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MARKOV TRACE ON THE YOKONUMA-HECKE ALGEBRA

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1. INTRODUCTION

The objective of this note is to prove that there exists a Markov trace on the Yokonuma-Hecke algebra. A motivation to define a Markov trace on algebras of type Hecke is to get polynomial invariants for knots in the sense of Jones construction [7].

The Yokonuma-Hecke algebra $Y_{r,n}(u)$ can be regarded as a u -deformation of the wreath product $C_r \wr S_n$, where C_r denotes the cyclic group with r elements and S_n is the symmetric group on n -symbols. In the case $u = p^n$ and $r = p^n - 1$, with p a prime number, the algebra $Y_{r,n}(q)$ is the endomorphism algebra for the permutation representation of $GL(n, q)$ respect to one unipotent maximal subgroup, see [8]. Thus one has a natural epimorphism ϕ from the algebra $Y_{r,n}(q)$ onto the Hecke algebra $H_n(q)$ (recall that $H_n(q)$ can be realized as the endomorphism algebra for the permutation representation of $GL(n, q)$ respect to one Borel subgroup).

The Yokonuma-Hecke algebra (of type A) is a quotient of the framed braid group $B_{r,n} := C_r \wr B_n$, where B_n is the braid group of type A , and B_n acts on C_r^n by permutation [9]. More precisely, one consider a presentation of $B_{r,n}$ by generators and relations (Proposition 1). Then the algebra $Y_{r,n}(u)$ is defined as the quotient of the group algebra of $B_{r,n}$, over $\mathbb{C}$, by the ideal generated by a certain family of quadratic expressions, see (11). Notice that these quadratic relations go, by ϕ , to the classical Hecke quadratic relations.

Let $Y_{r,0} = \mathbb{C}$ and put $Y_\infty = \cup_{n \geq 0} Y_{r,n}(u)$, where $Y_{r,n-1}(u)$ is regarded as a subalgebra of $Y_{r,n}(u)$. The goal of this note is to prove the theorem below (cf. [2, 10, 11]).

Theorem. (*Theorem 12*) *Let $z, \zeta_1, \dots, \zeta_{r-1}$ be in $\mathbb{C}$, and let us put $\zeta_r = 1$. There exists a unique linear map tr on Y_∞ with values in $\mathbb{C}$, such that $tr(1) = 1$, and*

$$\begin{aligned} tr(at_n^i) &= \zeta_i tr(a) & (1 \leq i \leq r, a \in Y_n) \\ tr(ag_nb) &= z tr(ab) & (a, b \in Y_n) \\ tr(ab) &= tr(ba) & (a, b \in Y_\infty). \end{aligned}$$

The theorem is proved using the same method developed to prove the existence of Markov trace on the Hecke algebra, see [3]. Thus, in particular, we have modeled some very known results (see Lemma 5 and Proposition 10) to our situation.

2. FRAMED BRAID GROUP

Let B_n be the braid group on n strings of type A . The group B_n can be presented (Artin's presentation) with generators $\sigma_1, \dots, \sigma_{n-1}$ and braid relations of type A :

$$(1) \quad \begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| > 1, \\ \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1. \end{aligned}$$

Set $C_r = \langle t \rangle$ the cyclic group with r elements. Set $C_r^n = C_r \times \dots \times C_r$ (n times) and $c_i = (1, \dots, t, \dots, 1) \in C_r^n$, where t is in the position i . Recall that C_r^n has a presentation with generators $c_1, \dots, c_n$ and relations

$$(2) \quad c_i^r = 1, \quad c_i c_j = c_j c_i \quad \text{for all } i, j.$$

Using the canonical projection from B_n onto the symmetric group S_n on n symbols, $\sigma_i \mapsto s_i := (i, i+1)$, we obtain that B_n acts by permuting the factors in C_r^n . We denote by $B_{r,n}$ the wreath product $C_r \wr B_n$, that is, $B_{r,n}$ is the semidirect product of C_r^n with B_n (notice that $B_{1,n}$ is B_n , and $B_{r,1}$ is C_r).

$$1 \longrightarrow C_r^n \xrightarrow{\iota} B_{r,n} \xrightarrow{\pi} B_n \longrightarrow 1$$

Let us denote by t_0 (or simply by t) the element $\iota(c_1) := (c_1, 1)$, by t_{i-1} (for $i > 1$) the element $\iota(c_i) := (c_i, 1)$, and by σ the elements $\pi^{-1}(\sigma) = (1, \sigma)$ of $B_{r,n}$. It is easy to check that the elements $t = t_0, \sigma_1, \dots, \sigma_{n-1}$ of $B_{r,n}$ satisfy the relations: $\sigma_i t = t \sigma_i$ ($i > 1$), $t \sigma_1 t \sigma_1^{-1} = \sigma_1 t \sigma_1^{-1} t$, and

$$(3) \quad \sigma_i \dots \sigma_1 t \sigma_1^{-1} \dots \sigma_i^{-1} = \sigma_i^{-1} \dots \sigma_1^{-1} t \sigma_1 \dots \sigma_i = t_i.$$

