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**ON WEAKLY HYPERBOLIC SPACES AND A CONVERGENCE-EXTENSION
THEOREM IN WEAKLY HYPERBOLIC SPACES**

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Abstract

In this article we prove some properties of the weakly hyperbolic spaces. Moreover, a convergence-extension theorem for analytic hypersurfaces (not necessarily with normal crossings) in weakly hyperbolic spaces is given.

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§ 1. INTRODUCTION

In [1] O. Alehyane and H. Amal presented the concept "*the weakly hyperbolic space*". The first aim of this paper is to prove some properties of the weakly hyperbolic spaces. Exactly, we study the weak hyperbolicity in holomorphic fiber bundles and the normalization of a complex space by the following results:

Theorem 1. *Let (Y, π, X) be a holomorphic fiber bundle with fiber F . If X is weakly hyperbolic and F is hyperbolic then Y is weakly hyperbolic.*

Theorem 2. *Let (Y, π, X) be a holomorphic fiber bundle and $f : X' \rightarrow X$ a holomorphic mapping. If (Y, π, X) is weakly hyperbolic, then its pull-back bundle $(f^{-1}Y, \pi', X')$ is also weakly hyperbolic.*

Theorem 3. *Let (Y, π) be a normalization of a complex space X . If X is weakly hyperbolic then Y is also weakly hyperbolic.*

At the end of Section 2 we show the Eastwood's Lemma for the weak hyperbolicity of complex spaces and the openness of the weak hyperbolicity.

Theorem 4. *Let $f : Y \rightarrow X$ be a holomorphic mapping of complex spaces. Assume that X is weakly hyperbolic and that to every point $a \in X$ there is a neighbourhood U of a such that $\pi^{-1}(U)$ is hyperbolic. Then Y is weakly hyperbolic.*

Proposition 5. *Let $f : Y \rightarrow X$ be a proper holomorphic mapping of complex spaces.*

i) If X is weakly hyperbolic and each fiber $f^{-1}(x)$ is weakly hyperbolic for all $x \in X$, then Y is weakly hyperbolic.

ii) If there is some $x_0 \in X$ such that $f^{-1}(x_0)$ is weakly hyperbolic, then there exists an open neighbourhood U of x_0 in X such that $f^{-1}(x)$ is weakly hyperbolic for all $x \in U$. (In fact, $f^{-1}(x)$ is hyperbolic for all $x \in U$.)

Similarly as the hyperbolicity of a complex space, the weak hyperbolicity also has applications in the problem of extending holomorphic mappings [1]. In the next section, we prove a convergence-extension theorem in weakly hyperbolic complex spaces. Since this is a new concept, we hope that one can find more applications.

The convergence-extension theorem of Noguchi (see [4] or [7]) is stated as follows

Theorem. *Let X be relatively compact and hyperbolically imbedded into a complex space Y . Let M be a complex manifold and A an analytic hypersurface of M with only normal crossings. Suppose that $\{f_j : M \setminus A \rightarrow X\}_{j=1}^{\infty}$ is a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \setminus A$ to a holomorphic mapping $f : M \setminus A \rightarrow X$.*

Then there are unique holomorphic extensions $\overline{f}_j : M \rightarrow Y$ of f_j and $\overline{f} : M \rightarrow Y$ of f

over M . Moreover, $\{\bar{f}_j\}_{j=1}^\infty$ converges uniformly on compact subsets of M to $\bar{f}$.

Recently, several Noguchi-type convergence-extension theorems for analytic hypersurfaces of complex manifolds have been obtained by various authors (see [5], [7], [13]). In particular, in [13] D.D. Thai and P. N. Mai have proved the above theorem in case that X is a complex subspace of a hyperbolic complex space Y such that X has Δ^* -EP for Y and A is an arbitrary analytic hypersurface of a complex manifold M .

The next aim of the article is to replace the hyperbolicity of the complex space Y in the above result by the weak hyperbolicity of X for Y . More precisely, we prove the following

Theorem 5. *Let X be a complex subspace of a complex space Y . Suppose that X is weakly hyperbolic for Y and has $\Delta^* - EP$ for Y . Let A be any analytic hypersurface of a complex manifold M . Let $\{f_j : M \setminus A \rightarrow X\}_{j=1}^\infty$ be a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \setminus A$ to a holomorphic mapping $f : M \setminus A \rightarrow X$.*

Then there are unique holomorphic extensions $\bar{f}_j : M \rightarrow Y$ and $\bar{f} : M \rightarrow Y$ of f_j and f respectively over M . Moreover, $\{\bar{f}_j\}_{j=1}^\infty$ converges uniformly on compact subsets of M to $\bar{f}$.

