MARKOV TRACE ON THE YOKONUMA-HECKE ALGEBRA

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1. INTRODUCTION

The objective of this note is to prove that there exists a Markov trace on the Yokonuma-Hecke algebra. A motivation to define a Markov trace on algebras of type Hecke is to get polynomial invariants for knots in the sense of Jones construction [7].

The Yokonuma-Hecke algebra $Y_{r,n}(u)$ can be regarded as a $u$-deformation of the wreath product $C_r \wr S_n$, where $C_r$ denotes the cyclic group with $r$ elements and $S_n$ is the symmetric group on $n$-symbols. In the case $u = p^n$ and $r = p^n - 1$, with $p$ a prime number, the algebra $Y_{r,n}(q)$ is the endomorphism algebra for the permutation representation of $GL(n,q)$ respect to one unipotent maximal subgroup, see [8]. Thus one has a natural epimorphism $\phi$ from the algebra $Y_{r,n}(q)$ onto the Hecke algebra $H_n(q)$ (recall that $H_n(q)$ can be realized as the endomorphism algebra for the permutation representation of $GL(n,q)$ respect to one Borel subgroup).

The Yokonuma-Hecke algebra (of type $A$) is a quotient of the framed braid group $B_{r,n} := C_r \wr B_n$, where $B_n$ is the braid group of type $A$, and $B_n$ acts on $C_{r^n}$ by permutation [9]. More precisely, one consider a presentation of $B_{r,n}$ by generators and relations (Proposition 1). Then the algebra $Y_{r,n}(u)$ is defined as the quotient of the group algebra of $B_{r,n}$, over $\mathbb{C}$, by the ideal generated by a certain family of quadratic expressions, see (11). Notice that those quadratic relations go by $\phi$, to the classical Hecke quadratic relations.

Let $Y_{r,0} = \mathbb{C}$ and put $Y_{r} = \bigcup_{n \geq 0} Y_{r,n}(u)$, where $Y_{r,n-1}(u)$ is regarded as a subalgebra of $Y_{r,n}(u)$. The goal of this note is to prove the theorem below (cf. [2, 10, 11]).

**Theorem.** (Theorem 12) Let $z, \zeta_1, \ldots, \zeta_{r-1}$ be in $\mathbb{C}$, and let us put $\zeta_r = 1$. There exists a unique linear map $tr$ on $Y_{\infty}$ with values in $\mathbb{C}$, such that $tr(1) = 1$, and

$$
\begin{align*}
tr(a_i^n) &= \zeta_i tr(a) \quad (1 \leq i \leq r, a \in Y_n) \\
tr(ab) &= z tr(ab) \quad (a, b \in Y_n) \\
tr(ab) &= tr(ba) \quad (a, b \in Y_{\infty}).
\end{align*}
$$

The theorem is proved using the same method developed to prove the existence of Markov trace on the Hecke algebra, see [3]. Thus, in particular, we have modeled some very known results (see Lemma 5 and Proposition 10) to our situation.

2. FRAMED BRAID GROUP

Let $B_n$ be the braid group on $n$ strings of type $A$. The group $B_n$ can be presented (Artin’s presentation) with generators $\sigma_1, \ldots, \sigma_{n-1}$ and braid relations of type $A$:

$$
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1, \\
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j \quad \text{for} \quad |i - j| = 1.
\end{align*}
$$

Set $C_r = \langle t \rangle$ the cyclic group with $r$ elements. Set $C_{r^n} = C_r \times \cdots \times C_r$ ($n$ times) and $c_i = (1, \ldots, t, \ldots, 1) \in C_{r^n}$, where $t$ is in the position $i$. Recall that $C_{r^n}$ has a presentation with generators $c_1, \ldots, c_n$, and relations

$$
c_i^2 = 1, \quad c_i c_j = c_j c_i \quad \text{for all} \quad i, j.
$$

Using the canonical projection from $B_n$ onto the symmetric group $S_n$ on $n$ symbols, $\sigma_i \mapsto s_i := (i, i+1)$, we obtain that $B_n$ acts by permuting the factors in $C_{r^n}$. We denote by $B_{r,n}$ the wreath product $C_r \wr B_n$, that is, $B_{r,n}$ is the semidirect product of $C_{r^n}$ with $B_n$ (notice that $B_{1,n}$ is $B_n$, and $B_{r,1}$ is $C_r$).

$$
1 \longrightarrow C_{r^n} \xrightarrow{\iota} B_{r,n} \xrightarrow{\pi} B_n \longrightarrow 1
$$

Let us denote by $t_0$ (or simply by $t$) the element $\iota(c_1) := (c_1, 1)$, by $t_{i-1}$ (for $i > 1$) the element $\iota(c_i) := (c_i, 1)$, and by $\sigma$ the elements $\pi^{-1} \iota(\sigma) = (1, \sigma)$ of $B_{r,n}$. It is easy to check that the elements $t = t_0, \sigma_1, \ldots, \sigma_{n-1}$ of $B_{r,n}$ satisfy the relations: $\sigma_i t = t \sigma_i (i > 1)$, $t \sigma_i t^{-1} = \sigma_1 t^{-1} \sigma_1 t$, and

$$
\sigma_i \cdots \sigma_1 t \sigma_1^{-1} \cdots \sigma_i^{-1} = \sigma_i^{-1} \cdots \sigma_1^{-1} t \sigma_1 \cdots \sigma_i = t_i.
$$
Proposition 1. The group $B_{r,n}$ has a presentation with the generators $t, \sigma_1, \ldots, \sigma_{n-1}$ and relations (1) for $\sigma_1, \ldots, \sigma_{n-1}$ and

\[ t^r = 1 \]
\[ \sigma_i t = t \sigma_i \quad \text{for} \quad i > 1 \]
\[ \sigma_i \ldots \sigma_1 t \sigma_1^{-1} \ldots \sigma_i^{-1} = \sigma_i^{-1} \ldots \sigma_1^{-1} t \sigma_i \ldots \sigma_i \]
\[ t \sigma_i t \sigma_i^{-1} = \sigma_i t \sigma_i^{-1} t. \]