Proposition 1. *The group $B_{r,n}$ has a presentation with the generators $t, \sigma_1, \dots, \sigma_{n-1}$ and relations (1) for $\sigma_1, \dots, \sigma_{n-1}$ and*

$$(4) \quad t^r = 1$$

$$(5) \quad \sigma_i t = t \sigma_i \quad \text{for} \quad i > 1$$

$$(6) \quad \sigma_i \cdots \sigma_1 t \sigma_1^{-1} \cdots \sigma_i^{-1} = \sigma_i^{-1} \cdots \sigma_1^{-1} t \sigma_1 \cdots \sigma_i$$

$$(7) \quad t \sigma_1 t \sigma_1^{-1} = \sigma_1 t \sigma_1^{-1} t.$$

Proof. Because $B_{r,n}$ is the semi-direct product of B_r and C_r^n , one gets a presentation for $B_{r,n}$ with generators $t = t_0, t_1, \dots, t_{n-1}$, $\sigma_1, \dots, \sigma_{n-1}$ and relations (1) for the σ_i 's, relations (2) for $\iota(c_j)$, and the following mixed relations:

$$\sigma_j^{-1} t_i \sigma_j = w(\sigma_j, t_i),$$

where $w(\sigma_j, t_i)$ is a word in the t_k 's. (see corollary 1, chap. 10 [6]). But it is straightforward to see that this last family of relations can be written as the family of relations constituted by relations (5), (6) and

$$(8) \quad \sigma_j t_i \sigma_j^{-1} = \begin{cases} t_{i-1} & \text{for } j = i \\ t_i & \text{for } j > i + 1 \\ t_i & \text{for } j < i, \end{cases}$$

for any j and $i \geq 1$.

Now relation (3) defines t_i ($i \geq 1$) in terms of t and σ_i 's. Then, one obtains that $B_{r,n}$ can be presented by $t, \sigma_1, \dots, \sigma_{n-1}$ with the relations given in Proposition 1, and with the relations in (8), $t_i^r = 1$ (for $i \geq 1$), and:

$$(9) \quad t_i t_j = t_j t_i.$$

To finish the proof of the proposition it remains to prove that $t_i^r = 1$ and relations (9), (8) are consequences of the relations given by Proposition 1.

It is trivial to see that $t_i^r = 1$ comes from relations (4) and (6). Now, using induction, one will see that (9) is a consequence of the relations given in Proposition 1. Thus if $i = 0$, one must prove that $t_0 t_j = t_j t_0$, for all j . Which is trivial for $j = 0, 1$ (see (7)). Let us suppose $j > 1$. One has $t_0 t_j = t(\sigma_j t_{j-1} \sigma_j^{-1}) = \sigma_j t t_{j-1} \sigma_j^{-1} = \sigma_j t_{j-1} t \sigma_j^{-1} = (\sigma_j t_{j-1} \sigma_j^{-1}) t = t_j t_0$. Therefore $t_0 t_j = t_j t_0$, for all j .

Now suppose that the affirmation is true for i . We must prove that for all j : $t_{i+1} t_j = t_j t_{i+1}$. Using the same argument as above, one can easily verify that it holds for $j \neq i + 2$. In the case $j = i + 2$, one has

$$\begin{aligned} t_{i+1} t_{i+2} &= (\sigma_{i+1} t_i \sigma_{i+1}^{-1}) (\sigma_{i+2} \sigma_{i+1} t_i \sigma_{i+1}^{-1} \sigma_{i+2}^{-1}) \\ &= \sigma_{i+1} t_i \sigma_{i+2} \sigma_{i+1} \sigma_{i+2}^{-1} t_i \sigma_{i+1}^{-1} \sigma_{i+2}^{-1} \\ &= \sigma_{i+1} t_i \sigma_{i+2} \sigma_{i+1} t_i \sigma_{i+2}^{-1} \sigma_{i+1}^{-1} \sigma_{i+2}^{-1} \\ &= \sigma_{i+1} \sigma_{i+2} t_i \sigma_{i+1} t_i \sigma_{i+1}^{-1} \sigma_{i+2}^{-1} \sigma_{i+1}^{-1} \\ &= \sigma_{i+1} \sigma_{i+2} \sigma_{i+1} t_i \sigma_{i+1}^{-1} t_i \sigma_{i+2}^{-1} \sigma_{i+1}^{-1} \quad (t_i t_{i+1} = t_{i+1} t_i) \\ &= \sigma_{i+2} \sigma_{i+1} \sigma_{i+2} t_i \sigma_{i+1}^{-1} \sigma_{i+2}^{-1} t_i \sigma_{i+1}^{-1} \\ &= \sigma_{i+2} \sigma_{i+1} t_i \sigma_{i+2} \sigma_{i+1}^{-1} \sigma_{i+2}^{-1} t_i \sigma_{i+1}^{-1} \\ &= (\sigma_{i+2} \sigma_{i+1} t_i \sigma_{i+1}^{-1} \sigma_{i+2}^{-1}) (\sigma_{i+1} t_i \sigma_{i+1}^{-1}) \\ &= t_{i+2} t_{i+1}. \end{aligned}$$

Finally, in (8) the cases $j = i$ and $j > i + 1$ follow trivially from the definition of t_i . In the case $j < i$, one has

$$\begin{aligned}
\sigma_j t_i \sigma_j^{-1} &= \sigma_i \cdots \sigma_j \sigma_{j+1} \sigma_j t_{j-1} \sigma_j^{-1} \sigma_{j+1}^{-1} \sigma_j^{-1} \cdots \sigma_i \\
&= \sigma_i \cdots \sigma_{j+1} \sigma_j (\sigma_{j+1} t_{j-1} \sigma_{j+1}^{-1}) \sigma_j^{-1} \sigma_{j+1}^{-1} \cdots \sigma_i \\
&= \sigma_i \cdots \sigma_{j+1} \sigma_j t_{j-1} \sigma_j^{-1} \sigma_{j+1}^{-1} \cdots \sigma_i \\
&= t_i.
\end{aligned}$$

□

Corollary 2. *For any $1 \leq m \leq r$, one has:*

$$\sigma_j t_i^m \sigma_j^{-1} = \begin{cases} t_{i+1}^m & \text{for } j = i + 1 \\ t_{i-1}^m & \text{for } j = i \\ t_i^m & \text{for } j > i + 1 \\ t_i^m & \text{for } j < i. \end{cases}$$

Remark 1. Let $B_{\infty,n}$ be the framed braid group [9], that is, the wreath product $C \wr B_n$, where C is the infinite cyclic group. Omitting the relation 4, Proposition 1 holds for $B_{\infty,n}$.

