Z_N Gauge Theories on a Lattice and Quantum Memory

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Abstract

In the present paper we shall study (2 + 1) dimensional Z_N gauge theories on a lattice. It is shown that the gauge theories have two phases, one is a Higgs phase and the other is a confinement phase. We investigate low-energy excitation modes in the Higgs phase and clarify relationship between the Z_N gauge theories and Kitaev’s model for quantum memory and quantum computations. Then we study effects of random gauge couplings (RGC) which are identified with noise and errors in quantum computations by Kitaev’s model. By using a duality transformation, it is shown that time-independent RGC give no significant effects on the phase structure and the stability of quantum memory and computations. Then by using the replica methods, we study Z_N gauge theories with time-dependent RGC and show that nontrivial phase transitions occur by the RGC.

1 Introduction

In the last few years, discrete gauge theories have got renewed interests as a possible device for the quantum computations, a quantum computer. This idea was first proposed by Kitaev in his seminal paper[1], and after that there appeared interesting works on this idea[2, 3, 4, 5]. One of the most difficult problem of making a quantum computer and performing quantum computations fault-tolerantly is the stability of the quantum states which participate in quantum memory and computations. There must be a (large) energy gap between these states and others in the system and also mixings of these states must be suppressed by certain effects or selection rules. Then one can conceive that topological interactions such as the Aharonov-Bohm (AB) effect may play an important role there. The AB effect in the two spatial dimensions gives nontrivial statistics to particles with gauge interactions, i.e., anyons. The groundstates of the anyons are degenerate if the space is a torus and almost no mixing occurs between them because of the topological quantum number. Whereas the gauge symmetry should be discrete in order to avoid long-range interactions

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Besides the topological ones, Kitaev’s model is based on the anyonic excitations in qubits system. However its detailed relationship to the gauge theory is still missing.

In this paper, we shall study discrete $Z_N$ gauge theories in $(2 + 1)$ dimensions. There are two phases in these model, one is a confinement phase and the other is a Higgs phase. We show that Kitaev’s system of qubits corresponds to some limit of the Higgs phase of the $Z_2$ gauge models. Stability of Kitaev’s model is closely related with the confinement-Higgs phase transition of the $Z_N$ gauge models.

This paper is organized as follows. In Sec.2, we study $Z_N$ gauge theories which appear as a result of spontaneous breakdown of $U(1)$ gauge symmetry. We clarify the relationship between the gauge system and Kitaev’s model for quantum memory and computations. In Sec.3, low-energy excitations in the Higgs phase are investigated. There appear anyonic excitations, magnetic vortices and dyons in a natural way as in the spontaneously broken gauge systems in the continuum space[6]. In Sec.4, phase structure and effects of the (static) random gauge couplings(RGC) are investigated by using a duality transformation. The $Z_N$ gauge systems are transformed to spin systems which are more tractable than the gauge systems. In Sec.5, effects of the time-dependent RGC are studied by the replica methods. It is found that nontrivial phase transitions occur as the RGC varies. Section 6 is devoted to conclusion.

2. $U(1)$ and $Z_N$ gauge theories

Let us start with the following $U(1)$ Abelian gauge-Higgs model on a 2-dimensional(2D) square lattice. Hamiltonian is given by,

$$H_{U(1)} = g^2 \sum_{\text{link}} E_{xi}^2 - 1g^2 \sum_{\text{plaquette}} UUUU + 1\kappa \sum_x (\Pi_x^\phi)^2 - \kappa \sum_{\text{link}} \phi_{x+1}^i U_{x+1}^N \phi_x - \gamma \sum_{\text{link}} \psi_{x+1}^i U_{x+1}^q \psi_x + M \sum_x \psi_x^\dagger \psi_x + \text{H.c.}$$

where $U_{xi}$ is the $U(1)$ gauge field on the link $(x,i)(x = \text{site}, i = \hat{1} \text{ or } \hat{2})$ and $E_{xi}$ is the conjugate electric field. The Higgs field $\phi_x \in U(1)$ carries $U(1)$ charge $N$ whereas the charge of the fermion field $\psi_x(\varphi_x)$ is $q(-q)$ which is an integer. $\Pi_x^\phi$ is the conjugate field of $\phi_x$, the gauge coupling is $g$ and the fermion mass is $M$. Other notations are standard. We are interested in the case $N \neq 1$. In this case there are two phases in the model, one is the Higgs phase and the other is the confinement phase. In particular in the limit $g^2 \to 0$, the gauge field $U_{xi}$ is restricted to the pure-gauge configuration and the model reduces to a Hamiltonian description of the classical $3D$ XY spin model plus the free

\footnote{We often call $\psi_x$ and $\varphi_x$ fermion because they satisfy fermionic anticommutation relations. As a result of the gauge interactions, they obey anyonic statistics in the Higgs phase. See later discussion.}
The classical 3D XY model exhibits a phase transition from the magnetized phase to the disordered phase at a critical coupling $\kappa_c$. On the other hand for large $\kappa$, quantum fluctuations of $\phi_x$ are suppressed and low-energy excitations of the gauge and Higgs fields are restricted as

$$U_{xi}^N \sim 1, \quad \phi_x \sim 1,$$

up to (time-independent) local gauge transformation. Then we can put

$$U_{xi} \sim Z_{xi},$$

where the $Z_N$ gauge operator $Z_{xi}$ is explicitly given as follows by $(N \times N)$ matrix,

$$Z_{xi} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & e^{2\pi N_i} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & e^{2\pi (N-1) N_i}
\end{pmatrix}.$$

Corresponding to the above representation of $Z_{xi}$, we introduce “conjugate matrix” $X_{xi}$ as follows,

$$X_{xi} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0
\end{pmatrix}.$$

One can easily verify the following commutation relations,

$$X_{xi}Z_{xi} = e^{2\pi N_i}Z_{xi}X_{xi}, \quad X_{xi}Z_{yj} = Z_{yj}X_{xi} \quad \text{for} \ (x, i) \neq (y, j).$$

The electric term in Eq.(1) is reduced to the following term in the reduced $Z_N$ gauge-field space,

$$E_{xi}^2 \sim -(X_{xi} + X_{xi}^\dagger).$$

The above result can be shown by using the eigenstates of the electric fields as basis vectors. Let us define an “empty state” $|0\rangle$ as

$$E|0\rangle = 0,$$

where we have omitted link index for notational simplicity. By using the following commutation relation,

$$[E, U] = U,$$
we can show
\[ E^2 U|0\rangle = U|0\rangle \equiv |1\rangle. \] (9)

Then the gauge field \( U \) is the rising operator of the electric field.