Finally, we generalise, partly, a theorem of Noguchi by the following

Corollary 2. *Let X be relatively compact and hyperbolically imbedded into a complex space Y . Let M be a complex manifold and A an analytic hypersurface of M . Then every holomorphic mapping $f : M \setminus A \rightarrow X$ extends to a holomorphic mapping $\bar{f} : M \rightarrow Y$.*

§ 2. SOME PROPERTIES OF THE WEAKLY HYPERBOLIC SPACES

For every positive number r , put $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$, $\Delta_1 = \Delta$.

Let X be a complex space and the Kobayashi pseudodistance on X is denoted by d_X .

Definition 1. A complex space X is said to be hyperbolic (in the sense of Kobayashi) if d_X is a distance on X , i.e., $d_X(x, y) > 0$ if $x \neq y$.

A complex subspace X is said to be hyperbolically imbedded in a complex space Y if for $x, y \in \bar{X}$ and $x \neq y$, there exist open neighbourhoods U of x and V of y in Y such that

$$d_X(U \cap X, V \cap X) > 0.$$

If $X = \Delta$, the Kobayashi distance coincides with Poincaré metric on Δ ,

$$d_\Delta(z, w) = \rho_\Delta(z, w).$$

If X is hyperbolic then d_X induces the natural topology on X .

For details concerning hyperbolic complex spaces we refer the readers to the books [3], [4], [5] and [7].

Definition 2. Let X be a complex space. X is called to be weakly hyperbolic if for every compact subset $K \subset X$, there exists an open neighbourhood U of K in X which satisfies the following two properties:

- i) U is hyperbolic;
- ii) for all $f \in \text{Hol}(\Delta, X) \cap C(\overline{\Delta}, X)$, if $f(\partial\Delta) \subset K$ then $f(\overline{\Delta}) \subset U$.

Here, we denote $\text{Hol}(\Delta, X)$ the space of holomorphic mappings from Δ into X which is equipped with the compact-open topology.

Examples. (see [1])

1. Every hyperbolic space is weakly hyperbolic.
2. If X is a complex space with a strictly plurisubharmonic function, then X is weakly hyperbolic.
3. If X is a compact complex space, then X is weakly hyperbolic if and only if X is hyperbolic.
4. If $\tilde{X}$ is a holomorphic covering of an analytic space X , then if X is weakly hyperbolic so is $\tilde{X}$.
5. Let $\{z_k\} \subset \Delta$, $\{z_k\} \rightarrow 0$, and $\{\alpha_k\}, \{\delta_k\} \subset (0, 1)$ such that

- (i) $\sum_{k \geq 1} \alpha_k \log \frac{1}{|z_k|} < \infty$, and
- (ii) $\lim_{k \rightarrow \infty} \alpha_k \log \frac{1}{\delta_k} = +\infty$.

In $\mathbb{C}^2$ consider the domain $X := \{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < e^{-u(z)}\}$, where $u(z) := \sum_k \alpha_k \log(\delta_k^2 + |z - z_k|^2)$. We have that X is weakly hyperbolic but not hyperbolic (see [1] and [14]).

6. $\mathbb{C}^n$ is weakly hyperbolic.

Indeed, take K a compact subset of $\mathbb{C}^n$. There is $R > 0$ such that $K \subset \Delta_R^n \subset \mathbb{C}^n$. We have that Δ_R^n is hyperbolic. Suppose that $f \in \text{Hol}(\Delta, \mathbb{C}^n) \cap C(\overline{\Delta}, \mathbb{C}^n)$ such that $f(\partial\Delta) \subset K \subset \Delta_R^n$. We can write $f = (f_1, \dots, f_n)$, where $f_i \in \text{Hol}(\Delta, \mathbb{C}) \cap C(\overline{\Delta}, \mathbb{C})$. Then we deduce $|f_i(\partial\Delta)| \leq R, \forall i = 1, \dots, n$. By the maximum principle it follows that $|f_i(\overline{\Delta})| \leq R, \forall i = 1, \dots, n$. Therefore $f(\overline{\Delta}) \subset \Delta_R^n$. Q.E.D

Now we prove some properties of weakly hyperbolic spaces which are useful for the next results.

Proposition 1. *Let X be a closed or open complex subspace of a complex space Y . If Y is weakly hyperbolic then X is also weakly hyperbolic.*

Proof. Let K be a compact subset of X . Then K is compact in Y . Since Y is weakly

hyperbolic, there exists an open hyperbolic neighbourhood U of K in Y such that U satisfies the condition ii) in Definition 2. Thus we get $U \cap X$ is an open hyperbolic neighbourhood of K in X . We have to prove that $U \cap X$ satisfies ii) in Definition 2. Let $f \in \text{Hol}(\Delta, X) \cap \mathcal{C}(\overline{\Delta}, X)$ and $f(\partial\Delta) \subset K$. Consider $f \in \text{Hol}(\Delta, Y) \cap \mathcal{C}(\overline{\Delta}, Y)$, we have $f(\overline{\Delta}) \subset U$, so $f(\overline{\Delta}) \subset U \cap X$. Hence X is weakly hyperbolic. Q.E.D

Remark. Every n - dimensional Stein manifold is weakly hyperbolic.