Remark 2. Let $S_{r,n}$ and $S_{\infty,n}$ be the finite versions of the framed braid groups $B_{r,n}$ and $B_{\infty,n}$, respectively. More precisely, $S_{r,n}$ is the group $C_r \wr S_n$, and $S_{\infty,n}$ is the group $C \wr S_n$ (notice that $S_{1,n}$ is S_n and $S_{2,n}$ is the Coxeter group of type B). One has that Proposition 1 holds for the group $S_{r,n}$, if we omit relation (7), and we add the relations $\sigma_i^2 = 1$. Using this presentation for $S_{r,n}$ one can prove that any element of $S_{r,n}$ can be written in the form:

$$(10) \quad w_0 \cdots w_{n-1},$$

where $P_0 := \{1, t^m; 1 \leq m \leq r-1\}$, $w_i \in P_i$, and

$$P_i := \{1, t_i^m, s_i w; w \in P_{i-1}, 1 \leq m \leq r-1\} \quad (1 \leq i \leq n-1).$$

Cf. section 2[2].

Notice that from the canonical epimorphisms $C \rightarrow C_r$ and $B_n \rightarrow S_n$, we get the commutative diagram

$$\begin{array}{ccccccc}
B_{\infty,n} & \longrightarrow & B_{r,n} & \longrightarrow & B_n & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
S_{\infty,n} & \longrightarrow & S_{r,n} & \longrightarrow & S_n & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & &
\end{array}$$

Finally, according to the diagram above, one can consider the lifting ℓ to $S_{r,n}$ of the length function ℓ' of S_n . For example: $\ell(ts_i) = \ell'(s_i) = 1$, $s_i = (i, i+1)$.

Lemma 3. *For any s_i, s_j and $g \in S_{r,n}$ such that $\ell(s_i g s_j) = \ell(g)$ and $\ell(s_i g) = \ell(g s_j)$, one has $s_i g = g s_j$.*

Proof. See Lemma §7.2[5].

□

3. HECKE ALGEBRAS

In the following we fix a parameter $u \in \mathbb{C}^\times$.

The Hecke algebra $H_n = H_n(u)$ at parameter u is defined as the quotient of the group algebra of B_n factoring out the quadratic relations $\sigma_i^2 = u + (u-1)\sigma_i$. In other words the Hecke algebra $H_n(u)$ is the associative algebra (over $\mathbb{C}$) with generators $1, h_1, \dots, h_{n-1}$ and relations: $h_i h_j = h_j h_i$ for $|i-j| > 1$, $h_i h_j h_i = h_j h_i h_j$ for $|i-j| = 1$, and

$$h_i^2 = u + (u-1)h_i \quad \text{for all } i.$$

Analogous to the Hecke algebra we are going to define the algebra $Y_{r,n} = Y_{r,n}(u)$ (of type A) as a quotient of the group algebra of $B_{r,n}$ factoring out certain exotic quadratic relations. To do this we introduce the following elements e_i in C_r^n ,

$$e_i := \frac{1}{r} \sum_{1 \leq m \leq r} t_{i-1}^m t_i^{r-m} \quad (1 \leq i \leq n-1).$$

Lemma 4. *For all i one has:*

$$(4.1) \quad \sigma_i e_i = e_i \sigma_i$$

$$(4.2) \quad e_i^2 = e_i.$$

$$(4.3) \quad e_i c^{\sigma_i} = e_i c, \quad \text{where } c^{\sigma_i} := \sigma_i c \sigma_i^{-1}, c \in C_r^n.$$

Proof. Let us prove (4.1). One has $\sigma_i t_{i-1} t_i^{-1} \sigma_i^{-1} = (\sigma_i t_{i-1} \sigma_i^{-1})(\sigma_i t_i^{-1} \sigma_i^{-1}) = t_i t_{i-1}^{-1}$. Therefore $\sigma_i K_i \sigma_i^{-1} = K_i$, where K_i is the cyclic group generated by $t_{i-1} t_i^{-1}$.

Now we can rewrite e_i as $e_i = r^{-1} \sum_{x \in K_i} x$, then it is clear that $\sigma_i e_i \sigma_i^{-1} = e_i$, i.e. $\sigma_i e_i = e_i \sigma_i$.

The proof of (4.2) and (4.3) is trivial. $\square$

Lemma 5. *For any s_i, s_j and $g \in S_{r,n}$ such that $\ell(s_i g s_j) = \ell(g)$ and $\ell(s_i g) = \ell(g s_j)$, one has (i) $s_i g e_j = e_i g s_j$ and (ii) $g e_i = e_j g$.*

Proof. (i) The proof is by induction on the length of g . Let us put $g = c s_r$, $\ell(g) = 1$. There are two cases: $i = r$ or $i \neq r$. In the first case the condition $\ell(s_i g) = \ell(g s_j)$ implies $j = r$, then the lemma is true. Suppose now that we are in the case $i \neq r$. Notice that the condition $\ell(s_i g s_j) = \ell(g)$ says that one must have $i = j$. Hence $|i - r| > 1$. Finally one can check that $s_i g e_i = e_i g s_i$. I. e. the lemma is true for any g of length 1.