For the \( Z_N \) case, we also define “empty” state for the \( X \) operator,
\[ X|0\rangle_X = |0\rangle_X, \quad X^\dagger|0\rangle_X = |0\rangle_X. \] (10)
The state \( |0\rangle_X \) can be expressed by the eigenstates of the \( Z \) operator
\[ |0\rangle_X = 1\sqrt{N} \left( |1\rangle_Z + |2\rangle_Z + \cdots + |N\rangle_Z \right). \] (11)

Then one can easily show,
\[ Z|0\rangle_X = |1\rangle_Z + e^{2\pi N i}|2\rangle_Z + \cdots + e^{2\pi N - 1 N i}|N\rangle_Z, \] (12)
and therefore
\[ XZ|0\rangle_X = e^{2\pi N i}Z|0\rangle_X, \] (13)
\[ (X + X^\dagger)Z|0\rangle_X = 2 \cos \left( 2\pi N \right) Z|0\rangle_X. \] (14)

From Eqs.(10) and (14), \( Z \) is the lowering operator of \( X + X^\dagger \) and therefore,
\[ E_{x_i}^2 \sim -(X_{x_i} + X_{x_i}^\dagger), \] (15)
up to irrelevant additive and multiplicative constants. Then for large \( \kappa \), the \( U(1) \) gauge theory (1) reduces to the following \( Z_N \) gauge theory,
\[
H_T = H_Z + H_Z^\psi + H_Z^\varphi,
H_Z = -\lambda_1 \sum X_{xi} - \lambda_2 \sum ZZZZ + \text{H.c.},
H_Z^\psi = -\gamma \sum \psi_{x+i}^\dagger Z_{x+i}^q \psi_x + M \sum \psi_{x+i}^\dagger \psi_x + \text{H.c.},
H_Z^\varphi = -\gamma \sum \varphi_{x+i}^\dagger Z_{x+i}^q \varphi_x + M \sum \varphi_{x+i}^\dagger \varphi_x + \text{H.c.},
\] (16)
where \( \lambda_1 \) and \( \lambda_2 \) are coupling constants of the \( Z_N \) gauge theory and they relate to the \( U(1) \) gauge coupling \( g^2 \) as \( \lambda_1 \sim g^2 \) and \( \lambda_2 \sim 1/g^2 \).

The above “derivation” of the \( Z_N \) gauge theory (16) from the \( U(1) \) gauge system (1) is rather sketchy but it might be useful for realization of discrete gauge systems in architecture of the quantum computers. For example, spontaneous breaking of \( U(1) \) gauge symmetry occurs in the superconductivity. In most of the superconductors including the high-temperature ones, the “Cooper pair”
carries electric charge $2e$. Then a discrete $Z_2$ gauge system close to the present one might be realized in some superconductors. The Hamiltonian (16) is directly obtained from the path-integral formalism of the $Z_N$ gauge theory on 3D lattice by taking the continuum limit of the time-like direction. In the 3D $Z_N$ gauge theory, there exist two phases, i.e., confinement and Higgs phases as we show later on. Phase transition occurs at a certain critical coupling $(\lambda_1/\lambda_2)c$. In the original $U(1)$ gauge theory, there exists a critical line connecting the XY phase transition at $(g = 0, \kappa = \kappa_c)$ and $Z_N$ gauge phase transition at $(g = g_c, \kappa = \infty)$ (see Fig.1)[7].

Physical state of the system (16) must be gauge-invariant and this condition is given as follows,

$$G_x \equiv \left( \prod_{(y,i) \in x} \tilde{X}_{yi} \right) e^{-2\pi q Ni(\psi^\dagger_x \psi_x - \phi^\dagger_x \phi_x)} , \quad G_x |phys\rangle = |phys\rangle,$$

(17)

where $(y, i) \in x$ denotes 4 links emanating from site $x$ and $\tilde{X}_{yi} = X_{yi}$ for $y = x$ whereas $\tilde{X}_{yi} = X^\dagger_{xi}$ for $y - i = x$. It is proved that $G_x$ is the gauge-transformation operator at site $x$ and the Hamiltonian $H_{\psi}Z + H_{\phi}Z$ in Eq.(16) commutes with $G_x$.

Recently Kitaev proposed a 2-dimensional qubits system for fault-tolerant quantum memory and computations[1]. This system is closely related to the $Z_2$ gauge theory and contains “anyonic excitations”. The system is defined on a torus and the Hamiltonian is given as follows in our notation,

$$H_K = -\sum_x \prod_{(y,i) \in x} X_{yi} - \sum_{pl} ZZ ZZ,$$

(18)

where $Z_x$ and $X_x$ are explicitly given by the Pauli matrices in the $Z_2$ case, i.e., $Z = \sigma^z$ and $X = \sigma^x$. The groundstates and excited states of the Hamiltonian (18) are easily obtained since the first and second terms of (18) commute with each other. The groundstates satisfy

$$\prod_{(y,i) \in x} X_{yi} |GS\rangle_K = |GS\rangle_K , \quad \prod_{pl} Z |GS\rangle_K = |GS\rangle_K ,$$

(19)

for all sites and plaquettes. The groundstates are four-fold degenerate on the torus, as we explain in the following section. These degenerate lowest-energy states form basis for quantum memory[1].