Indeed, by the Embedding Theorem in [2] X can be biholomorphically mapped onto a closed complex submanifold of $\mathbb{C}^{2n+1}$. So Proposition 1 and Example 6 imply that X is weakly hyperbolic. Q.E.D

Proposition 2. *If X_1, X_2 are weakly hyperbolic then $X_1 \times X_2$ is also weakly hyperbolic.*

Proof. Given K a compact subset of $X_1 \times X_2$. Then $K_i := \text{pr}_i(K)$ are compact subsets of X_i ($i=1,2$) respectively, where $\text{pr}_i : X_1 \times X_2 \rightarrow X_i$ are projections. Hence, there exist open hyperbolic neighbourhoods U_i of K_i in X_i respectively such that the condition ii) in Definition 2 is satisfied (*). Clearly $U_1 \times U_2$ is an open hyperbolic neighbourhood of $K_1 \times K_2$ in $X_1 \times X_2$. Let $f \in \text{Hol}(\Delta, X_1 \times X_2) \cap \mathcal{C}(\overline{\Delta}, X_1 \times X_2)$ and $f(\partial\Delta) \subset K$. Then $f_i(\partial\Delta) \subset K_i$, where $f_i(\partial\Delta) := \text{pr}_i(f(\partial\Delta))$, $i = 1, 2$. By (*) we have $f_i(\overline{\Delta}) \subset U_i$, $i = 1, 2$. Thus $f(\overline{\Delta}) \subset f_1(\overline{\Delta}) \times f_2(\overline{\Delta}) \subset U_1 \times U_2$. This implies that $X_1 \times X_2$ is weakly hyperbolic. Q.E.D

Now we study the weak hyperbolicity of holomorphic fiber bundles.

Theorem 1. *Let (Y, π, X) be a holomorphic fiber bundle with fiber F . If X is weakly hyperbolic and F is hyperbolic then Y is weakly hyperbolic.*

Proof. Let K be a compact subset of the complex space Y . Since $\pi : Y \rightarrow X$ is holomorphic, then $\pi(K) \subset X$ is compact. By the weak hyperbolicity of X , there exists an open neighbourhood U of $\pi(K)$ in X which satisfies conditions i) and ii) in Definition 2 (*).

Since $\pi(K) \subset U$, then $K \subset \pi^{-1}(U)$ and we have $\pi^{-1}(U)$ is hyperbolic: indeed, for each point $x \in U$ there exists an open neighbourhood $V(x) \subset U$ such that $\pi^{-1}(V) = V \times F$. Since U is hyperbolic we have that V is hyperbolic. By hypothesis F is hyperbolic, it implies that $\pi^{-1}(V)$ is hyperbolic. By Eastwood's Lemma [5] we get that $\pi^{-1}(U)$ is hyperbolic.

In order to prove condition ii) of Definition 2, let $f \in \text{Hol}(\Delta, Y) \cap \mathcal{C}(\overline{\Delta}, Y)$ and suppose $f(\partial\Delta) \subset K$. We have to prove that

$$f(\overline{\Delta}) \subset \pi^{-1}(U).$$

Since $f(\partial\Delta) \subset K$, then $\pi \circ f(\partial\Delta) \subset \pi(K)$. By (*) we deduce

$$\pi \circ f(\overline{\Delta}) \subset U.$$

Hence

$$f(\overline{\Delta}) \subset \pi^{-1}(\pi \circ f(\overline{\Delta}) \subset \pi^{-1}(U)).$$

Therefore Y is weakly hyperbolic. Q.E.D

Remark. In the case where (Y, π, X) is trivial we have that X and F are weakly hyperbolic if and only if Y is weakly hyperbolic.

This follows from Propositions 1 and 2. Q.E.D

Definition 3. A holomorphic fiber bundle (Y, π, X) is said to be weakly hyperbolic at $x \in X$ if x has a neighbourhood U in X such that $Y_U = \pi^{-1}(U)$ is weakly hyperbolic, and it is weakly hyperbolic if it is weakly hyperbolic at every point $x \in X$.

Theorem 2. Let (Y, π, X) be a holomorphic fiber bundle and $f : X' \rightarrow X$ a holomorphic mapping. If (Y, π, X) is weakly hyperbolic, then its pull-back bundle $(f^{-1}Y, \pi', X')$ is also weakly hyperbolic.

Proof. Set $Y' = f^{-1}Y$. Given a point $x' \in X'$, by hypothesis, let U be a neighbourhood of $f(x')$ in X such that Y_U is weakly hyperbolic. Let U' be a hyperbolic neighbourhood of x' in X' such that $U' \subset f^{-1}(U)$. We shall prove that $Y'_{U'} = \pi'^{-1}(U')$ is weakly hyperbolic.