Now, suppose the lemma is true for the elements of length less than m . Set $g = c s_{k_1} \cdots s_{k_m}$ a reduced expression for g . One has two possibilities: $\ell(g) > \ell(s_i g)$ or $\ell(g) < \ell(s_i g)$. For the first case one has $s_i g = s_i c s_{k_1} \cdots s_{k_m} = c^{s_i} s_i s_{k_1} \cdots s_{k_m}$, and in virtue of Matsumoto's lemma (see §1.7 [4]) one can write $g = c s_i g'$, where $\ell(g') < m$. Then

$$\begin{aligned} s_i g e_j &= c^{s_i} s_i s_i g' e_j \\ &= c^{s_i} s_i e_i g' s_j \quad (\text{induction hypothesis}) \\ &= c^{s_i} e_i s_i g' s_j \\ &= e_i c s_i g' s_j = e_i g s_j. \end{aligned}$$

Suppose now that $\ell(g) < \ell(s_i g)$. Then one has $\ell(s_i g s_j) = \ell(g) < \ell(s_i g) = \ell(g s_j)$. Thus one apply the first case on the elements $g s_j$ and s_i : $s_i (g s_j) e_j = e_i (g s_j) s_j$. Hence $s_i g e_j = e_i g s_j$.

The proof of (ii) follows directly from (i) and Lemma 3. $\square$

Definition 1. The algebra $Y_{r,n} = Y_{r,n}(u)$ at parameter u is defined as the quotient of group algebra of $B_{r,n}$ by factoring through the ideal generated by the “quadratic expressions”:

$$(11) \quad \sigma_i^2 - 1 - (u-1)e_i + (u-1)e_i \sigma_i.$$

We denote the image of σ_i under the canonical epimorphism $B_{r,n} \rightarrow Y_{r,n}$ by g_i , and the image of t_i (respectively e_i) again by t_i (respectively e_i).

In other terms $Y_{r,n}$ is the algebra generated by $1, g_1, \dots, g_{n-1}, t$ and defining relations formed by the braid relations (1) for the g_i 's, and:

$$(12) \quad t^r = 1$$

$$(13) \quad g_i t = t g_i \quad \text{for } i > 1$$

$$(14) \quad g_i \cdots g_1 t g_1^{-1} \cdots g_i^{-1} = g_i^{-1} \cdots g_1^{-1} t g_1 \cdots g_i$$

$$(15) \quad t g_1 t g_1^{-1} = g_1 t g_1^{-1} t$$

$$(16) \quad g_i^2 = 1 - (u-1)e_i(1 - g_i).$$

Now the element $t_i \in Y_{r,n}$ is

$$t_i = g_i t_{i-1} g_i^{-1} = g_i^{-1} t_{i-1} g_i.$$

One can now prove without difficulty the following proposition.

Proposition 6. *The algebra $Y_{r,n}$ has a presentation with generators $1, \dots, g_{n-1}, \tau_1, \dots, \tau_n$ and relations:*

$$\begin{aligned}\tau_i^r &= 1 \\ \tau_i \tau_j &= \tau_j \tau_i \quad \text{for all } i, j \\ \tau_j g_i &= g_i \tau_{g_i(j)} \\ g_i g_j &= g_j g_i \quad \text{for } |i - j| > 1 \\ g_i g_j g_i &= g_j g_i g_j \quad \text{for } |i - j| = 1 \\ g_i^2 &= 1 - (u - 1)e_i(1 - g_i),\end{aligned}$$

where $g_i(j)$ is the result of the action of $(i, i + 1)$ on j .

Now, notice that Lemma 4 holds in $Y_{r,n}$, then we get

$$g_i^{-1} = g_i - (u^{-1} - 1)e_i + (u^{-1} - 1)e_i g_i.$$

Furthermore, the natural embedding $B_{r,n} \subseteq B_{r,n+1}$ induces the tower of algebras

$$\mathbb{C} \subseteq Y_{r,1} \subseteq \dots \subseteq Y_{r,n} \subseteq Y_{r,n+1} \subseteq \dots$$

Lemma 7. *For all i , one has:*

$$(7.1) \quad g_i g_{i-1} t_i^m = t_{i-1}^m g_i g_{i-1}$$

$$(7.2) \quad g_i e_i = e_i g = r^{-1} \sum_{1 \leq m \leq r} t_{i-1}^m g_i t_{i-1}^{r-m}$$

$$(7.3) \quad t_n \text{ belong to the center of } Y_{r,n}.$$

Proof. We prove (7.1)

$$\begin{aligned}g_i g_{i-1} t_i^m &= g_i g_{i-1} g_i t_{i-1}^m g_i^{-1} \\ &= g_{i-1} g_i g_{i-1} t_{i-1}^m g_i^{-1} \\ &= g_{i-1} g_i t_{i-2}^m g_{i-1} g_i^{-1} \\ &= g_{i-1} t_{i-2}^m g_i g_{i-1} g_i^{-1} \\ &= g_{i-1} t_{i-2}^m g_{i-1}^{-1} g_i g_{i-1} \\ &= t_{i-1}^m g_i g_{i-1}.\end{aligned}$$

One has $t_i t_{i-1}^m g_i t_{i-1}^{-m} = t_{i-1}^m (t_i g_i) t_{i-1}^{-m} = t_{i-1}^m g_i t_{i-1} t_{i-1}^{-m}$, then (7.2) follows.