The first excited states are explicitly given by

$$\prod_{(y,i) \in x} X_{yi} |1st\rangle_K = -|1st\rangle_K , \quad \text{or} \quad \prod_{pl} Z |1st\rangle_K = -|1st\rangle_K ,$$

(20)

for some specific site or plaquette and otherwise they satisfy Eq.(19). It is not so difficult to see that Kitaev’s model is equivalent to the model (16) with $N = 2, \gamma = 0, M = 2, q = 1$ and $\lambda_1 = 0, \lambda_2 = 1$. With these parameters and the physical state condition (17), the groundstates of the gauge model are given as

$$\prod_{pl} Z |GS\rangle_Z = |GS\rangle_Z , \quad \psi^\dagger_x \psi_x |GS\rangle_Z = 0 , \quad \phi^\dagger_x \phi_x |GS\rangle_Z = 0 ,$$

(21)
for all plaquettes and sites. From (17), the second and third conditions of (21) mean
\[ \prod_{(y,i) \in x} X_{yi}|GS\rangle_Z = |GS\rangle_Z \]. On the other hand, the first excited states of the gauge system are given by,

\[ \prod_{pl} Z|1st\rangle^V_Z = -|1st\rangle^V_Z, \text{ or } \psi^\dagger_x \psi_x |1st\rangle^V_Z = |1st\rangle^V_Z, \]

(22)

for some specific plaquette or site. From Eq.(17), the second condition in (22) is equivalent to

\[ \prod_{(y,i) \in x} X_{yi}|1st\rangle^\psi_Z = -|1st\rangle^\psi_Z \text{ for } e^{\pi i \psi^\dagger_x \psi_x} |1st\rangle^\psi_Z = -|1st\rangle^\psi_Z \text{ and energy increases by } 2 \text{ because of the mass term in (16) with } M = 2. \]

Similarly for the other fermion \( \varphi_x \). In the original paper by Kitaev[1], relationship between his model and gauge theories was slightly discussed but full relationship was missing. In the following sections we shall study phase structure of the present \( Z_N \) gauge model, low-energy excitations, effects of random gauge couplings, etc. All these discussions give an important insight to the stability problem of Kitaev’s model.

### 3 Low-energy excitations in the Higgs phase

As we show in the following section, there are two phases in the \( Z_N \) gauge theory in \((2+1)\) dimensions \( H_Z \) in (16). For large \( \lambda_2/\lambda_1 \), fluctuation of the gauge field \( Z_{xi} \) is small and the Higgs phase is realized whereas for small \( \lambda_2/\lambda_1 \), the gauge field \( Z_{xi} \) fluctuates strongly and the confinement phase is realized. The Higgs phase of the model can be used for a quantum memory. Coupling of the matter fields \( \psi_x \) etc. enhances the Higgs phase.

Let us study the model on the torus and focus on the Higgs phase for large \( \lambda_2/\lambda_1 \). In particular for \( \lambda_1 = 0 \), the groundstates are given by Eq.(21) and low-energy excited states are particle states of \( \psi_x, \varphi_x \) and states of plaquette magnetic excitation or vortex, i.e., \( \prod_{pl} Z|1st\rangle^V_Z = -|1st\rangle^V_Z \) for specific plaquette. As we study the model on the torus, we have the following “trivial” identities

\[ \prod_{\text{all sites}} \prod_{(y,i) \in x} X_{yi} = 1, \quad \prod_{\text{all pl's}} \prod_{pl} Z = 1, \]

(23)

and therefore the above excitations must appear in pairs. As the groundstates satisfy Eq.(21), there is no magnetic flux in each plaquette. Then one may think that the groundstate is unique. However this is not the case. There are two nontrivial cycles on the torus, and let us call them a-cycle and b-cycle, i.e., noncontractible closed paths. We introduce the dual lattice in the usual way, and choose certain noncontractible closed loops on the original and dual lattices. We use notations such that \( C_a^Z(C_b^Z) \) for a suitably chosen closed loop corresponding to the a-cycle(b-cycle) on the original lattice and \( C_a^X(C_b^X) \) for a loop corresponding to the a-cycle(b-cycle) on the dual lattice. Later discussion does not depend on the choice of the loops. Then we define the following operators, \( Z_a, Z_b, X_a \) and
X_b,

\[
Z_a = \prod_{C_a^x} Z_{xi}, \quad Z_b = \prod_{C_b^x} Z_{xi}, \\
X_a = \prod_{C_x^a} X_{xi}, \quad X_b = \prod_{C_x^b} X_{xi},
\]

(24)

where \(X_{xi}'s\) in \(X_a\) cross \(C_a^x\) and similarly for \(X_b\). These operators are obviously invariant under gauge transformation and commute with \(H_Z\) when \(\lambda_1 = 0\). Furthermore they satisfy the following commutation relations,

\[
X_a Z_b = e^{2\pi N_i} Z_b X_a \quad X_b Z_a = e^{2\pi N_i} Z_a X_b,
\]

(25)

and otherwise commute. Therefore the groundstates are eigenstate of the the operators, e.g., \(Z_a\) and \(Z_b\) and they are \(N^2\)-fold degenerate. This result holds even in the presence of the fermions \(\psi_x\) and \(\varphi_x\) since \(Z_a\) and \(Z_b\) commute with \(H_T\) for vanishing \(\lambda_1\) in Eq.(16).

Fermions \(\psi_x\) and \(\varphi_x\) move in an unfluctuating “background” field of \(Z_{xi}'s\) with vanishing magnetic field. However they distinguish the above \(N^2\)-fold degenerate \(Z\)’s groundstates. In fact while \(\psi_x\) (or \(\varphi_x\)) fermion moves along a closed loop of the a-cycle, it acquires phase factor which is an eigenvalue of \((Z_a)^q\), and similarly for the b-cycle. Then the Higgs phase is a “topologically ordered” phase. The \(N^2\) groundstates work as qudit for quantum memory and the quantum states of the qudit are distinguishable by using matter fields like \(\psi_x\).