Let K be a compact subset in $Y'_{U'}$, then $K \subset K_1 \times K_2$ where K_1, K_2 are compact subsets in U' and Y_U respectively. Since Y_U is weakly hyperbolic, there exists a hyperbolic neighbourhood V of K_2 in Y_U such that the condition ii) of Definition 2 is satisfied (*). Therefore $(U' \times V) \cap Y'_{U'}$ is a hyperbolic neighbourhood of K in $Y'_{U'}$.

Suppose that $g \in \text{Hol}(\Delta, Y'_{U'}) \cap C(\overline{\Delta}, Y'_{U'})$ and $g(\partial\Delta) \subset K$. Then we can write $g(\partial\Delta) = (g_1(\partial\Delta), g_2(\partial\Delta)) \subset K_1 \times K_2$. Since U' is hyperbolic, and hence weakly hyperbolic, and by (*) we have that $(g_1(\overline{\Delta}), g_2(\overline{\Delta})) \subset (U' \times V) \cap Y'_{U'}$. Hence $g(\overline{\Delta}) \subset (U' \times V) \cap Y'_{U'}$.

It implies that (Y', π', X') is weakly hyperbolic. Q.E.D

Proposition 3. Let (V, π_V, X) and (W, π_W, X) be holomorphic fiber bundles. If they are weakly hyperbolic then the Whitney sum $(V \oplus W, \pi, X)$ is also weakly hyperbolic.

Proof. For every $x \in X$, by hypothesis, there is a neighbourhood U of x in X such that $V_U = \pi_V^{-1}(U)$ and $W_U = \pi_W^{-1}(U)$ are weakly hyperbolic. Since $(V \oplus W)_U$ is a complex subspace of $V_U \times W_U$, then by Propositions 1 and 2 we deduce that $(V \oplus W)_U$ is weakly hyperbolic. It follows that $(V \oplus W, \pi, X)$ is weakly hyperbolic. Q.E.D

Definition 4.(see [3]) A complex space X is said to be normal at a point $a \in X$ if the ring $\mathcal{O}_{X,a}$ of germs of holomorphic functions at a is integrally closed in its ring of quotients. If X is normal at every point of X , then X is said to be normal.

A normalization of a complex space X is a pair (Y, π) consisting of a normal complex space Y and a surjective holomorphic mapping $\pi : Y \rightarrow X$ such that

- i) $\pi : Y \rightarrow X$ is proper and $\pi^{-1}(a)$ is finite for every $a \in X$;

ii) If S is the set of singular points of X , then $Y \setminus \pi^{-1}(S)$ is dense in Y and $\pi : Y \setminus \pi^{-1}(S) \rightarrow X \setminus S$ is biholomorphic.

Every complex space X has a unique (up to an isomorphism) normalization (Y, π) (see [8], [6, p.118]).

The following Theorem shows that the weak hyperbolicity is preserved by passing to the normalization of a complex space.

Theorem 3. *Let (Y, π) be a normalization of a complex space X . If X is weakly hyperbolic then Y is also weakly hyperbolic.*

This Theorem is deduced immediately from the following Proposition.

Proposition 4. *Let $\pi : Y \rightarrow X$ be a proper holomorphic mapping of complex spaces such that $\pi^{-1}(a)$ is finite for every $a \in X$. If X is weakly hyperbolic, so is Y .*

Proof. Let K be a compact subset of the complex space Y . Since $\pi : Y \rightarrow X$ is holomorphic, we have that $\pi(K) \subset X$ is compact. By hypothesis X is weakly hyperbolic, there exists an open neighbourhood U of $\pi(K)$ in X which satisfies conditions i) and ii) of Definition 2 (*).

Since $\pi(K) \subset U$, then $K \subset \pi^{-1}(U)$. By Theorem 4.1 [3, p.101] and since U is hyperbolic, we get that $\pi^{-1}(U)$ is hyperbolic.

Let $f \in \text{Hol}(\Delta, Y) \cap \mathcal{C}(\overline{\Delta}, Y)$ and suppose $f(\partial\Delta) \subset K$. Then $\pi \circ f(\partial\Delta) \subset \pi(K)$. By (*) we deduce that $\pi \circ f(\overline{\Delta}) \subset U$. It follows that $\pi^{-1}(\pi \circ f(\overline{\Delta})) \subset \pi^{-1}(U)$. Hence

$$f(\overline{\Delta}) \subset \pi^{-1}(\pi \circ f(\overline{\Delta})) \subset \pi^{-1}(U).$$

Therefore Y is weakly hyperbolic. Q.E.D

Theorem 4. (Eastwood's Lemma for the weak hyperbolicity)(see [5])

Let $\pi : Y \rightarrow X$ be a holomorphic mapping of complex spaces. Assume that X is weakly hyperbolic and that to every point $a \in X$ there is a neighbourhood U of a such that $\pi^{-1}(U)$ is hyperbolic. Then Y is weakly hyperbolic.