The assertion (7.3) follows directly from Corollary 2. $\square$

Proposition 8. *All elements in $Y_{r,n}$ can be written as a linear combination of words in $t, g_1, \dots, g_{n-1}$, having at most one $\alpha_{n-1} \in \{g_{n-1}, t_{n-1}^m; 1 \leq m \leq r\}$.*

Proof. The proof of the proposition uses the same argument of induction used in Lemma 3.1[1].

From (11) and Lemma 7 it is easy to check that the assertion is true for $n = 2$.

Suppose the assertion is true for n . Now, any element in $Y_{r,n+1}$ is a linear combination of words in $t, g_1, \dots, g_n$. Let M be such a word, we must prove that M is a linear combination of words having at most one $\alpha_n \in \{g_n, t_n^m; 1 \leq m \leq r - 1\}$. For that, it is sufficient to prove that if M has two occurrences of α_n , then M is a linear combination of words with at most one α_n . Thus, suppose that $M = M_1 \alpha_n M_2 \alpha_n M_3$, where M_i belongs to $Y_{r,n}$. Furthermore, one can suppose that M_2 is a word in $t, g_1, \dots, g_{n-1}$ with at most one α_{n-1} .

Now, if M_2 does not have α_{n-1} , then $M = M_1 M_2 \alpha_n^2 M_3$. From (11) and Lemma 7 one gets

$$M = M_1 M_2 M_3 + r^{-1}(u - 1) \sum_{1 \leq m \leq r} M_1 M_2 t_{n-1}^m t_n^m M_3 + r^{-1}(u - 1) \sum_{1 \leq m \leq r} M_1 M_2 t_{n-1}^m g_n t_{n-1}^{-m} M_3.$$

Thus, the assertion is true in this case.

In the case that M_2 has exactly once α_{n-1} , one can rewrite M like $M'_1 \alpha_n \alpha_{n-1} \alpha_n M_2$, where M'_1, M_2 are words in $t, g_1 \dots g_{n-1}$. Thus, we need only to prove that the words below are a linear combination of the words having at most one α_n (we shall say that the word can be reduced),

$$\begin{array}{llll} (i) g_n g_{n-1} g_n & (ii) g_n t_{n-1}^m g_n & (iii) t_n^m g_{n-1} g_n & (iv) t_n^{m_1} t_{n-1}^{m_2} g_n \\ (v) g_n g_{n-1} t_n^m & (vi) g_n t_{n-1}^{m_1} t_n^{m_2} & (vii) t_n^{m_1} g_{n-1} t_n^{m_2} & (viii) t_n^{m_1} t_{n-1}^{m_2} t_n^{m_3}, \end{array}$$

where $m, m_1, m_2, m_3 \in \{1, 2, \dots, r-1\}$.

Trivially the words in (i) and (viii) can be reduced.

The reduction of words (iii) and (v) is direct from Lemma 7.

Using $t_n g_n = g_n t_{n-1}$, we deduce that the word in (iv) can be reduced. In a similar way a reduction for (vi) can be taken.

We are now going to reduce the word in (vii). One has

$$\begin{aligned} t_n^{m_1} g_{n-1} t_n^{m_2} &= g_n^{-1} t_{n-1}^{m_1} g_n g_{n-1} g_n t_{n-1}^{m_2} g_n^{-1} \\ &= g_n^{-1} t_{n-1}^{m_1} g_{n-1} g_n g_{n-1} t_{n-1}^{m_2} g_n^{-1} \\ &= g_n^{-1} g_{n-1} g_n t_{n-2}^{m_1} t_{n-2}^{m_2} g_{n-1} g_n^{-1} \\ &= g_n^{-1} g_{n-1} g_n t_{n-2}^{m_1+m_2} g_{n-1} g_n^{-1} \\ &= g_{n-1} g_n g_{n-1} t_{n-2}^{m_1+m_2} g_{n-1} g_n^{-1} \\ &= g_{n-1} g_n t_{n-1}^{m_1+m_2} g_n^{-1} \\ &= g_{n-1} t_n^{m_1+m_2}. \end{aligned}$$

Thus the word in (vii) was reduced.

Finally to reduce the word in (ii), let us write:

$$(17) \quad g_n t_{n-1}^m g_n = g_n t_{n-1}^m g_n^{-1} g_n^2 = t_n^m g_n^2 = t_n^m (1 + (u-1)e_n - (u-1)e_n g_n).$$

Now using Lemma 7 we deduce that $g_n t_{n-1} g_n$ can be reduced. $\square$

The next task is to prove that $Y_{r,n}$ admits a basis formed by words in normal form. By definition the *normal words* of $Y_{r,n}$ are the words

$$(18) \quad v_0 v_1 \dots v_{n-1},$$

where $v_i \in R_i$, $R_0 := \{1, t^m; 1 \leq m \leq r-1\}$, and

$$R_i := \{1, t_i^m, g_i v; v \in R_{i-1}, 1 \leq m \leq r-1\} \quad (1 \leq i \leq n-1).$$

Proposition 9. $Y_{r,n}$ is linearly spanned by the normal words.

Proof. The assertion is true for $n = 1$ (recall that $Y_{r,1}(u) = \mathbb{C}C_r$). Suppose that the proposition for n is true. From Proposition 8 the algebra $Y_{r,n+1}$ is linearly generated by the words in the form

$$(i) M_1, \quad (ii) M_2 t_n^m M_3, \quad (iii) M_4 g_n M_5,$$

where the M_i 's are words in $t, g_1, \dots, g_{n-1}$.