Let us discuss excitations in detail. As we explained above, the fermions must appear in a pair. Two-fermion state at sites \(x\) and \(y\) is explicitly given as,

\[
|F; C_{xy} \rangle = \psi_y^\dagger \left( \prod_{C_{xy}} Z^q \right) \varphi_x^\dagger |GS\rangle_Z,
\]

(26)

where \(C_{xy}\) is a certain path on the original lattice connecting \(x\) and \(y\), and the state (26) obviously satisfies the physical-state condition (17). On the other hand two-vortex state at dual sites \(x^*\) and \(y^*\) is given as,

\[
|V; \tilde{C}_{x^*y^*} \rangle = \left( \prod_{\tilde{C}_{x^*y^*}} X \right) |GS\rangle_Z,
\]

(27)

where \(\tilde{C}_{x^*y}\) is a certain path on the dual lattice connecting \(x^*\) and \(y^*\) and \(X's\) in (27) are on the links crossing \(\tilde{C}_{x^*y}\) (see Fig.2). This state is also a physical state. Other physical excitations are produced by applying the gauge-invariant operators in Eqs.(26) and (27) successively on the groundstates.

Fermionic excitations and magnetic vortices satisfy a nontrivial statistics. This is an Aharonov-Bohm effect of the \(Z_N\) gauge theory. To see this, we consider the state like

\[
\psi_{y_1}^\dagger \left( \prod_{C_{y_1y_1}} Z^q \right) \varphi_{x_1}^\dagger \cdot \left( \prod_{\tilde{C}_{x_2^*y_2^*}} X \right) |GS\rangle_Z,
\]

(28)
and assume that the paths \( C_{x_1y_1} \) and \( \tilde{C}_{x_2y_2} \) do not entangle with each other. Let us move the \( \varphi_x \) fermion at \( x_1 \) around the vortex at \( x_2^* \) once counterclockwise (and not \( y_2^* \)) and then return it to the original position \( x_1 \). The resultant path \( C'_{x_1y_1} \) encircles \( x_2^* \) once and \( \tilde{C}_{x_2y_2} \) cross with each other. Then the state can be written as

\[
\psi_y \left( \prod_{C_{x_1y_1}} Z^q \cdot \prod_{\tilde{C}_{x_2y_2}} X \right) |GS\rangle_Z = \psi_y \left( \prod_{C_{x_1y_1}} Z^q \cdot \prod_{C_{\text{closed}}} Z^q \right) \varphi_{x_1} \cdot \left( \prod_{\tilde{C}_{x_2y_2}} X \right) |GS\rangle_Z, \tag{29}
\]

where \( C_{\text{closed}} \) is the closed path \( (C_{x_1y_1}^{-1} \cdot C_{x_1y_1}) \) which encircles \( x_2^* \) once and has a single common link (or odd number of links) with \( \tilde{C}_{x_2y_2} \). Because of the nontrivial commutation relation between \( Z_{xi} \) and \( X_{xi} \) and (21), the resultant state differs from the original one by the phase factor \( e^{2q\pi N_i} \),

\[
\psi_y \left( \prod_{C_{x_1y_1}} Z^q \cdot \prod_{C_{\text{closed}}} Z^q \right) \varphi_{x_1} \cdot \left( \prod_{\tilde{C}_{x_2y_2}} X \right) |GS\rangle_Z = e^{2q\pi N_i} \psi_y \left( \prod_{C_{x_1y_1}} Z^q \right) \varphi_{x_1} \cdot \left( \prod_{\tilde{C}_{x_2y_2}} X \right) |GS\rangle_Z. \tag{30}
\]

The above anyonic properties of the low-energy excitations are closely related with the ground-state degeneracy. In the continuum spacetime, a Chern-Simons (CS) gauge theory is often employed for describing anyons which are a nontrivial representation of the braid group. In anyon systems on a torus, movement of an anyon along noncontractible loops like the \( a \)-cycle and/or \( b \)-cycle is a nontrivial element of the braid group. On the torus, the zero modes of the CS gauge field play an important role and the groundstate wave function of anyons becomes multi-component because of the zero modes\[8\]. Similar phenomenon occurs in the present \( Z_N \)-gauge system as we explained above.

One may conceive that the system has dyonic excitations as in the continuum theories\[6\]. The answer is positive. Dyon \( d_x \) with “electric charge” \( Q_E \) and “magnetic charge” \( R \) is described by the following Hamiltonian,

\[
H_D = - \sum_{x+i_j} d_{x+i_j}^\dagger Z_{x+i_j}^{Q_E} X_{x+i_j}^R d_x + \text{H.c.,} \tag{31}
\]

where we assume that the fields \( d_x \) and \( d_{x+i_j}^\dagger \) themselves satisfy the fermionic commutation relations for simplicity. The link \((\vec{x},i)\) is associated with the link \((x,i)\) and defined as follows,

\[
\text{link } (\vec{x},i) = \begin{cases} 
(x + 1, 2) & \text{for } i = 1, \\
(x + 2, 1) & \text{for } i = 2.
\end{cases} \tag{32}
\]

From the above definition (31) and (32), it is obvious that the electric charge \( Q_E \) of the dyon \( d_x \) is located at the site \( x \) whereas its magnetic charge \( R \) is located at the nearest-neighbor plaquette (see Fig.3). Regularization is naturally introduced by the spatial lattice. It is not so difficult to show that
the above dyon satisfies nontrivial representation of the braid group and there appears the phase factor like $-\exp(\pm 2(Q_E + R)\pi i N)$ when two dyons interchange with each other.