Proof. Let K be a compact subset in Y . Then $\pi(K)$ is compact in X . By the weak hyperbolicity of X , there exists an open hyperbolic neighbourhood U of $\pi(K)$ in X which satisfies condition ii) of Definition 2.

We have that $\pi^{-1}(U)$ is an open hyperbolic neighbourhood of K in Y : Indeed, by hypothesis and applying Eastwood's Lemma to the hyperbolic neighbourhood U we deduce that $\pi^{-1}(U)$ is hyperbolic.

In the same way as in the Proof of Proposition 4 we have that condition ii) of Definition 2 is satisfied. Hence Y is weakly hyperbolic. Q.E.D

The following Proposition shows that the weak hyperbolicity is open under proper holomorphic mappings.

Proposition 5. *Let $f : Y \rightarrow X$ be a proper holomorphic mapping of complex spaces.*

i) If X is weakly hyperbolic and each fiber $f^{-1}(x)$ is weakly hyperbolic for all $x \in X$, then Y is weakly hyperbolic.

ii) If there is some $x_0 \in X$ such that $f^{-1}(x_0)$ is weakly hyperbolic, then there exists an open neighbourhood U of x_0 in X such that $f^{-1}(x)$ is weakly hyperbolic for all $x \in U$. (In fact, $f^{-1}(x)$ is hyperbolic for all $x \in U$.)

Proof. i) Let K be a compact subset of Y . Then $f(K)$ is compact in X . By hypothesis, there exists an open neighbourhood U of $f(K)$ in X such that U satisfies conditions i) and ii) of Definition 2.

First we prove that $f^{-1}(U)$ is hyperbolic. By hypothesis and Example 4 §2, $f^{-1}(x)$ is hyperbolic for all $x \in U$. Since U is hyperbolic, applying Proposition 3.1 [5, p.72] we deduce that $f^{-1}(U)$ is hyperbolic. The condition ii) is implied by the same arguments as in Proof of Proposition 4. Thus Y is weakly hyperbolic.

ii) Since f is proper, $f^{-1}(x_0)$ is compact. Combining with the hypothesis $f^{-1}(x_0)$ is weakly hyperbolic we get that $f^{-1}(x_0)$ is hyperbolic. By Theorem 2.2 ([5], p.68), there is an open hyperbolic neighbourhood V of $f^{-1}(x_0)$. It implies that there is some open neighbourhood U of x_0 in X such that $f^{-1}(U) \subset V$: Indeed, if it is not, then there exists a sequence $\{y_n\} \subset Y \setminus V$ (*) such that $f(y_n) = x_n \rightarrow x_0$. Let K be a compact neighbourhood of x_0 in X . Then by the properness of f , $f^{-1}(K)$ is compact in Y . Since $x_n \rightarrow x_0$ we have $x_n \in K, \forall n \geq n_0$. Hence there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that $y_{n_k} \rightarrow y_0$. Since f is continuous, $f(y_0) = \lim_{k \rightarrow \infty} f(y_{n_k}) = \lim_{k \rightarrow \infty} x_{n_k} = x_0$. Therefore $y_0 \in f^{-1}(x_0) \subset V$. We have $y_{n_k} \rightarrow y_0 \in V$, then there exists $k_0 \in \mathbb{N}$ such that $y_{n_k} \in V$ for all $k \geq k_0$, which is in contradiction to (*). Therefore, $f^{-1}(x) \subset f^{-1}(U) \subset V$ is hyperbolic for all $x \in U$, thus concluding the proof of the theorem.

Q.E.D

§ 3. A CONVERGENCE-EXTENSION THEOREM IN WEAKLY HYPERBOLIC SPACES

Definition 5. Let X be a complex subspace of a complex space Y and $\Delta^* := \Delta \setminus \{0\}$. We say that X has the Δ^* -EP for Y if every holomorphic mapping f from Δ^* into X extends to a holomorphic mapping $\overline{f}$ from Δ into Y . If X has the Δ^* -EP for itself then X is said to have the Δ^* -EP.

Examples. 1) By a theorem of Kobayashi [4, Thm 6.3.7, p.284], if X is relatively compact and

hyperbolically imbedded into Y then X has the Δ^* -EP for Y . In particular, if X is compact hyperbolic then X has Δ^* -EP.

2) It is easy to see from the Riemann extension theorem that if D is a bounded domain in $\mathbb{C}^n$ and Ω is an open neighbourhood of $\overline{D}$ in $\mathbb{C}^n$ then D has the Δ^* -EP for Ω .

For details concerning the Δ^* -EP we refer the readers to [10], [11] and [14].

Definition 6. An open complex subspace X of a complex space Y is said to be weakly disc-convex for Y if every sequence $\{f_n\} \subset \text{Hol}(\Delta, Y)$ converges in $\text{Hol}(\Delta, Y)$ whenever the sequence $\{f_n|_{\Delta^*}\} \subset \text{Hol}(\Delta^*, X)$ converges in $\text{Hol}(\Delta^*, X)$. If X is weakly disc-convex for itself then X is said to be weakly disc-convex.

