.

In order to attempt the canonical simulation, we add the volume fixing term $\delta S$ to the total action,

$$\delta S(N_4,V_4) = \delta \kappa_4 (N_4 - V_4)^2,$$  \hspace{1cm} (5)

where $\delta \kappa_4$ is the parameter to fix the volume $N_4$ and $V_4$ denotes the size of the target volume. At $N_4 = V_4$, we take a sample of $\mathcal{M}_{\text{DT}}(4D)$ and attempt to measure.

2. Fractal dimension based on the geodesic distance

In order to analyze the geometry of $\mathcal{M}_{\text{DT}}(4D)$, we discuss the fractal properties based on the geodesic distance with measuring the fractal dimension and the correlation functions of the additional fields.

We consider some different definitions of the geodesic distance on $\mathcal{M}_{\text{DT}}(4D)$. One of such the definitions is the shortest distance between two 4-simplices along the 4-simplices, so-called the simplex-simplex geodesic distance. However, we expect that the expansion of the volume along the simplex-simplex geodesic distance is restricted, since the number of 4-simplices boarding on the 4-simplex is fixed. Another definition is the shortest distance between two vertices along the link, so-called the vertex-vertex geodesic distance.

In order to discuss the difference between the two distances, we introduce the box-counting dimension $\text{dim}_{\text{BC}}(\mathcal{M}_{\text{DT}})$ on $\mathcal{M}_{\text{DT}}$,

$$\text{dim}_{\text{BC}}(\mathcal{M}_{\text{DT}}) = \frac{-\log(N(S(r(D))))}{\log r(D)},$$  \hspace{1cm} (6)

where $D$ denotes the scale of the geodesic distance, $N(S(r(D)))$ denotes the number of the closed-surface $S(r(D))$ with the radius $r(D)$. The box-counting method is known as one of the standard methods to define the fractal dimension.

In Figure 1, we show the $\text{dim}_{\text{BC}}(\mathcal{M}_{\text{DT}}(4D))$ based on each geodesic distances in the case of pure gravity. The numerical results suggest that each geodesic distance need rescaling in order to be equivalent to the other definition of geodesic distance. We consider that this rescaling factor is caused by the difference of the degree of freedom of each distances on the discretized manifold.

From the behavior of the box-counting dimension in the branched polymer region, the simplex-simplex geodesic distance is the intrinsic scale and that it needs the rescaling factor for agreement with the unique shortest distance on the manifold. On the contrary, numerical results suggest that the vertex-vertex geodesic distance doesn’t need the extra rescaling factor and that it is close to the unique shortest distance. Then, we choose the vertex-vertex geodesic distance for analysis of the geometry of $\mathcal{M}_{\text{DT}}(4D)$.

3. Behavior of the correlation function

We analyze the correlation functions on the vertex-vertex geodesic distance instead of the conventional geodesic distance of simplex-simplex
distance. In order to analyze the behavior of the correlation function, we introduce the correlation length $\xi$ for the operator $O(x)$ on the vertex $x$,

$$\xi^{-1} = -\frac{\log(<O(x)O(y)>)}{|x-y|}. \quad (7)$$

This value diverges to infinity at the critical point of second order phase transition.

In Figure 2, we show the numerical results in the case of the pure gravity and the additional one vector field $N_A = 1$. The numerical result suggests that the value of $\xi$ indicates the property of manifold and the phase structure. Moreover, from the finite-size-scaling of the value of $\xi$ at the critical point between the crumpled phase and smooth phase in the case of $N_A = 1$, $\xi$ diverges to infinity for the large $N_4$ limit.

While the divergence of $\xi$ is not found both at the critical point of the pure gravity case and at the obscure transition point between the smooth phase and the branched polymer. The numerical results suggest that the type of transition at the critical point of the additional matter case is different from the type of both the critical point of the pure gravity and the obscure transition point.

Actually, we found the power low behavior of the correlation functions at the critical point in the case of $N_A = 1$. In Figure 3, we show the correlation functions for the local scalar curvature. The deviation from the power low can be considered as the finite size effect.

### 4. Summary and Discussion

In order to discuss the geometry on $M_{DT}(4D)$, we analyze the geodesic distances defined along the lattice building blocks. From the numerical result of the box-counting dimension, the simplex-simplex geodesic distance need extra in order to be equivalent to the vertex-vertex geodesic distance. Moreover, we show the behavior of the correlation function numerically. The correlation length realizes the phase diagram and its peak value diverges to the infinity at the critical point for the large $N_A$ limit. Then, the numerical results of the box-counting dimension and the correlation function suggest that the massless mode propagates on the simplicial manifold which has the fractal dimension nearly 4.

In conclusion, the geometry on the simplicial manifold can be analyzed on the basis of the vertex-vertex geodesic distance.

### REFERENCES