Proof. Because $B_{r,n}$ is the semi-direct product of $B_r$ and $C^n_r$, one gets a presentation for $B_{r,n}$ with generators $t = t_0, t_1, \ldots, t_{n-1}, \sigma_1, \ldots, \sigma_{n-1}$ and relations (1) for the $\sigma_i$'s, relations (2) for $\iota(c_j)$, and the following mixed relations:

\[ \sigma_j^{-1} t_i \sigma_j = w(\sigma_j, t_i), \]

where $w(\sigma_j, t_i)$ is a word in the $t_k$'s. (see corollary 1, chap. 10 [6]). But it is straightforward to see that this last family of relations can be written as the family of relations constituted by relations (5), (6) and

\[ \sigma_j t_i \sigma_j^{-1} = \begin{cases} t_{i-1} & \text{for} \ j = i \\ t_i & \text{for} \ j > i + 1 \\ t_i & \text{for} \ j < i, \end{cases} \]

for any $j$ and $i \geq 1$.

Now relation (3) defines $t_i$ ($i \geq 1$) in terms of $t$ and $\sigma_i$'s. Then, one obtains that $B_{r,n}$ can be presented by $t, \sigma_1, \ldots, \sigma_{n-1}$ with the relations given in Proposition 1, and with the relations in (8), $t_i^r = 1$ (for $i \geq 1$), and:

\[ t_i t_j = t_j t_i. \]

To finish the proof of the proposition it remains to prove that $t_i^r = 1$ and relations (9), (8) are consequences of the relations given by Proposition 1.

It is trivial to see that $t_i^r = 1$ comes from relations (4) and (6). Now, using induction, one will see that (9) is a consequence of the relations given in Proposition 1. Thus if $i = 0$, one must prove that $t_0 t_j = t_j t_0$, for all $j$. Which is trivial for $j = 0, 1$ (see (7)). Let us suppose $j > 1$. One has $t_0 t_j = t(\sigma_j t_{j-1} \sigma_j^{-1}) = \sigma_j t_{j-1} \sigma_j^{-1} = \sigma_j t_{j-1} \sigma_j^{-1} = (\sigma_j t_{j-1} \sigma_j^{-1}) t = t_j t_0$. Therefore $t_0 t_j = t_j t_0$, for all $j$.

Now suppose that the affirmation is true for $i$. We must prove that for all $j$: $t_{i+1} t_j = t_j t_{i+1}$. Using the same argument as above, one can easily verify that it holds for $j \neq i + 2$. In the case $j = i + 2$, one has

\[
\begin{align*}
t_{i+1} t_{i+2} & = (\sigma_i t_1 \sigma_i^{-1}) (\sigma_i t_2 \sigma_i^{-1}) \ldots (\sigma_i t_{i+1} \sigma_i^{-1}) \sigma_{i+2} \\
& = \sigma_i t_1 \sigma_i t_2 \sigma_i t_{i+2} \sigma_{i+2}^{-1} \sigma_{i+1}^{-1} \\
& = \sigma_i t_1 \sigma_i t_2 \sigma_i t_{i+2} \sigma_{i+2}^{-1} \sigma_{i+1}^{-1} \\
& = \sigma_i t_1 \sigma_i t_2 \sigma_i t_{i+2} \sigma_{i+2}^{-1} \sigma_{i+1}^{-1} \\
& = (\sigma_i t_{i+1} t_{i+2}) (\sigma_{i+2} t_{i+1} \sigma_{i+2}^{-1}) t_{i+1} \\
& = t_{i+2} t_{i+1}.
\end{align*}
\]
Finally, in (8) the cases \( j = i \) and \( j > i + 1 \) follow trivially from the definition of \( t_i \). In the case \( j < i \), one has
\[
\sigma_j t^m_i \sigma_j^{-1} = \sigma_i \cdots \sigma_j \sigma_{j+1} \sigma_{j-1} \sigma_j^{-1} \sigma_{j+1}^{-1} \cdots \sigma_i
\]
\[
= \sigma_i \cdots \sigma_j (\sigma_{j+1} \sigma_{j-1} \sigma_j^{-1}) \sigma_{j+1}^{-1} \cdots \sigma_i
\]
\[
= \sigma_i \cdots \sigma_{j+1} \sigma_{j-1} \sigma_j^{-1} \sigma_{j+1}^{-1} \cdots \sigma_i
\]
\[
= t_i.
\]

Corollary 2. For any \( 1 \leq m \leq r \), one has:
\[
\sigma_j t^m_i \sigma_j^{-1} = \begin{cases} 
  t^m_{i+1} & \text{for } j = i + 1 \\
  t^m_{i-1} & \text{for } j = i \\
  t^m_i & \text{for } j > i + 1 \\
  t^m_i & \text{for } j < i.
\end{cases}
\]

Remark 1. Let \( B_{\infty,n} \) be the framed braid group [9], that is, the wreath product \( C \wr B_n \), where \( C \) is the infinite cyclic group. Omitting the relation 4, Proposition 1 holds for \( B_{\infty,n} \).