Remarks. a) From the theorems of Montel and Kiernan (see [9]) we get the following implications:

Complete hyperbolic $\implies$ taut $\implies$ weakly disc-convex.

The converse assertions are not true in general (see [9]).

For details concerning the (weak) disc-convexity we refer the readers to [12].

b) If X is relatively compact and hyperbolically imbedded in Y , then X is weakly disc-convex for Y .

Indeed, this is deduced from Noguchi's theorem on Δ (see [5, Thm. 4.1, p.56]) and Definition 6. Q.E.D

Definition 7. Let X be an open complex subspace of a complex space Y . X is said to be weakly hyperbolic for Y if for every compact subset $K \subset \overline{X}$, there exists an open neighbourhood U of K in Y which satisfies the following

- i) U is hyperbolic;
- ii) for all $f \in \text{Hol}(\Delta, Y) \cap \mathcal{C}(\overline{\Delta}, Y)$, if $f(\partial\Delta) \subset K$ then $f(\overline{\Delta}) \subset U$.

Remark. When $X = Y$ we get the concept of the weak hyperbolicity of the complex space X .

The following is a relation between the weak disc-convexity and the Δ^* -EP in a weakly hyperbolic complex subspace X for a complex space Y .

Proposition 6. Let X be a weakly hyperbolic complex subspace for a complex space Y . If X has the Δ^* -EP for Y then X is weakly disc-convex for Y .

Proof. Let $\{f_n\}_{n \geq 1} \subset \text{Hol}(\Delta, Y)$ converge in $\text{Hol}(\Delta^*, X)$ to $f \in \text{Hol}(\Delta^*, X)$. Since X has the Δ^* -EP for Y , there exists $g \in \text{Hol}(\Delta, Y)$ such that $g = f$ on Δ^* .

Put $x_0 = g(0)$. By the continuity of g we get that $x_0 \in \overline{X}$. Put $K := \overline{V \cap X}$, where V is a relatively compact neighbourhood of x_0 in Y . By the hypothesis that X is weakly hyperbolic for Y , there is an open hyperbolic neighbourhood $U \subset Y$ of K which satisfies ii) in Definition 7.

Take $s \in (0, 1)$ such that $g(\overline{\Delta}_s) \subset V$. So we have $g(\partial\Delta_s) \subset g(\overline{\Delta}_s) \subset V$. Therefore

$$f_n(\partial\Delta_s) \rightarrow f(\partial\Delta_s) = g(\partial\Delta_s) \subset V.$$

On the other hand, $f_n, f \in \text{Hol}(\Delta^*, X)$ then $f_n(\partial\Delta_s) \rightarrow f(\partial\Delta_s) \subset V \cap X \subset K$. It implies that there exists $n_0 \geq 1$ such that $f_n(\partial\Delta_s) \subset K$ for all $n \geq n_0$. Indeed, since $f(\partial\Delta_s)$ is compact and $V \cap X$ is open in Y we have $\text{dist}(f(\partial\Delta_s), \partial(V \cap X)) = \delta > 0$, where dist is a distance function with respect to some metric in Y . By hypothesis that $f_n(\partial\Delta_s)$ converges uniformly to $f(\partial\Delta_s)$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$ we have $\text{dist}(f_n(x), f(x)) < \delta \quad \forall x \in \partial\Delta_s$. Therefore $f_n(\partial\Delta_s) \subset V \cap X$ for all $n \geq n_0$. Since X is weakly hyperbolic for Y , we have

$$f_n(\overline{\Delta}_s) \subset U \text{ for all } n \geq n_0.$$

By abuse of notation let us denote by f_n also the mapping $f_n|_{\Delta_s}$, then we get $\{f_n\} \subset \text{Hol}(\Delta_s, U)$. Since U is hyperbolic, by Ascoli's theorem, to prove the relatively compactness of $\{f_n\}_{n \geq 1}$ in $\text{Hol}(\Delta_s, U)$ it suffices to prove that the subset $X_t := \{f_n(t), n \geq 1\}$ is relatively compact in U for all $t \in \Delta_s$.

If $t \in \Delta_s^*$, the uniformly convergence on the compact subsets of Δ_s^* implies $X_t \Subset U$.

For $t = 0$, suppose that X_0 is not relatively compact in U . Take $\{K_i\}_{i \in \mathbb{N}^*}$ an exhaustive sequence of compact subsets of U , i.e., $K_i \subset \text{int}(K_{i+1})$ for all i and $U = \bigcup_i K_i$. Then there exist two increasing sequences $n_i, m_i \in \mathbb{N}^*$ such that $f_{n_i}(0) \in K_{m_{i+1}} \setminus K_{m_i}$ for all $i \geq 1$.