When we turn on the parameter $\lambda_1$ in $H_Z$ (16), the operator $Z_a$ and $Z_b$ do not commute with $H_T$ anymore and therefore degeneracy of the groundstate disappears. This stems from the fact that because of the term $\lambda_1 \sum X_{x,i}$, $Z_{x,i}$ becomes dynamical and it fluctuates quantum mechanically and then genuine anyonic properties of the low-energy excitations break down. However for small $\lambda_1$, there is still an energy gap between the $N^2$ “groundstates” with fine structure and the other excited states. Furthermore, these $N^2$ states are far apart with each other in the quantum-mechanical configuration space and are hardly mixed if the torus is sufficiently large. Therefore the system with small value of $\lambda_1$ is still suited for a quantum memory as Kitaev suggested first. However as $\lambda_1$ increases, a phase transition occurs as we show in the following section. In the new phase, a confinement phase, the gauge field fluctuates randomly and the system is useless as a quantum memory.\footnote{It is very interesting to see that similar gauge-theory argument can be applied to neural network models for brain\cite{9}. There Higgs phase corresponds to good brains and the confinement phase to dementia.}

4 Duality transformation, phase transition and random gauge couplings

In the previous section, we discussed that for the quantum memory and computations the Higgs phase must be realized in the present system. In this section we shall study the phase structure of the gauge-theory model $H_Z$ in (16). To this end, the system is defined on a large spatial square lattice. We shall perform a duality transformation which transforms the gauge-theory model into a more tractable spin model. For the $Z_2$ gauge theory, the duality transformation is discussed in Kogut’s review article\cite{10}.

Let us consider the pure gauge system $H_Z$ in (16) with the physical state condition,

$$\prod_{(y,i) \in x} \tilde{X}_{yi} = X_{x,1} X_{x,-1} X_{x,2} X_{x,-2} = X_{x,1} X_{x,-1,1} X_{x,2} X_{x,-2,2} = 1. \quad (33)$$

By solving the above condition (33), the operator $X_{x,2}$ is given as follows by the remaining operators,

$$X_{x,2} = X_{x,1}^{\dagger} X_{x,-1}^{\dagger} X_{x,-2,1} X_{x,-2,2}^{\dagger} \ldots \quad (34)$$

As the “conjugate” operators $Z_{x,2}$ of $X_{x,2}$ commute with the Hamiltonian $H_Z$, we can set it as a constant, $Z_{x,2} = 1$.\footnote{It is very interesting to see that similar gauge-theory argument can be applied to neural network models for brain\cite{9}. There Higgs phase corresponds to good brains and the confinement phase to dementia.}
Then we introduce the following dual operators \( W_{x^*} \) and \( V_{x^*} \) which reside on sites of the dual lattice,

\[
W_{x^*} = \prod_{(yi) \in x^*} Z_{yi},
\]

\[
V_{x^*} = \prod_{l \geq 0} X_{x - 2l, 1},
\]

where \((yi) \in x^*\) denotes 4 links to the plaquette on the original lattice which is dual to the site \(x^*\) of the dual lattice. From the definition (35), one can easily verify relations like,

\[
W_N x^* = V_N x^* = 1,
\]

\[
V_{x^*} W_{x^*} = e^{i2\pi N} W_{x^*} V_{x^*},
\]

\[
V_{x^*} W_y^\dagger x^* = W_y^\dagger x^* V_{x^*}, \quad \text{for } x^* \neq y^*,
\]

\[
V_{x^*} W_{x^* - 2} = X_1 x^*, \quad V^\dagger_{x^*} V_{x^* - 1} = X_2 x^*.
\]

From Eqs.(36) and (37), the Hamiltonian \( H_Z \) in (16) can be rewritten in terms of \( V_{x^*} \) and \( W_{x^*} \),

\[
H_Z = -\lambda_1 \sum_{x^*, i=1,2} V_{x^*} V^\dagger_{x^* - i} - \lambda_2 \sum_{x^*} W_{x^*} + \text{H.c.} \quad (38)
\]

The above quantum Hamiltonian (38) is nothing but that of the 3D classical \( Z_N \) Ising model (the clock model) which is obtained by the transfer-matrix methods and taking the continuum limit of one direction.

The Hamiltonian (38) is more tractable than the original one (16). There are two phases, i.e., ordered and disordered phases, and a phase transition occurs as the value \( \lambda_1/\lambda_2 \) varies. For small \( \lambda_1/\lambda_2 \) limit, the groundstate is given by

\[
W_{x^*} |0\rangle_S = W^\dagger_{x^*} |0\rangle_S = |0\rangle_S.
\]

In the representation,

\[
W_{x^*} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & e^{2\pi Ni} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & e^{2\pi(N-1)Ni}
\end{pmatrix},
\]

(40)
the above groundstate $|0\rangle_S$ is explicitly given as,

$$|0\rangle_S = \prod_{x^*} |0\rangle_{x^*}, \quad |0\rangle_{x^*} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}. \quad (41)$$

For small but nonvanishing $\lambda_1/\lambda_2$, the groundstate is obtained by the usual perturbative calculation, and the term $V_{x^*}V_{x^*}^\dagger$ tilts nearest-neighbor $W_{x^*}$ and $W_{x^* - i}$ by $\pm 2\pi N$, respectively. In this phase, there is no “magnetization”, i.e.,

$$S\langle 0|V_{x^*}|0\rangle_S = 0. \quad (42)$$

Low-energy excitations are given by,

$$1/\sqrt{N^2} \sum_{x^*} e^{ip\cdot x^*} V_{x^*}, |0\rangle_S, \quad 1/\sqrt{N^2} \sum_{x^*} e^{ip\cdot x^*} V_{x^*}^\dagger, |0\rangle_S, \quad (43)$$

where 2-vector $p$ is a momentum and $N^2$ is the number of the sites. Excited energy of the above states (43) can be easily calculated and obtained as follows for small $\lambda_1/\lambda_2$,

$$E = 2\lambda_2 \left( 1 - \cos(2\pi N) \right) + \cdots. \quad (44)$$

From (44), the energy gap is a decreasing function of $N$.