Now from the hypothesis of induction the M_i 's are a linear combination of normal words of $Y_{r,n}$. Therefore, the proof is reduced to prove that the words in (ii) and (iii) are a linear combination of normal words, where now the M_i 's are normal words in $Y_{r,n}$.

Let us put $M_3 = v_0 \dots v_{n-1}$, where $v_i \in R_i$. Then in (ii) we have

$$(iv) \quad M_2 t_n^m M_3 = M_2 t_n^m v_0 \dots v_{n-1} = M_2 v_0 \dots v_{n-2} t_n^m v_{n-1}.$$

If $v_{n-1} = t_{n-1}^{m'}$, we get $M_2 t_n^m M_3 = M_2 v_0 \dots v_{n-2} v_{n-1} t_n^m$, which is a normal word in $Y_{r,n+1}$.

If $v_{n-1} = g_{n-1} v'_{n-2}$, where $v'_{n-2} \in R_{n-2}$, one has

$$\begin{aligned} M_2 t_n^m M_3 &= M_2 v_0 \dots v_{n-2} t_n^m g_{n-1} v'_{n-2} \\ &= M_2 v_0 \dots v_{n-2} g_{n-1} t_n^m v'_{n-2} \\ &= (M_2 v_0 \dots v_{n-2} g_{n-1} v'_{n-2}) t_n^m. \end{aligned}$$

Because the words above belong to $Y_{r,n}$, we deduce from the hypothesis of induction that (iv) is a linear combination of normal words.

Let us put $M_5 = v_0 \cdots v_{n-1}$, where $v_i \in R_i$. Then in (iii) we have

$$M_4 g_n M_5 = M_4 g_n v_0 \cdots v_{n-1} = (M_4 v_0 \cdots v_{n-2}) g_n v_{n-1}.$$

Now from the fact that $g_n v_{n-1} \in R_n$, and as the word between parenthesis belongs to $Y_{r,n}$, we get that $M_4 g_n M_5$ is a linear combination of normal words. $\square$

Before concluding this section we shall show that the set of normal words is linearly independent in $Y_{r,n}$. For that we may use the same method used by J. Tits to prove the analogous claim for the Hecke algebra, see Appendix [5] and §7.1-7.3[4]. The sketch of the proof is as follows. Set $V = \mathbb{C}S_{r,n}$, we are going to define certain homomorphisms of $\mathbb{C}$ -algebras ρ and λ from the Yokonuma-Hecke algebra to the algebra $\text{End}(V)$. Thus for any $t \in C_r^n$ one can define ρ_t and λ_t as the homotheties $\rho_t g = tg$ and $\lambda_t g = gt$, and for g_i one put:

$$\begin{aligned} \rho_{g_i} g &:= \begin{cases} s_i g & \text{for } \ell(s_i g) > \ell(g) \\ s_i g - (u-1)e_i(1-s_i)g & \text{for } \ell(s_i g) < \ell(g) \end{cases} \\ \lambda_{g_i} g &:= \begin{cases} g s_i & \text{for } \ell(g s_i) > \ell(g) \\ g s_i - (u-1)g(1-s_i)e_i & \text{for } \ell(g s_i) < \ell(g) \end{cases} \end{aligned}$$

(cf. Proposition 2.14[8]).

It is obvious that $\rho_t \lambda_{t'} = \lambda_{t'} \rho_t$, for any $t, t' \in C_r^n$. Now, using essentially Lemma 3 and Lemma 5 one can prove that $\rho_{g_i} \lambda_{g_j} = \lambda_{g_j} \rho_{g_i}$, for any i, j . More precisely, one makes the proof by distinguishing according to the length of g , $s_i g$, $g s_j$ and $s_i g s_j$; for instance, suppose that $\ell(s_i g) > \ell(g)$, $\ell(g s_j) > \ell(g)$ and $\ell(s_i g s_j) < \ell(s_i g)$, then one has:

$$\begin{aligned} g &\xrightarrow{\rho} s_i g \xrightarrow{\lambda} (s_i g) s_j - (u-1)s_i g(1-s_j)e_j, \\ g &\xrightarrow{\lambda} g s_j \xrightarrow{\rho} s_i(g s_j) - (u-1)e_i(1-s_i)g s_j. \end{aligned}$$

Now using Lemma 5 one obtains that these expressions are equals. Thus one can prove that the map $t \mapsto \rho_t$, $g_i \mapsto \rho_{g_i}$ defines a homomorphism of $\mathbb{C}$ -algebras $\rho : Y_{r,n} \longrightarrow \text{End}(V)$. In particular one can deduce that $\rho(g)$ is $\mathbb{C}$ -linear. Furthermore, one has that,

$$\rho(v_0 \cdots v_{n-1})(1) = w_{n-1}^{-1} \cdots w_0^{-1}.$$

Thus any linear combination $\sum_g a_g g = 0$ (where g run the normal words (18)) becomes a linear combination $\sum_w a_g w^{-1} = 0$, where now w run the elements of $S_{r,n}$ (see (10)). Therefore $a_g = 0$, for all g , that means that the normal words is a set linearly independent. Hence

Proposition 10. *The set of the normal words is a $\mathbb{C}$ -basis for $Y_{r,n}$. Hence the dimension of $Y_{r,n}$ is $r^n n!$.*

4. MARKOV TRACE

In the following we write $Y_n = Y_{r,n}$, and let us put

$$Y_\infty = \bigcup_{n \geq 0} Y_n,$$

where, by definition, $Y_0 = \mathbb{C}$.