Remark 2. Let \( S_{r,n} \) and \( S_{\infty,n} \) be the finite versions of the framed braid groups \( B_{r,n} \) and \( B_{\infty,n} \), respectively. More precisely, \( S_{r,n} \) is the group \( C_r \wr S_n \), and \( S_{\infty,n} \) is the group \( C \wr S_n \) (notice that \( S_{1,n} = S_n \) and \( S_{2,n} \) is the Coxeter group of type \( B \)). One has that Proposition 1 holds for the group \( S_{r,n} \), if we omit relation (7), and we add the relations \( \sigma_i^2 = 1 \). Using this presentation for \( S_{r,n} \) one can prove that any element of \( S_{r,n} \) can be written in the form:
\[
(10) \quad w_0 \cdots w_{n-1},
\]
where \( P_0 := \{1, t^m; 1 \leq m \leq r - 1\} \), \( w_i \in P_i \), and
\[
P_i := \{1, t^m_i, s_i w; w \in P_{i-1}, 1 \leq m \leq r - 1\} \quad (1 \leq i \leq n - 1).
\]

Cf. section 2[2].

Notice that from the canonical epimorphisms \( C \twoheadrightarrow C_r \) and \( B_n \twoheadrightarrow S_n \), we get the commutative diagram
\[
\begin{array}{cccc}
B_{\infty,n} & \rightarrow & B_{r,n} & \rightarrow & B_n & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
S_{\infty,n} & \rightarrow & S_{r,n} & \rightarrow & S_n & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & & \\
\end{array}
\]

Finally, according to the diagram above, one can consider the lifting \( \ell \) to \( S_{r,n} \) of the length function \( \ell \) of \( S_n \). For example: \( \ell(t s_i) = \ell(s_i) = 1, s_i = (i, i + 1) \).

Lemma 3. For any \( s_i, s_j \) and \( g \in S_{r,n} \) such that \( \ell(s_i g s_j) = \ell(g) \) and \( \ell(s_i g) = \ell(g s_j) \), one has \( s_i g = g s_j \).

Proof. See Lemma §7.2[5].

\[\square\]

3. HECKE ALGEBRAS

In the following we fix a parameter \( u \in C^\times \).

The Hecke algebra \( H_n = H_n(u) \) at parameter \( u \) is defined as the quotient of the group algebra of \( B_n \) factoring out the quadratic relations \( \sigma_i^2 = u + (u - 1) \sigma_i \). In other words the Hecke algebra \( H_n(u) \) is the associative algebra (over \( C \)) with generators \( 1, h_1, \ldots, h_{n-1} \) and relations: \( h_i h_j = h_j h_i \) for \( |i - j| > 1 \), \( h_i h_j h_i = h_j h_i h_j \) for \( |i - j| = 1 \), and
\[
h_i^2 = u + (u - 1) h_i \quad \text{for all } i.
\]
Analogous to the Hecke algebra we are going to define the algebra $Y_{r,n} = Y_{r,n}(u)$ (of type $A$) as a quotient of the group algebra of $B_{r,n}$ factoring out certain exotic quadratic relations. To do this we introduce the following elements $e_i$ in $C^n$:

$$e_i := \frac{1}{r} \sum_{1 \leq m \leq r} t_i^{m-1} t_i^{-m} \quad (1 \leq i \leq n - 1).$$

**Lemma 4.** For all $i$ one has:

(4.1) $\sigma_i e_i = e_i \sigma_i$

(4.2) $e_i^2 = e_i$.

(4.3) $e_i c_i = e_i c$,  where  $c_i := \sigma_i \sigma_i^{-1}, c \in C^n$.

**Proof.** Let us prove (4.1). One has $\sigma_i t_i^{-1} \sigma_i^{-1} = (\sigma_i t_i^{-1} \sigma_i^{-1})(\sigma_i t_i^{-1} \sigma_i^{-1}) = t_i t_i^{-1}$. Therefore $\sigma_i t_i^{-1} = c_i$, where $c_i$ is the cyclic group generated by $t_i t_i^{-1}$.

Now we can rewrite $e_i$ as $e_i = r^{-1} \sum_{x \in K} x$, then it is clear that $\sigma_i e_i = e_i$, i.e. $\sigma_i e_i = e_i \sigma_i$.

The proof of (4.2) and (4.3) is trivial. \qed

**Lemma 5.** For any $s_i, s_j$ and $g \in S_{r,n}$ such that $\ell(s_i s_j) = \ell(g)$ and $\ell(s_i g) = \ell(g s_j)$, one has (i) $s_i g e_j = e_i g s_j$ and (ii) $g e_j = e_j g$.

**Proof.** (i) The proof is by induction on the length of $g$. Let us put $g = c r$, $\ell(g) = 1$. There are two cases: $i = r$ or $i \neq r$. In the first case the condition $\ell(s_i g) = \ell(g s_j)$ implies $j = r$, then the lemma is true. Suppose now that we are in the case $i \neq r$. Notice that the condition $\ell(s_i g s_j) = \ell(g s_j)$ says that one must have $i = j$. Hence $|i - r| > 1$. Finally one can check that $s_i g e_j = e_i g s_j$. I.e. the lemma is true for any $g$ of length 1.