For large $\lambda_1/\lambda_2$, on the other hand, the groundstate of the spin system (38) is given by

$$V_{x^*}|\tilde{0}\rangle_S = e^{i\alpha}|\tilde{0}\rangle_S, \quad V_{x^*}^\dagger|\tilde{0}\rangle_S = e^{-i\alpha}|\tilde{0}\rangle_S, \quad (45)$$

where $e^{i\alpha} \in Z_N$ and therefore it is $N$-fold degenerate.\(^3\) There is a nonvanishing magnetization for large $\lambda_1/\lambda_2$,

$$S\langle \tilde{0}|V_{x^*}|\tilde{0}\rangle_S \neq 0. \quad (46)$$

From Eqs.(35), (46) and the discussion in the previous section, it is obvious that vortex condensation occurs in the gauge-system state corresponding to $|\tilde{0}\rangle_S$. This means that for large $\lambda_1/\lambda_2$ the confinement phase is realized and therefore the gauge system does not work as a quantum memory.

This result is important for the architecture of the quantum computer.

\(^3\)This result of the spin system does not mean that the original gauge system has the degenerate groundstates. Actually from (16), the groundstate satisfies $X_{xi}|\text{GS; gauge}\rangle = 1$ for all links $(xi)$ in the large $\lambda_1/\lambda_2$ limit.
It is interesting and also important to study another type of disturbance for realization of the
Higgs phase, or, a good quantum memory, i.e., the effect of random gauge couplings which corre-
sponds to noise and errors in quantum computations. In this section, we consider static random
gauge coupling(RGC) with random variables $\tau_x \in \mathbb{Z}_N$, and the Hamiltonian is given by,

$$H^R_Z = -\lambda_1 \sum_{\text{link}} X_{x_i} - \lambda_2 \sum_{pl} \tau_x ZZZZ. \quad (47)$$

We assume a simple local correlation for the random variables,

$$[\tau_x \tau_{y^*}] \propto \delta_{x^*y^*}, \quad (48)$$

where $[\cdots]$ denotes the ensemble average.

It seems rather difficult to study the above random gauge system (47). However by using the
duality transformation (35), we can rewrite the Hamiltonian $H^R_Z$ as in the nonrandom case which
we studied above,

$$H_Z = -\lambda_1 \sum_{x^*,i=1,2} V_{x^*,V_{x^*}^i} - \lambda_2 \sum_{x^*} \tau_{x^*} W_{x^*} + \text{H.c.} \quad (49)$$

Then we redefine the dual operators $W_{x^*}$ as follows,

$$\tilde{W}_{x^*} \equiv \tau_{x^*} W_{x^*}. \quad (50)$$

One can easily verify that the new operator $\tilde{W}_{x^*}$ and the old one $W_{x^*}$ satisfy exactly the same
operator equations in (36), and also there are no spatial correlations of $\tilde{W}_{x^*}$ because of (48). Then
$H^R_Z$ is equivalent to the original $H_Z$ and the random system has the same phase structure with
the nonrandom one. The groundstate, excitated states, etc. are different in the two systems but
there exists one-to-one correspondence between them. This result can be partly seen in the original
gauge system (47). For static RGC $\tau_x$ and $\lambda_1 = 0$, there is a unique Z-field configuration of
the lowest energy up to local gauge transformations. Vortex excitations are generated by applying
the string operator ($\prod_{x} X_{x_i}$) on this lowest-energy configuration as in the nonrandom case. Then
there is one-to-one correspondence. The perturbative term $\lambda_1 X_{x_i}$ generates a pair of vortices in a
nearest-neighbor plaquettes from the lowest-energy state as in the nonrandom system.

In the following section, we shall study random $\mathbb{Z}_2$ gauge system with full RGC by the replica
methods and show that nontrivial phase structure appears.

5 Replica mean-field theory

In this section we shall study the $d$-dimensional random $\mathbb{Z}_2$ gauge theories by the replica mean-field
theory(RMFT). RMFT has been often applied to the random spin systems and spin-glass problems.
The random gauge theories have been less studied and as far as we know there is no systematic studies on the random gauge theories by the replica methods. Numerical Monte-Carlo simulations are also useful to obtain phase diagram in the $p-T$ plane, where $p$ is the concentration of plaquettes of “wrong sign” and $T$ is the temperature[11, 12]. In the previous sections we used the Hamiltonian formalism, but in this section we employ the path-integral Lagrangian formalism since the path-integral formalism is more suitable for the MFT.

Let us first study the nonrandom $Z_2$ gauge theory on a $d$-dimensional lattice by the MFT[13]. The partition function $Z$ is given by,

$$Z = \text{Tr} e^{-\beta S}, \quad S = -\sum_{pl} \sigma \sigma \sigma - h \sum_{\text{link}} \sigma_{xi}, \quad (51)$$

where the $Z_2$ gauge variables $\sigma_{xi}(i = 1, \cdots, d)$ take $\pm 1$, $\text{Tr}$ means $\sum_{\sigma_{xi}=\pm 1}$, $\beta$ is inverse temperature and $h$ is an external “magnetic field”. It is not so difficult to drive MFT action $S_M$. To this end, let us decompose $\sigma_{xi}$ as $\sigma_{xi} = U_0 + \delta \sigma_{xi}$ where $U_0$ is the MF for $\sigma_{xi}$ and $\delta \sigma_{xi}$ is the fluctuation from it[14]. In terms of the new variables,

$$S = -U_0^4 N_P - U_0^3 \cdot 2(d - 1) \sum_{\text{link}} \delta \sigma_{xi} - h \sum_{\text{link}} \sigma_{xi} + O((\delta \sigma_{xi})^2)$$

$$= -U_0^4 N_P - U_0^3 \cdot 2(d - 1) \sum_{\text{link}} \sigma_{xi} + 2(d - 1)U_0^4 N_L - h \sum_{\text{link}} \sigma_{xi} + O((\delta \sigma_{xi})^2), \quad (52)$$

where $N_P$ and $N_L$ are the numbers of plaquettes and links of the lattice, respectively and $N_L = 2d - 1N_P$. From (52), $S_M$ is obtained as,

$$S_M = 3U_0^4 N_P - \{2(d - 1)U_0^3 + h\} \sum_{\text{link}} \sigma_{xi}. \quad (53)$$