Since g is continuous, by the same argument as above, there is $r \in (0, 1)$ such that $g(\Delta_r) \subset K$. By the compactness of $K \subset U$, there is $p \in \mathbb{N}^*$ such that $K \subset K_{m_p}$. Then for all $q \geq p + 2$ we have

$$f_{n_q}(0) \in U \setminus K_{m_{p+1}} \text{ and } \mathbb{B}(f_{n_q}(0), \varepsilon) \subset U \setminus K_{m_{p+1}},$$

where $\varepsilon := \frac{1}{2}d_X(\partial K_{m_{p+1}}, \partial K_{m_{p+2}})$. Since U is hyperbolic, the family $\{f_n; n \geq n_0\}$ is equicontinuous. Hence there exists $s' \in (0, s)$ such that

$$d_X(f_n(z), f_n(0)) < \varepsilon \quad \forall z \in \Delta_{s'}, \forall n \geq 1.$$

Put $\delta := \min\left(\frac{s'}{2}, \frac{r}{2}\right)$, then we have

$$g(\partial\Delta_\delta) = f(\partial\Delta_\delta) \subset K \text{ and } f_{n_q}(\partial\Delta_\delta) \subset U \setminus K_{m_{p+1}} \quad \forall q \geq p + 2.$$

This is impossible because the subsequence $\{f_{n_q}\}_q \rightarrow f$ in $\text{Hol}(\Delta^*, X)$ and $K \subset K_{m_p} \subset \text{int}(K_{m_{p+1}}) \subset K_{m_{p+1}}$, so $\text{dist}(K_{m_p}, U \setminus K_{m_{p+1}}) > 0$. Hence $X_0 \Subset U$. Thus $\{f_n; n \geq 1\}$ is relatively compact in $\text{Hol}(\Delta_s, U)$. It follows that $\{f_n\}$ converges uniformly on compact subsets of Δ_s to the mapping g in $\text{Hol}(\Delta_s, Y)$. Therefore X is weakly-disc convex for Y . Q.E.D

Theorem 5. *Let X be an open complex subspace of a complex space Y . Suppose that X is weakly hyperbolic for Y and has $\Delta^* - EP$ for Y . Let A be any analytic hypersurface of a*

complex manifold M . Let $\{f_j : M \setminus A \rightarrow X\}_{j=1}^\infty$ be a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \setminus A$ to a holomorphic mapping $f : M \setminus A \rightarrow X$.

Then there are unique holomorphic extensions $\bar{f}_j : M \rightarrow Y$ and $\bar{f} : M \rightarrow Y$ of f_j and f respectively over M . Moreover, $\{\bar{f}_j\}_{j=1}^\infty$ converges uniformly on compact subsets of M to $\bar{f}$.

Proof. Take any $f \in \text{Hol}(M \setminus A, X)$. We may assume that A is nonsingular; i.e., we extend f to $M \setminus S(A)$ then to $M \setminus S(S(A))$ and so on. Here $S(Z)$ denotes the singular locus of Z .

By localizing the mapping f , we may assume that $M = \Delta^m = \Delta^{m-1} \times \Delta$ and $A = \Delta^{m-1} \times \{0\}$.

For each $z' \in \Delta^{m-1}$, consider the holomorphic mapping $f_{z'} : \Delta^* \rightarrow X$ given by $f_{z'}(z) = f(z', z)$ for all $z \in \Delta^*$. Since X has the Δ^* -EP for Y , there exists a holomorphic extension $\bar{f}_{z'} : \Delta \rightarrow Y$ of $f_{z'}$ for each $z' \in \Delta^{m-1}$. Define the mapping $\bar{f} : \Delta^{m-1} \times \Delta \rightarrow Y$ by $\bar{f}(z', z) = \bar{f}_{z'}(z)$ for all $(z', z) \in \Delta^{m-1} \times \Delta$. It suffices to prove that $\bar{f}$ is continuous at $(z'_0, 0) \in \Delta^{m-1} \times \Delta$.

Indeed, assume that $\{(z'_k, z_k)\} \in \Delta^{m-1} \times \Delta$ such that $\{(z'_k, z_k)\} \rightarrow (z'_0, 0)$. Put $\sigma_k = \bar{f}_{z'_k}$ for each $k \geq 1$ and $\sigma_0 = \bar{f}_{z'_0}$. Then the sequence $\{\sigma_k|_{\Delta^*}\}$ converges uniformly on compact subsets to the mapping $\{\sigma_0|_{\Delta^*}\}$ in $\text{Hol}(\Delta^*, X)$. Since X is weakly hyperbolic for Y and has the Δ^* -EP for Y , by Proposition 6, X is weakly disc-convex for Y . It implies that the sequence $\{\sigma_k\}$ converges uniformly to the mapping σ_0 in $\text{Hol}(\Delta, Y)$. Therefore, $\{\sigma_k(z_k) = \bar{f}(z'_k, z_k)\} \rightarrow \sigma_0(0) = \bar{f}(z'_0, 0)$. Hence, $\bar{f}$ is continuous at $(z'_0, 0)$. It implies that every holomorphic mapping $f : M \setminus A \rightarrow X$ extends holomorphically over M .