Now, suppose the lemma is true for the elements of length less than $n$. Let $g = c s_{k_1} \cdots s_{k_m}$ a reduced expression for $g$. One has two possibilities: $\ell(g) > \ell(s_i g)$ or $\ell(g) < \ell(s_i g)$. For the first case one has $s_i g = s_i c s_{k_1} \cdots s_{k_m} = c s_i s_{k_1} \cdots s_{k_m}$, and in virtue of Matsumoto’s lemma (see §1.7 [4]) one can write $g = c s_i g'$, where $\ell(g') < m$. Then

$$s_i g e_j = c s_i s_i g' e_j = c s_i e_i g' s_j \quad \text{(induction hypothesis)}$$

$$= c s_i e_i g' s_j = e_i c s_i g' s_j = e_i g s_j.$$

Suppose now that $\ell(g) < \ell(s_i g)$. Then one has $\ell(s_i g s_j) = \ell(g) < \ell(s_i g) = \ell(g s_j)$. Thus one apply the first case on the elements $g s_j$ and $s_i$: $s_i (g s_j) e_j = e_i (g s_j) s_j$. Hence $s_i g e_j = e_i g s_j$.

The proof of (ii) follows directly from (i) and Lemma 3. \qed

**Definition 1.** The algebra $Y_{r,n} = Y_{r,n}(u)$ at parameter $u$ is defined as the quotient of group algebra of $B_{r,n}$ by factoring through the ideal generated by the “quadratic expressions”:

$$\sigma_i^2 - 1 - (u - 1) \epsilon + (u - 1) \epsilon \sigma_i.$$

We denote the image of $\sigma_i$ under the canonical epimorphism $B_{r,n} \to Y_{r,n}$ by $g_i$, and the image of $t_i$ (respectively $e_i$) again by $t_i$ (respectively $e_i$).

In other terms $Y_{r,n}$ is the algebra generated by $1, g_1, \ldots, g_{n-1}, t$ and defining relations formed by the braid relations (1) for the $g_i$’s, and:

$$t_i^r = 1$$

$$g_{it} = t g_i \quad \text{for} \quad i > 1$$

$$g_i \cdots g_{i+1} t_i^{-1} \cdots g_i^{-1} = g_i^{-1} \cdots g_i^{-1} t_i \cdots g_i$$

$$t g_i g_i^{-1} t = g_i t g_i^{-1} t$$

$$g_i^2 = 1 - (u - 1) \epsilon (1 - g_i).$$

Now the element $t_i \in Y_{r,n}$ is

$$t_i = g_i t_{i-1} g_i^{-1} = g_i^{-1} t_i - 1 g_i.$$
One can now prove without difficulty the following proposition.

**Proposition 6.** The algebra $Y_{r,n}$ has a presentation with generators $1, \ldots, g_{n-1}$, $\tau_1, \ldots, \tau_n$ and relations:

\[
\begin{align*}
\tau_i^r &= 1 \\
\tau_i \tau_j &= \tau_j \tau_i \quad \text{for all } i, j \\
\tau_j \tau_i &= \tau_i \tau_{g_i(j)} \\
g_i \tau_j &= g_{j} \tau_i \\
g_i \tau_{g_i(j)} &= g_{j} \tau_i \\
g_i \tau_{g_i(j)} &= g_{j} \tau_i \\
g_i^2 &= 1 - (u - 1)e_i(1 - g_i),
\end{align*}
\]

where $g_i(j)$ is the result of the action of $(i, i + 1)$ on $j$.

Now, notice that Lemma 4 holds in $Y_{r,n}$, then we get

\[
g_i^{-1} = g_i - (u^{-1} - 1)e_i + (u^{-1} - 1)e_i g_i.
\]

Furthermore, the natural embedding $B_{r,n} \subseteq B_{r,n+1}$ induces the tower of algebras

\[
\mathbb{C} \subseteq Y_{r,1} \subseteq \cdots \subseteq Y_{r,n} \subseteq Y_{r,n+1} \subseteq \cdots
\]

**Lemma 7.** For all $i$, one has:

(1) $g_i g_{i - 1} t_i^m = t_{i - 1}^m g_i g_{i - 1}$

(2) $g_i e_i = e_i g = r^{-1} \sum_{1 \leq m \leq r} t_{i - 1}^m g_i t_i^{-m}$

(3) $t_i$ belong to the center of $Y_{r,n}$.

**Proof.** We prove (7.1)

\[
g_i g_{i - 1} t_i^m = g_i g_{i - 1} t_{i - 1}^m g_{i - 1}^{-1} = g_{i - 1} g_i g_{i - 1} t_{i - 1}^m g_i^{-1} = g_{i - 1} g_i t_{i - 2}^m g_i^{-1} = g_{i - 1} t_{i - 2}^m g_i g_{i - 1}^{-1} = g_{i - 1} t_{i - 2}^m g_i g_{i - 1}^{-1} = t_{i - 1}^m g_i g_{i - 1}.
\]

One has $t_i t_{i - 1}^m g_i t_i^{-m} = t_{i - 1}^m (t_i g_i) t_{i - 1}^{-m} = t_{i - 1}^m g_i t_{i - 1}^m t_i^{-m}$, then (7.2) follows.

The assertion (7.3) follows directly from Corollary 2. \qed

**Proposition 8.** All elements in $Y_{r,n}$ can be written as a linear combination of words in $t, g_1, \ldots, g_{n-1}$, having at most one $\alpha_{n-1} \in \{g_{n-1}, t_{n-1}^m; 1 \leq m \leq r\}$.