Proposition 11. *The map ϕ :*

$$\oplus_{i=1}^r \alpha_i \oplus (a \otimes b) \mapsto \sum_{i=1}^r \alpha_i t_n^i + a g_n b$$

defines an isomorphism of (Y_n, Y_n) -bimodules $\phi : (Y_n \oplus \cdots \oplus Y_n) \oplus Y_n \otimes_{Y_{n-1}} Y_n \longrightarrow Y_{n+1}$.

Proof. ϕ is surjective because any word in Y_{n+1} is a linear combination of words involving (only) the words:

$$vg_nw \quad \text{and} \quad w_i t_n^m \quad (1 \leq m \leq r),$$

where v, w and $w_i \in Y_n$.

On the other hand, using Proposition 10 one deduces that $Y_n \otimes_{Y_{n-1}} Y_n$ is spanned linearly by the set

$$\{t_{n-1}^m \otimes v, g_j \cdots g_{n-1} t_{n-1}^m \otimes v; 1 \leq m \leq r, 1 \leq j \leq n-1, v \text{ is a normal word in } Y_n\}.$$

Therefore the dimension of $(Y_n \oplus \cdots \oplus Y_n) \oplus Y_n \otimes_{Y_{n-1}} Y_n$ is less or equal than

$$rr^n n! + rr^n n! + (n-1)r(r^n n!) = r^{n+1}(n+1)!.$$

Then ϕ is an isomorphism. □

Theorem 12. *Let $z, \zeta_1, \dots, \zeta_{r-1}$ be in $\mathbb{C}$, and let us put $\zeta_r = 1$. There exists a unique linear map tr on Y_∞ with values in $\mathbb{C}$, such that $tr(1) = 1$, and*

$$\begin{aligned} tr(at_n^i) &= \zeta_i tr(a) & (1 \leq i \leq r, a \in Y_n) \\ tr(ag_nb) &= z tr(ab) & (a, b \in Y_n) \\ tr(ab) &= tr(ba) & (a, b \in Y). \end{aligned}$$

Proof. One defines tr recursively as follows. First, one defines tr on $Y_0 = \mathbb{C}$ as the identity and on $Y_1 = \mathbb{C}C_r$ by $tr(t^m) = \zeta_m$, $1 \leq m \leq r$. After, using the proposition above one defines tr on Y_{n+1} by

$$tr(x) = \sum_i \zeta_i tr(\alpha_i) + z tr(ab),$$

where $\phi^{-1}(x) = \sum_i \alpha_i + (a \otimes b)$.

From the construction of tr , it is clear that it only remains to prove that tr satisfies the trace condition, that is

$$(19) \quad tr(xy) = tr(yx), \quad \text{for all } x, y \in Y_{n+1}.$$

According to Proposition 8 one can suppose that x and y are words having at most one $\gamma_n \in \{g_n, t_n^m; 0 < m < r\}$.

In the case where y does not contain γ_n , it is easy to check the trace condition. In fact, set $x = a\gamma_nb$ ($a, b \in Y_n$) and let us distinguish between $\gamma_n = g_n$ or $\gamma_n = t_n^m$. If $\gamma_n = g_n$, one has

$$tr(xy) = tr(ag_nby) = z tr(aby) = z tr(yab) = tr(yag_nb) = tr(yx),$$

and if $\gamma_n = t_n^m$,

$$tr(xy) = tr(at_n^mby) = tr(abyt_n^m) = \zeta_m tr(aby) = tr(yabt_n^m) = tr(yx).$$

We will now prove that the trace condition holds in the case where y does contain γ_n . Notice now that one can suppose that $y = \gamma_n$. One can also suppose that a and b are words having at most one γ_{n-1} . Then let us write $a = a'\gamma_{n-1}a''$ and $b = b'\gamma_{n-1}b''$, where a', a'', b' and b'' belong to Y_{n-1} . Notice that in the case $\gamma_{n-1} = t_{n-1}^m$ one can put $a'' = 1$; idem for b'' .

To prove the trace condition, one shall make it according to the distinct $\gamma_n = g_n, t_n^m$ and $\gamma_{n-1} = g_{n-1}, t_{n-1}^m$. Thus one obtains sixteen cases to be analyzed. In fact one can check that for all these cases the trace condition (19) holds. The checking of these cases is only a large computation. For that, we will analyze only three of the more representatives cases.

Case $x = ag_nb, y = g_n, a = a'g_{n-1}a''$ and $b = b'g_{n-1}b''$:

One has

$$\begin{aligned}
tr(xy) &= tr(ag_nb'g_{n-1}g_nb'') \\
&= tr(ab'g_ng_{n-1}g_nb'') \\
&= tr(ab'g_{n-1}g_ng_{n-1}b'') \\
&= ztr(ab'g_{n-1}^2b'') \\
&= ztr(ab'b'') + z(u-1)tr(ab'e_{n-1}b'') - z(u-1)tr(ab'e_{n-1}g_nb'').
\end{aligned}$$

On the other hand,

$$\begin{aligned}
tr(yx) &= tr(g_na'g_{n-1}a''g_nb) \\
&= tr(a'g_ng_{n-1}g_na''b) \\
&= tr(a'g_{n-1}g_ng_{n-1}a''b) \\
&= ztr(a'g_{n-1}^2a''b) \\
&= ztr(a'a''b) + z(u-1)tr(a'e_{n-1}a''b) - z(u-1)tr(a'e_{n-1}g_na''b).
\end{aligned}$$

It is clear that

$$ztr(ab'b'') = ztr(a'g_na''b'b'') = z^2tr(a'a''b'b'') = ztr(a'a''b).$$

From the induction hypothesis easily one gets:

$$tr(ab'e_{n-1}b'') = tr(a'e_{n-1}a''b) \quad \text{and} \quad tr(ab'e_{n-1}g_{n-1}b'') = tr(a'e_{n-1}g_{n-1}a''b).$$

Therefore in this case one gets the trace condition.