Then it is straightforward to calculate the partition function from $S_M$ in (53),

$$Z_{MF} = \text{Tr} e^{-\beta S_M}$$

$$= e^{-3U_0^4 N_P \left[2 \cosh \beta \{2(d - 1)U_0^3 + h\}\right]^{N_L}}. \quad (54)$$

The “magnetization” $m$ per link is calculated from (54) as

$$m = 1Z_{MF} N_L \partial Z_{MF} / \partial h$$

$$= \tanh \beta \{2(d - 1)U_0^3 + h\}. \quad (55)$$

Similarly the free energy is obtained as,

$$F = -1/\beta d - 1N_P \log \left[2 \cosh \beta \{2(d - 1)U_0^3 + h\}\right] + 3U_0^4 N_P. \quad (56)$$
Numerical calculation of the free energy \( F \) in (56) is given in Fig.4 as a function \( U_0 \) for vanishing \( h \) and at various inverse temperatures \( \beta \). From Eqs.(55) and (56), it is verified that the magnetization \( m \) is equal to the value of \( U_0 \) at stationary points of \( F = F(U_0) \). Result in Fig.4 shows that there is a first-order phase transition as the temperature varies and at low temperature the magnetization \( m \) is nonvanishing.

It is known that there is a second-order phase transition in 3D \( Z_2 \) gauge theory which is dual to the 3D Ising model as we showed in the previous section[10]. This means that the MFT gives correct results only at large spatial dimensions as it is well known for the spin systems, etc. However we believe that the MFT is still useful for obtaining rough estimations of the physical quantities, phase structure of systems, etc.

Let us turn to random gauge theories(RGT). We study the random \( Z_2 \) theory with the following action,

\[
S_R = -\sum_{pl} J_p \sigma \sigma \sigma, \quad (57)
\]

where we assume that the RGC \( J_p \) has the probability distribution like,

\[
P(J_p) = 1J \sqrt{2\pi} \exp \left\{ -12J^2(J_p - J_0)^2 \right\}, \quad (58)
\]

with positive parameters \( J \) and \( J_0 \). We choose (58) in which \( J_p \) takes continuous real value instead of the discrete distribution \( J_p = \pm J_0 \), because it is more tractable.

We apply replica tricks to the above RGT and then the partition function is given as,

\[
[Z^n] = \int \left\{ \prod_p dJ_p P(J_p) \right\} \text{Tr} \exp \left( \beta \sum_p J_p \sum_{\alpha=1}^n \prod_p \sigma^\alpha + \beta h \sum_{\text{link}} \sigma^\alpha \right), \quad (59)
\]

where \( \alpha \) is the replica index which takes \( \alpha = 1, \cdots, n \), and we shall take the limit \( n \to 0 \) in the final stage of the calculation. Because of the replica tricks, integration over \( J_p \) can be done for each \( p \),

\[
\int dJ_p e^{-12J^2(J_p - J_0)^2} e^{\beta J \sum_{\alpha=1}^n \Pi_p \sigma^\alpha} = e^{12\beta^2 J^2 \sum_{\alpha,\beta} \Pi_p \sigma^\alpha \Pi_p \sigma^\beta + J_0 \beta \sum_{\alpha} \Pi_p \sigma^\alpha} \quad (60)
\]

We introduce the MF \( U_{0\alpha} \) for \( \sigma^\alpha_{xi} \) and the glass MF(GMF) \( Q_{\alpha\beta} \) for \( \sigma^\alpha_{xi} \sigma^\beta_{xi} \). Then the terms in the action (60) can be rewritten as follows as in the MFT for the nonrandom case (53),

\[
\sum_{pl} \prod_p \sigma^\alpha \rightarrow -3N_p U_{0\alpha}^4 + 4CU_{0\alpha}^3 \sum_{\text{link}} \sigma^\alpha_{xi},
\]

\[
\sum_{pl} \prod_p \sigma^\alpha \sum_p \sigma^\beta \rightarrow -3N_p Q_{\alpha\beta}^4 + 4CQ_{\alpha\beta}^3 \sum_{\text{link}} \sigma^\alpha_{xi} \sigma^\beta_{xi}, \quad (61)
\]

where we have put \( C = d - 12 \).
From Eqs. (59), (60) and (61),

\[
[Z^n] = \exp \left( -3 \beta^2 J^2 N_P \sum_{\alpha<\beta} Q_{\alpha\beta}^4 - 3J_0 \beta N_P \sum_{\alpha} U_{\alpha 0}^4 + N_L \log \text{Tr} e^L \right),
\]

(62)

where

\[
L = 4 \beta^2 J^2 C \sum_{\alpha<\beta} Q_{\alpha\beta}^3 \sigma_{\alpha x}^\beta \sigma_{\beta x}^\alpha + \beta \sum_{\alpha} (4J_0 C U_{\alpha 0}^3 + h) \sigma_{\alpha x}^\alpha.
\]

(63)

We assume a replica symmetric (RS) solution for \( U_{\alpha 0} = U_0 \) and \( Q_{\alpha\beta} = Q \). In the RS case, \( \log \text{Tr} e^L \) can be evaluated as follows,

\[
\log \text{Tr} e^L = \log \text{Tr} \sqrt{4 \beta^2 J^2 C Q^3} 2\pi \int_{-\infty}^{\infty} dz \exp \left( -4 \beta^2 J^2 C Q^3 z^2 + 4 \beta^2 J^2 C Q^3 \sum_{\alpha} \sigma_{\alpha x}^\alpha - 2 \beta^2 J^2 C \sigma_{\alpha x}^\alpha \right)
\]

\[
\quad \quad \quad - 2 \beta^2 J^2 C n Q^3 + \beta \sum_{\alpha} (4J_0 C U_{\alpha 0}^3 + h) \sigma_{\alpha x}^\alpha \right)
\]

\[
= \log \left( 1 + n \int_{-\infty}^{\infty} \frac{Dz}{\sqrt{2\pi}} \log (2 \cosh \beta \tilde{H}(z)) - 2 \beta^2 J^2 C Q^3 + O(n^2) \right),
\]