**Proof.** The proof of the proposition uses the same argument of induction used in Lemma 3.1[1]. From (11) and Lemma 7 it is easy to check that the assertion is true for $n = 2$.

Suppose the assertion is true for $n$. Now, any element in $Y_{r,n+1}$ is a linear combination of words in $t, g_1, \ldots, g_{n}$. Let $M$ be such a word, we must prove that $M$ is a linear combination of words having at most one $\alpha_n \in \{g_n, t_n^m; 1 \leq m \leq r - 1\}$. For that, it is sufficient to prove that if $M$ has two occurrences of $\alpha_n$, then $M$ is a linear combination of words with at most one $\alpha_n$. Thus, suppose that $M = M_1 \alpha_n M_2 \alpha_n M_3$, where $M_i$ belongs to $Y_{r,n}$. Furthermore, one can suppose that $M_2$ is a word in $t, g_1, \ldots, g_{n-1}$ with at most one $\alpha_{n-1}$.

Now, if $M_2$ does not have $\alpha_{n-1}$, then $M = M_1 M_2 \alpha_n^2 M_3$. From (11) and Lemma 7 one gets

\[
M = M_1 M_2 M_3 + r^{-1}(u - 1) \sum_{1 \leq m \leq r} M_1 M_2 t_{n-1}^m M_3 + r^{-1}(u - 1) \sum_{1 \leq m \leq r} M_1 M_2 t_{n-1}^m g_n t_{n-1}^{-m} M_3.
\]

Thus, the assertion is true in this case.
In the case that $M_2$ has exactly once $\alpha_{n-1}$, one can rewrite $M$ like $M'_1\alpha_n\alpha_{n-1}\alpha_nM_2$, where $M'_1$, $M_2$ are words in $t, g_1\ldots g_{n-1}$. Thus, we need only to prove that the words below are a linear combination of the words having at most one $\alpha_n$ (we shall say that the word can be reduced),

\[
\begin{align*}
(i) & \quad g_ng_{n-1}g_n & (ii) & \quad g_n t_{n-1}^m g_n & (iii) & \quad t_{n-1}^m g_{n-1} g_n & (iv) & \quad t_{n-1}^m t_{n-2}^m g_n \\
(v) & \quad g_n g_{n-1} t_{n-1}^m & (vi) & \quad g_n t_{n-1}^m t_{n-2}^m g_n & (vii) & \quad t_{n-1}^m g_{n-1} t_{n-1}^m & (viii) & \quad t_{n-1}^m t_{n-2}^m t_{n-1}^m,
\end{align*}
\]

where $m, m_1, m_2, m_3 \in \{1, 2, \ldots, r-1\}$.

Trivially the words in (i) and (viii) can be reduced.

The reduction of words (iii) and (v) is direct from Lemma 7.

Using $t_n g_n = g_{n+1}$, we deduce that the word in (iv) can be reduced. In a similar way a reduction for (vi) can be taken.

We are now going to reduce the word in (vii). One has

\[
t_{n-1}^m g_{n-1} t_{n-1}^m = g_{n-1}^{-1} t_{n-1}^m g_{n-1}^{-1} t_{n-1}^m g_{n-1}^{-1} = g_{n-1}^{-1} t_{n-1}^m g_{n-1}^{-1} t_{n-1}^m g_{n-1}^{-1} = g_{n-1}^{-1} g_{n-1} t_{n-2}^m g_{n-1}^{-1} = g_{n-1}^{-1} g_{n-1} t_{n-2}^m g_{n-1}^{-1} = g_{n-1}^{-1} t_{n-2}^m g_{n-1}^{-1} = g_{n-1}^{-1} t_{n-1}^m g_{n-1}^{-1}.
\]

Thus the word in (vii) was reduced.

Finally to reduce the word in (ii), let us write:

\[
g_n t_{n-1}^m g_n = g_n t_{n-1}^m g_n = t_{n}^m g_n = t_{n}^m(1 + (u-1)e_n - (u-1)e_n g_n).
\]

Now using Lemma 7 we deduce that $g_n t_{n-1}^m g_n$ can be reduced. □

The next task is to prove that $Y_{r,n}$ admits a basis formed by words in normal form. By definition the normal words of $Y_{r,n}$ are the words

\[
v_0 v_1 \cdots v_{n-1},
\]

where $v_i \in R_i$, $R_0 := \{1, t^m \ ; \ 1 \leq m \leq r-1\}$, and

\[
R_i := \{1, t^m \ ; \ g_i v \ ; \ v \in R_{i-1}, \ 1 \leq m \leq r-1\} \quad (1 \leq i \leq n-1).
\]

**Proposition 9.** $Y_{r,n}$ is linearly spanned by the normal words.

**Proof.** The assertion is true for $n = 1$ (recall that $Y_{r,1}(u) = CC_r$). Suppose that the proposition for $n$ is true. From Proposition 8 the algebra $Y_{r,n+1}$ is linearly generated by the words in the form

\[
(i) \quad M_1, \quad (ii) \quad M_2 t_{n-1}^m M_3, \quad (iii) \quad M_4 g_n M_5,
\]

where the $M_i$’s are words in $t, g_1, \ldots, g_{n-1}$.

Now from the hypothesis of induction the $M_i$’s are a linear combination of normal words of $Y_{r,n}$. Therefore, the proof is reduced to prove that the words in (ii) and (iii) are a linear combination of normal words, where now the $M_i$’s are normal words in $Y_{r,n}$.