Case $x = ag_nb$, $y = g_n$, $a = a'g_{n-1}a''$ and $b = b't_{n-1}^m$:

One has

$$\begin{aligned}
tr(xy) &= tr(ag_nb't_{n-1}^mg_n) \\
&= tr(ab'g_nt_{n-1}^mg_n) \\
&= tr(ab'g_n^2t_n^m) \\
&= tr(ab't_n^m) + (u-1)tr(ab'e_nt_n^m) - (u-1)tr(ab'e_ng_nt_n^m).
\end{aligned}$$

On the other hand

$$\begin{aligned}
tr(yx) &= tr(g_na'g_{n-1}a''g_nb) \\
&= tr(a'g_ng_{n-1}g_na''b) \\
&= tr(a'g_{n-1}g_ng_{n-1}a''b) \\
&= ztr(a'g_{n-1}^2a''b) \\
&= ztr(a'a''b) + z(u-1)tr(a'e_{n-1}a''b) - z(u-1)tr(a'e_{n-1}g_na''b).
\end{aligned}$$

Now, $ztr(a'a''b) = ztr(a'a''b't_{n-1}^m) = z\zeta_m tr(a'a''b') = \zeta_m tr(a'g_{n-1}a''b') = \zeta_m tr(ab')$. Therefore

$$(20) \quad tr(ab't_n^m) = ztr(a'a''b).$$

One has $ztr(a'e_{n-1}a''b) = ztr(a'e_{n-1}a''b't_{n-1}^m) = ztr(a'a''b'e_{n-1}t_{n-1}^m)$. Now, if we denote by μ the corresponding $m + r - i$ reduced mod r to the form $1 \leq \mu \leq r$, then we get

$$\begin{aligned}
ztr(a'e_{n-1}a''b) &= r^{-1} \sum_i z\zeta_\mu tr(a'a''b't_{n-2}^i) \\
&= r^{-1} \sum_i z\zeta_\mu tr(a't_{n-2}^i a''b') \\
&= r^{-1} \sum_i \zeta_\mu tr(a't_{n-2}^i g_{n-1} a''b') \\
&= r^{-1} \sum_i \zeta_\mu tr(a'g_{n-1} a''t_{n-1}^i b') \\
&= r^{-1} \sum_i \zeta_\mu tr(ab't_{n-1}^i) \\
&= tr(ab'(r^{-1} \sum_i t_{n-1}^i t_n^{-i}) t_n^m).
\end{aligned}$$

Therefore

$$(21) \quad (u-1)tr(ab'e_n t_n^m) = z(u-1)tr(a'e_{n-1}a''b).$$

One has

$$\begin{aligned}
ztr(a'e_{n-1}g_{n-1}a''b) &= ztr(a'e_{n-1}g_{n-1}a''b't_{n-1}^m) \\
&= ztr(a'e_{n-1}g_{n-1}t_{n-1}^m a''b') \\
&= ztr(a'(r^{-1} \sum_i t_{n-2}^{i+m} g_{n-1} t_{n-2}^{-i}) a''b') \\
&= z^2 tr(a't_{n-2}^m a''b') \\
&= ztr(a't_{n-2}^m g_{n-1} a''b') \\
&= ztr(a'g_{n-1} t_{n-1}^m a''b') \\
&= ztr(a'g_{n-1} a''b't_{n-1}^m) = ztr(ab).
\end{aligned}$$

One has

$$\begin{aligned}
tr(ab'e_n g_n t_n^m) &= tr(ab'r^{-1} \sum_i t_{n-1}^i g_n t_{n-1}^{-i} t_n^m) \\
&= tr(ab'r^{-1} \sum_i t_{n-1}^{i+m} g_n t_{n-1}^{-i}) \\
&= ztr(ab'r^{-1} \sum_i t_{n-1}^{i+m} t_{n-1}^{-i}) \\
&= ztr(ab't_{n-1}^m) = ztr(ab).
\end{aligned}$$

Hence

$$ztr(a'e_{n-1}g_{n-1}a''b) = tr(ab'e_n g_n t_n^m).$$

From this equality, (20) and (21) one gets the trace condition.

Case $x = ag_n b$, $y = g_n$, $a = a't_{n-1}^\mu$ and $b = b't_{n-1}^\nu$:

One has $tr(xy) = tr(ag_n b't_{n-1}^\nu g_n) = tr(ab'g_n t_{n-1}^\nu g_n) = tr(ab'g_n^2 t_n^\nu)$. Then,

$$\begin{aligned}
tr(xy) &= tr(a't_{n-1}^\mu b'g_n^2 t_n^\nu) \\
&= tr(a'b't_{n-1}^\mu g_n^2 t_n^\nu) \quad (t_{n-1}^\mu g_n^2 t_n^\nu = g_n^2 t_{n-1}^\mu t_n^\nu) \\
&= tr(a'b'g_n^2 t_n^\nu t_{n-1}^\mu).
\end{aligned}$$

Therefore $tr(xy) = tr(a'g_n^2 t_{n-1}^\mu b't_{n-1}^\nu) = tr(g_n a't_{n-1}^\mu g_n b't_{n-1}^\nu) = tr(yx)$. □

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