(64)

where

\[
Dz = dz \ e^{-z^2} \sqrt{2\pi}, \ \tilde{H}(z) = 2J \sqrt{C Q^3} z + 4J_0 C U_{\alpha 0}^3 + h.
\]

(65)

From (64), the free energy \( F_R \) is evaluated as

\[-\beta F_R = \lim_{n \to 0} [Z^n] - 1n
\]

\[
= N_P \left( -3 \beta^2 J^2 2(n - 1) Q^4 - 3J_0 \beta U_0^4 + 1C \log \text{Tr} e^L \right)
\]

\[
= N_P \left( 32 \beta^2 J^2 Q^4 - 3J_0 \beta U_0^4 + 1C \int Dz \ \log (2 \cosh \beta \tilde{H}(z)) - 2 \beta^2 J^2 Q^3 \right).
\]

(66)

The values of MF’s \( U_0 \) and \( Q \) are determined by the stationary condition of \( F_R \),

\[
\partial F_R / \partial U_0 = 0, \ \partial F_R / \partial Q = 0.
\]

(67)

Numerical calculation is necessary for solving Eq. (67), and the result is given in Fig. 5.

Let us explain physical meanings of the “order parameters” \( U_{\alpha 0} \) and \( Q_{\alpha\beta} \). In the ordinary gauge theories with constant gauge coupling, the confinement and deconfinement phases are distinguished by the expectation value of the Wilson loop operator \( W(C) \),

\[
W(C) = \prod_{(xi) \in C} \sigma_{xi},
\]

(68)

where \( C \) is a large closed loop on the original lattice. If the system is in the confinement phase, \( \langle W(C) \rangle \propto e^{-\text{Area}(C)} \), whereas in the deconfinement phase, \( \langle W(C) \rangle \propto e^{-\text{Perimeter}(C)} \).
In the RGT, on the other hand, the ensemble average must be taken in order to obtain physical quantities. Then order parameter is given by $⟨W(C)⟩$. Nonvanishing of the MF $U_{0α}$ means the perimeter law $⟨W(C)⟩ \propto e^{-\text{Perimeter}(C)}$, which indicates that the system is in the deconfinement phase, the Higgs phase in the present case. As shown in Fig.5, the RMFT predicts that the Higgs phase exists in the RGT if the fluctuation of the RGC $J_p$ is not so large[15]. Result in Fig.5 also indicates the existence of a “gauge glass” phase[4]. In this phase, $U_{0α} = 0$ whereas $Q_{αβ} \neq 0$. This means $⟨W(C)⟩ \propto e^{-\text{Area}(C)}$ whereas $[⟨W(C)⟩^2] \propto e^{-\text{Perimeter}(C)}$. This prediction of the gauge glass is itself interesting but the spatial dimension must be probably large for its realization.4

Reliability of the RMFT can be studied as in the usual spin glass models like the Sherrington-Kirkpatrick model. Parisi-type solutions for replica-symmetry breaking are also interesting. These problems are under study and results will be reported in a future publication.

6 Conclusion

In this paper we explicitly showed the relationship between $Z_N$ gauge theories and Kitaev’s model for quantum memory and computations. $Z_N$ gauge theories appear as a result of the spontaneous breakdown of the $U(1)$ gauge theory with the “Higgs field” of charge $N$. The Higgs-phase limit of the $Z_N$ gauge systems corresponds to Kitaev’s model. Stability of Kitaev’s model was discussed and it was shown that the errors or noise represented by the term like $\sum X_{x_i}$ induce the phase transition to the confinement phase in the gauge-theory terminology. In that phase, quantum memory and quantum computations are impossible. Then we studied effects of the RGC which are also regarded as noise and errors in quantum computations. Static RGC gives no significant effect on the phase structure whereas time-dependent RGC induces phase transitions including that to the gauge-glass phase. Application of the present studies to non-Abelian discrete gauge theory is interesting and important for quantum computations.

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References


4In this paper we are considering systems only on the simple square or cubic lattice. If we consider more complicated lattice or networks, effective spatial dimensions increase and the gauge glass phase may appear.


[12] Properties of the three-dimensional random $Z_2$ gauge theory at zero temperature are numerically investigated in Ref.[4].


[14] As it is well-known, Elitzur’s theorem forbids the nonvanishing expectation value of the gauge-variant quantities like $\sigma_{xi}$. Discussion on the compatibility of the MFT and Elitzur’s theorem is given in Ref.[13].
Recent numerical studies on 3D $Z_2$ random gauge theory in Refs.[4] and [11] indicate that the Higgs phase at low temperature disappears quite rapidly as the concentration $p$ of the “wrong-sign” plaquettes increases.
Figure 1: Phase diagram of the gauge-Higgs model

Figure 2: Path $\tilde{c}_{x^* y^*}$ connecting dual sites $x^*$ and $y^*$
Figure 3: The links (\(\bar{x}_i\)) on which \(X^R_{\bar{x}_i}\) operators. Dyon is composed of electric charge on site and magnetic vortex on plaquette.

Figure 4: Free energy of the \(\mathbb{Z}_2\) gauge system by the MFT.
Figure 5: Phase diagram of the RGT obtained by the RMFT. The Higgs phase exists at low temperature (T) and small fluctuation of the RGC. Gauge-glass phase appears at low T and large fluctuation of the RGC. In the Higgs phase, the MF’s $U_0 \neq 0$ and also $Q \neq 0$, whereas in the gauge-glass phase $U_0 = 0$ and $Q \neq 0$. 