Let us put $M_3 = v_0 \cdots v_{n-1}$, where $v_i \in R_i$. Then in (ii) we have

\[
(iv) \quad M_2 t_{n-1}^m M_3 = M_2 t_{n-1}^m v_0 \cdots v_{n-1} = M_2 v_0 \cdots v_{n-2} t_{n}^m v_{n-1}.
\]

If $v_{n-1} = t_{n-1}^m$, we get $M_2 t_{n-1}^m M_3 = M_2 v_0 \cdots v_{n-2} v_{n-1} t_{n}^m$, which is a normal word in $Y_{r,n+1}$.

If $v_{n-1} = g_{n-1} t_{n-2}^m$, where $v_{n-2} \in R_{n-2}$, one has

\[
M_2 t_{n-1}^m M_3 = M_2 v_0 \cdots v_{n-2} t_{n}^m g_{n-1} t_{n-2}^m = M_2 v_0 \cdots v_{n-2} g_{n-1} t_{n-2}^m v_{n-2} = (M_2 v_0 \cdots v_{n-2} g_{n-1} t_{n-2}^m) t_{n}^m.
\]
Because the words above belong to \( Y_{r,n} \), we deduce from the hypothesis of induction that (iv) is a linear combination of normal words.

Let us put \( M_5 = v_0 \cdots v_{n-1} \), where \( v_i \in R_q \). Then in (iii) we have
\[
M_{4g_n} M_5 = M_{4g_n} v_0 \cdots v_{n-1} = (M_4 v_0 \cdots v_{n-2}) g_n v_{n-1}.
\]
Now from the fact that \( g_n v_{n-1} \in R_n \), and as the word between parenthesis belongs to \( Y_{r,n} \), we get that \( M_{4g_n} M_5 \) is a linear combination of normal words. 

Before concluding this section we shall show that the set of normal words is linearly independent in \( Y_{r,n} \). For that we may use the same method used by J. Tits to prove the analogous claim for the Hecke algebra, see Appendix [5] and §7.1-7.3[4]. The sketch of the proof is as follows. Set \( V = \mathbb{C} S_{r,n} \), we are going to define certain homomorphisms of \( \mathbb{C} \)-algebras \( \rho \) and \( \lambda \) from the Yokonuma-Hecke algebra to the algebra \( \text{End}(V) \). Thus for any \( t \in C^\circ_n \) one can define \( \rho_t \) and \( \lambda_t \) as the homotheties \( \rho_t g = tg \) and \( \lambda_t g = gt \), and for \( g_i \) one put:
\[
\rho_g g := \begin{cases} 
  s_i g & \text{for } \ell(s_i g) > \ell(g) \\
  s_i g - (u - 1)e_i(1 - s_i)g & \text{for } \ell(s_i g) < \ell(g)
\end{cases}
\]
\[
\lambda_g g := \begin{cases} 
  g s_i & \text{for } \ell(g s_i) > \ell(g) \\
  g s_i - (u - 1)(1 - s_i) e_i g s_i & \text{for } \ell(g s_i) < \ell(g)
\end{cases}
\]
(cf. Proposition 2.14[8]).

It is obvious that \( \rho_t \lambda_{t'} = \lambda_{t'} \rho_t \), for any \( t, t' \in C^\circ_n \). Now, using essentially Lemma 3 and Lemma 5 one can prove that \( \rho_{t_i} \lambda_{s_j} = \lambda_{s_j} \rho_{t_i} \), for any \( i, j \). More precisely, one makes the proof by distinguishing according to the length of \( g, s_i g, g s_j \) and \( s_i g s_j \); for instance, suppose that \( \ell(s_i g) > \ell(g) \), \( \ell(g s_i) > \ell(g) \) and \( \ell(s_i g s_j) < \ell(s_i g) \), then one has:
\[
g \overset{\rho}{\rightarrow} (s_i g)s_j - (u - 1)s_i g(1 - s_i)e_i,
\]
\[
g \overset{\lambda}{\rightarrow} g s_j \overset{\rho}{\rightarrow} s_i(g s_j) - (u - 1)e_i(1 - s_i)g s_j.
\]

Now using Lemma 5 one obtains that these expressions are equals. Thus one can prove that the map \( t \mapsto \rho_t, g_i \mapsto \rho_{g_i} \) defines a homomorphism of \( \mathbb{C} \)-algebras \( \rho : Y_{r,n} \rightarrow \text{End}(V) \). In particular one can deduce that \( \rho(g) \) is \( \mathbb{C} \)-linear. Furthermore, one has that,
\[
\rho(v_0 \cdots v_{n-1})(1) = w_{n-1}^{-1} \cdots w_0^{-1}.
\]
Thus any linear combination \( \sum g a_g g = 0 \) (where \( g \) run the normal words (18)) becomes a linear combination \( \sum_w a_g w^{-1} = 0 \), where now \( w \) run the elements of \( S_{r,n} \) (see (10)). Therefore \( a_g = 0 \), for all \( g \), that means that the normal words is a set linearly independent. Hence

**Proposition 10.** The set of the normal words is a \( \mathbb{C} \)-basis for \( Y_{r,n} \). Hence the dimension of \( Y_{r,n} \) is \( r^n n! \).

4. **Markov Trace**

In the following we write \( Y_n = Y_{r,n} \), and let us put
\[
Y_\infty = \bigcup_{n \geq 0} Y_n,
\]
where, by definition, \( Y_0 = \mathbb{C} \).