Let $\{f_k\} \in \text{Hol}(M \setminus A, X)$ be such that $\{f_k\} \rightarrow f_0$ in $\text{Hol}(M \setminus A, X)$. We will show that $\{\bar{f}_k\} \rightarrow \bar{f}_0$ in $\text{Hol}(M, Y)$.

Let z_0 be an arbitrary point of A . By localizing our assertion, we may assume that $M = \Delta^m = \Delta^{m-1} \times \Delta$ and $A = \Delta^{m-1} \times \{0\}$, $z_0 = (0, 0)$. Put $a_0 = \bar{f}_0(z_0)$. We have that $\bar{f}_0(\overline{\Delta^m(1/2)}) = K$ is compact in $\overline{X}$. Then by the hypothesis that X is weakly hyperbolic for Y , there exists a hyperbolic neighbourhood U of K in Y and $a_0 \in U$.

We first show that for an arbitrary number $0 < \varepsilon < \frac{1}{2}$, there exists a neighbourhood V_0 of z_0 such that $\bar{f}_0(V_0) \subset \mathbb{B}_U(a_0; \varepsilon) := \{y \in U : d_U(y, a_0) < \varepsilon\}$ and $\bar{f}_k(V_0) \subset \mathbb{B}_U(a_0; \varepsilon)$ for all $k \geq k_0$. Take a point $z_1 \in \mathbb{B}_{\Delta^m(\frac{1}{2})}(z_0; \frac{\varepsilon}{3}) \setminus A$, where $\mathbb{B}_{\Delta^m(\frac{1}{2})}(z_0; \frac{\varepsilon}{3}) := \{z \in \Delta^m(\frac{1}{2}) : d_{\Delta^m(\frac{1}{2})}(z, z_0) < \frac{\varepsilon}{3}\}$. Then $f_0(z_1) \in \mathbb{B}_U(a_0; \frac{\varepsilon}{3})$. There is an integer $k_0 = k_0(z_1)$ such that $f_k(z_1) \in \mathbb{B}_U(a_0; \frac{2\varepsilon}{3})$ for all $k \geq k_0$. Hence we have

$$\bar{f}_k(\mathbb{B}_{\Delta^m(\frac{1}{2})}(z_1; \frac{\varepsilon}{3})) \subset \mathbb{B}_U(a_0; \varepsilon).$$

Put $V_0 = \mathbb{B}_{\Delta^m(\frac{1}{2})}(z_0; \frac{\varepsilon}{3}) \cap \mathbb{B}_{\Delta^m(\frac{1}{2})}(z_1; \frac{\varepsilon}{3})$. Then $z_0 \in V_0$ and $\bar{f}_0(V_0) \subset \mathbb{B}_U(a_0; \varepsilon)$, $\bar{f}_k(V_0) \subset \mathbb{B}_U(a_0; \varepsilon)$ for all $k \geq k_0$.

Take $\varepsilon > 0$ so small that $\mathbb{B}_U(a_0; \varepsilon)$ is contained in a holomorphic local coordinate neigh-

bourhood of a_0 in Y . Choose $\delta > 0$ small enough such that $\overline{\Delta}_\delta^m \subset V_0$. Since $\{\overline{f}_k|_{(\partial\Delta_\delta)^m}\}_{k=1}^\infty$ converges uniformly on compact subsets to $\overline{f}_0|_{(\partial\Delta_\delta)^m}$, the maximum principle implies the uniform convergence of $\{\overline{f}_k|_{\Delta_\delta^m}\}_{k=1}^\infty$ with limit $\overline{f}_0|_{\Delta_\delta^m}$. Q.E.D

Theorem 5 contains the following result of D. D. Thai and P. N. Mai [13, Thm. 2], which was proved by a different method.

Corollary 1. *Let X be a complex subspace of a hyperbolic complex space Y such that X has Δ^* -EP for Y . Let A be an analytic hypersurface of a complex manifold M . Let $\{f_j : M \setminus A \rightarrow X\}_{j=1}^\infty$ be a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \setminus A$ to a holomorphic mapping $f : M \setminus A \rightarrow X$.*

Then there are unique holomorphic extensions $\overline{f}_j : M \rightarrow Y$ of f_j and $\overline{f} : M \rightarrow Y$ of f over M , and $\{\overline{f}_j\}_{j=1}^\infty$ converges uniformly on compact subsets of M to $\overline{f}$.

By the remark b) of Definition 6 and the proof of Theorem 5 we have

Corollary 2. *Let X be a relatively compact and hyperbolically imbedded into a complex space Y . Let M be a complex manifold and A an analytic hypersurface of M . Then every holomorphic mapping $f : M \setminus A \rightarrow X$ extends to a holomorphic mapping $\overline{f} : M \rightarrow Y$.*

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