**Proposition 11.** The map \( \phi : \oplus_{i=1}^r a_i \otimes (a \otimes b) \mapsto \sum_{i=1} a_i t_n^i + a g_n b \)

defines an isomorphism of \((Y_n, Y_n)\)-bimodules \( \phi : (Y_n \oplus \cdots \oplus Y_n) \otimes Y_n \otimes Y_{n-1} \rightarrow Y_{n+1} \).
Proof. $\phi$ is surjective because any word in $Y_{n+1}$ is a linear combination of words involving (only) the words:

$$v g_n w \quad \text{and} \quad w_i t_i^m \quad (1 \leq m \leq r),$$

where $v$, $w$ and $w_i \in Y_n$.

On the other hand, using Proposition 10 one deduces that $Y_n \otimes Y_{n-1} Y_n$ is spanned linearly by the set

$$\{ t_i^m \otimes v, g_j \ldots g_{n-1} t_i^m \otimes v; \ 1 \leq m \leq r, 1 \leq j \leq n-1, v \text{ is a normal word in } Y_n \}.$$

Therefore the dimension of $(Y_n \otimes \cdots \otimes Y_n) \otimes Y_n \otimes Y_{n-1} Y_n$ is less or equal than

$$r r^n n! + r r^n n! + (n-1)r (r^n n!) = r^{n+1} (n+1)!.$$ 

Then $\phi$ is an isomorphism. $\square$

\textbf{Theorem 12.} Let $z$, $\zeta_1$, $\ldots$, $\zeta_{r-1}$ be in $\mathbb{C}$, and let us put $\zeta_r = 1$. There exists a unique linear map $tr$ on $Y_\infty$ with values in $\mathbb{C}$, such that $tr(1) = 1$, and

$$tr(\alpha^i_n) = \zeta_i tr(a) \quad (1 \leq i \leq r, a \in Y_n)$$

$$tr(\alpha b) = ztr(ab) \quad (a, b \in Y_n)$$

$$tr(ab) = tr(ba) \quad (a, b \in Y).$$

\textbf{Proof.} One defines $tr$ recursively as follows. First, one defines $tr$ on $Y_0 = \mathbb{C}$ as the identity and on $Y_1 = \mathbb{C}C_r$ by $tr(t^n) = \zeta_n$, $1 \leq m \leq r$. After, using the proposition above one defines $tr$ on $Y_{n+1}$ by

$$tr(x) = \sum_i \zeta_i tr(\alpha_i) + ztr(ab),$$

where $\phi^{-1}(x) = \sum_i \alpha_i + (a \otimes b)$.

From the construction of $tr$, it is clear that it only remains to prove that $tr$ satisfies the trace condition, that is

$$(19) \quad tr(xy) = tr(yx), \quad \text{for all} \ x, y \in Y_{n+1}.$$ 

According to Proposition 8 one can suppose that $x$ and $y$ are words having at most one $\gamma_n \in \{g_n, t_i^m; 0 \leq m < r\}$.

In the case where $y$ does not contain $\gamma_n$, it is easy to check the trace condition. In fact, set $x = a \gamma b \ (a, b \in Y_n)$ and let us distinguish between $\gamma_n = g_n$ or $\gamma_n = t_i^m$. If $\gamma_n = g_n$, one has

$$tr(xy) = tr(\alpha g_0 b) = ztr(ab) = ztr(ab) = tr(yab) = yag_n b = tr(yx),$$

and if $\gamma_n = t_i^m$,

$$tr(xy) = tr(\alpha g_n b) = tr(\alpha t_i^m b) = \zeta_i tr(ab) = tr(ya t_i^m b) = tr(yx).$$

We will now prove that the trace condition holds in the case where $y$ does contain $\gamma_n$. Notice now that one can suppose that $y = \gamma_n$. One can also suppose that $a$ and $b$ are words having at most one $\gamma_n$. Then let us write $a = a' \gamma_{n-1} a''$ and $b = b' \gamma_{n-1} b''$, where $a', a'', b'$ and $b''$ belong to $Y_{n-1}$.

To prove the trace condition, one shall make it according to the distinct $\gamma_n = g_n, t_i^m$ and $\gamma_n = g_{n-1}, t_i^m$. Thus it is obtained sixteen cases to be analyzed. In fact one can check that for all these cases the trace condition (19) holds. The checking of these cases is only a large computation. For that, we will analyze only three of the more representatives cases.

\textbf{Case} $x = ag_n b$, $y = \gamma_n$, $a = a' g_{n-1} a''$ and $b = b' g_{n-1} b''$:
One has

\[
\text{tr}(xy) = \text{tr}(ag_nb'g_{n-1}g_nb'') \\
= \text{tr}(ab'_n g_{n-1}g_nb'') \\
= \text{tr}(ab'g_{n-1}g_n g_{n-1}b'') \\
= z\text{tr}(ab'_n g_{n-2}b'') \\
= z\text{tr}(ab'b'') + z(u-1)\text{tr}(ab'e_{n-1}b'') - z(u-1)\text{tr}(ab'e_{n-1}g_{n-1}b'').
\]

On the other hand,

\[
\text{tr}(yx) = \text{tr}(g_n a' g_{n-1}a''g_n b) \\
= \text{tr}(a' g_{n-1}g_n a''b) \\
= \text{tr}(a' g_{n-1}g_n g_{n-1}a''b) \\
= z\text{tr}(a'^2 g_{n-1}a''b) \\
= z\text{tr}(a'a''b) + z(u-1)\text{tr}(a'e_{n-1}a''b) - z(u-1)\text{tr}(a'e_{n-1}g_{n-1}a''b).
\]

It is clear that

\[
z\text{tr}(ab'b'') = z\text{tr}(a' g_n a''b'') = z^2\text{tr}(a'a''b'') = z\text{tr}(a'a''b).
\]

From the induction hypothesis easily one gets:

\[
\text{tr}(ab'e_{n-1}b'') = \text{tr}(a'e_{n-1}a''b) \quad \text{and} \quad \text{tr}(ab'e_{n-1}g_{n-1}b'') = \text{tr}(a'e_{n-1}g_{n-1}a''b).
\]

Therefore in this case one gets the trace condition.

Case \( x = ag_n b, y = g_n, a = a' g_{n-1}a'' \) and \( b = b' t_{n-1}^m \): One has

\[
\text{tr}(xy) = \text{tr}(ag_n b't_{n-1}^m) \\
= \text{tr}(ab' g_{n-1} t_{n-1}^m) \\
= \text{tr}(ab' g_{n} t_{n}^m) \\
= \text{tr}(ab' t_{n}^m) + (u-1)\text{tr}(ab'e_{n} t_{n}^m) - (u-1)\text{tr}(ab'e_{n}g_{n} t_{n}^m).
\]

On the other hand

\[
\text{tr}(yx) = \text{tr}(g_n a' g_{n-1}a''g_n b) \\
= \text{tr}(a' g_{n-1}g_n a''b) \\
= \text{tr}(a' g_{n-1}g_n g_{n-1}a''b) \\
= z\text{tr}(a'^2 g_{n-1}a''b) \\
= z\text{tr}(a'a''b) + z(u-1)\text{tr}(a'e_{n-1}a''b) - z(u-1)\text{tr}(a'e_{n-1}g_{n-1}a''b).
\]

Now, \( z\text{tr}(a'a''b) = z\text{tr}(a'a''b't_{n-1}^m) = z\zeta_m \text{tr}(a'a''b') = \zeta_m \text{tr}(a'g_{n-1}a''b') = \zeta_m \text{tr}(ab'). \) Therefore

\[(20) \quad \text{tr}(ab't_{n}^m) = z\text{tr}(a'a''b).
\]
One has $ztr(a'e_{n-1}a''b) = ztr(a'e_{n-1}a''b t^m_{n-1}) = ztr(a'a''b e_{n-1}t^m_{n-1})$. Now, if we denote by $\mu$ the corresponding $m + r - i$ reduced mod $r$ to the form $1 \leq \mu \leq r$, then we get

$$ztr(a'e_{n-1}a''b) = r^{-1} \sum_i \zeta_\mu tr(a' t^i_{n-2}a''b)$$

$$= r^{-1} \sum_i \zeta_\mu tr(a' t^i_{n-2}a''b')$$

$$= r^{-1} \sum_i \zeta_\mu tr(a' g_{n-1}a'' t^i_{n-1}b')$$

$$= r^{-1} \sum_i \zeta_\mu tr(ab' t^i_{n-1})$$

$$= tr(ab' (r^{-1} \sum_i t^i_{n-1}r^{-i}) t^m_{n-1}).$$

Therefore

$$\tag{21} (u - 1)tr(ab'e_{n} t^m_{n}) = z(u - 1)tr(a'e_{n-1}a''b).$$

One has

$$ztr(a'e_{n-1}g_{n-1}a''b) = ztr(a'e_{n-1}g_{n-1}a''b t^m_{n-1})$$

$$= ztr(a'e_{n-1}g_{n-1}t^m_{n-1}a''b')$$

$$= ztr(a' (r^{-1} \sum_i t^m_{n-2}g_{n-1}t^{-i}_{n-2})a''b')$$

$$= z^2 tr(a' t^m_{n-2}a''b')$$

$$= ztr(a' t^m_{n-2}g_{n-1}a''b')$$

$$= ztr(a' g_{n-1}t^m_{n-1}a''b')$$

$$= ztr(a' g_{n-1}a''b' t^m_{n-1}) = ztr(ab).$$

One has

$$tr(ab'e_{n} g_{n} t^m_{n}) = tr(ab' r^{-1} \sum_i t^i_{n-1}g_{n}t^{-i}_{n-1} t^m_{n})$$

$$= tr(ab' r^{-1} \sum_i t^i_{n-1}g_{n}t^{-i}_{n-1})$$

$$= tr(ab' r^{-1} \sum_i t^i_{n-1}t^{-i}_{n-1})$$

$$= ztr(ab' t^m_{n-1}) = ztr(ab).$$

Hence

$$ztr(a'e_{n-1}g_{n-1}a''b) = tr(ab'e_{n} g_{n} t^m_{n}).$$

From this equality, (20) and (21) one gets the trace condition.

Case $x = ag_{n}b$, $y = g_{n}$, $a = a't^\mu_{n-1}$ and $b = b't^\nu_{n-1}$:

One has $tr(xy) = tr(ag_{n} b' t^\mu_{n-1} g_{n}) = tr(ab' g_{n} t^\nu_{n}) = tr(ab' g_{n} t^\nu_{n})$. Then,

$$tr(xy) = tr(a't^\mu_{n-1}b'g^2_{n} t^\nu_{n})$$

$$= tr(ab' g^2_{n} t^\nu_{n})$$

Therefore $tr(xy) = tr(a' g^2_{n} t^\mu_{n-1} b' t^\nu_{n-1}) = tr(g_{n} a't^\mu_{n-1} g_{n} b' t^\nu_{n-1}) = tr(yx)$.
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