Quantum Markov Process on a Lattice

T. Hashimoto, M. Horibe and A. Hayashi

Department of Applied Physics, Fukui University, Bunkyo 3-9-1, Fukui 910-8507, Japan

We develop a systematic description of Weyl and Fano operators on a lattice phase space. Introducing the so-called ghost variable even on an odd lattice, odd and even lattices can be treated in a symmetric way. The Wigner function is defined using these operators on the quantum phase space, which can be interpreted as a spin phase space. If we extend the space with a dichotomic variable, a positive distribution function can be defined so-called "ghost variable" [2]. The Weyl function on the space can be constructed in a similar manner.

1. Introduction

The quantum Markov process for an integer spin system was constructed on the lattice phase space by Cohendet et al. [1]. The time evolution of the pseudo-distribution function, i.e., the Wigner function on the space can be derived from the process. However, the extension of their method to a half integer spin system is not straightforward. There are difficulties in the construction of the Wigner function on an even lattice. A way to avoid this difficulties is to introduce the so-called "ghost variable" [2]. The Weyl and Fano operators on a lattice can be constructed on an even lattice if we consider fictitious lattice points between the original ones (see also [3]). In the present paper, we develop the systematic treatment of the Wigner function on both odd and even lattices and show that a quantum Markov process for a half integer spin system can also be constructed in a similar manner.

2. Weyl and Fano operators with ghost variables on a lattice

We consider a lattice composed of $N$ lattice points. A lattice with odd or even lattice points is called odd or even lattice, respectively. We denote the set of integers and half integers by $\mathbb{Z}$. For odd $N$, we define $J^o$, $J^p$ as the sets composed of integers and half integers between $-(N/2)$ and $(N/2)+1$, respectively, and $\bar{J}^o$ be the non-negative part of $\bar{J}^o$. For even $N$, we define $J^e, J^e_-$ similarly between $-(N/2)$ and $(N-1)/2$. The symbols $J, J, J_+$ are the abbreviations of $J^o, J^p, J^e$ for odd $N$ and $J^e, J^e_-$ for even $N$. We consider a Kronecker’s delta function $\bar{\delta}(n,m)$ on $\mathbb{Z}$, defined by $\bar{\delta}(n,m) = \bar{\delta}_{2m,2n}$ in mod $2N$, where $m, n \in \mathbb{Z}$. We denote one of primitive roots of unity of order $N$ by $\omega$, e.g., $\omega = \exp\left(\frac{2\pi i}{N}\right)$.

The $(n,m)$ components of basic phase, shift and skew operators $Q, P, T$ are defined as

$$Q \equiv \omega^n \bar{\delta}(n,m), P \equiv \bar{\delta}(n+1,m), T \equiv \bar{\delta}(n,m+1),$$

(1)

for odd $N$, and

$$Q \equiv \omega^n \bar{\delta}(n,m), P \equiv (1-2\bar{\delta}(n,m+1))\bar{\delta}(n+1,m),$$

$$T \equiv \bar{\delta}(n,m+1),$$

(2)

for even $N$, where $m, n \in J$. The Froquet or Bloch angle is $\pi$ in the latter case [4].

3. Wigner function on a lattice

3.1. Weyl operators

We define the Weyl operators by

$$W_{m,n} \equiv \omega^{-2mn} Q^{2m} P^{-2m} = \omega^{2mn} P^{-2m} Q^{2m},$$

(3)

for both odd and even $N$, where $m, n \in \bar{J}$. We can see the Weyl operators satisfy the relations for reflection and shift:

$$W_{m,n} = W_{-m,-n} = TW_{m,n}T,$$

(4)
\[ W_{m,n}W_{m',n'} = \omega^{-2(m'n' - mn')} W_{m+m',n+n'}. \] (5)

The \( N^2 \) operators \( W_{m,n} \) with \( (m,n) \) in the basic region \( J_+ \times J_+ \) are complete and orthogonal in the trace norm. Generally, \( W_{m,n} \) with \( (m,n) \in \mathbb{Z} \times \mathbb{Z} \) and \( W_{\text{mod}(m, \frac{N}{2}), \text{mod}(n, \frac{N}{2})} \) in the basic region are related as

\[ W_{m,n} = \theta(m, n) W_{\text{mod}(m, \frac{N}{2}), \text{mod}(n, \frac{N}{2})}, \tag{6} \]

where \( \theta(m, n) = (-1)^{\tau(m,n)} \), and

\[ \tau(m, n) = 2n \left[ \frac{2m}{N} \right] + 2m \left[ \frac{2n}{N} \right] + \left[ \frac{2n}{N} \right], \quad (7) \]

for odd \( N \),

\[ \tau(m, n) = 2n \left[ \frac{2m}{N} \right] + 2m \left[ \frac{2n}{N} \right] + \left[ \frac{2m}{N} \right] + \left[ \frac{2n}{N} \right], \quad (8) \]

for even \( N \).

The Weyl operators have the next symplectic property, i.e., they are rotated by \( \pi/2 \) anticlockwise under the Fourier transformation,

\[ \mathcal{F} W_{m,n} \mathcal{F}^\dagger = W_{-n,m}, \quad \mathcal{F} = \frac{1}{\sqrt{N}} \omega^{mn}. \tag{9} \]

### 3.2. Fano operators

We define the Fano operators by

\[ \Delta_{m,n} = W_{m,n}T = \omega^{-2mn} Q^{2n} T P^{2m}, \tag{10} \]

for both odd and even \( N \), where \( m, n \in J \). They are hermite but over complete. Those in the basic region are orthogonal and complete. The Fano operators have the same symplectic property as the Weyl operators,

\[ \mathcal{F} \Delta_{m,n} \mathcal{F}^\dagger = \Delta_{-n,m}. \tag{11} \]

We can see that the Fano operators have proper marginal properties:

\[ \sum_{n \in J} \Delta_{m,n} = 0, \text{ for half integer } m \text{ (ghost) } \tag{12} \]

\[ \sum_{n \in J} \Delta_{m,n} = 2N \delta_{i,m} \delta_{j,n}, \text{ for integer } m \tag{13} \]

\[ \sum_{m \in J} \Delta_{m,n} = 0, \text{ for half integer } n \text{ (ghost) } \tag{14} \]

\[ \sum_{m \in J} \Delta_{m,n} = 2(\omega^{m(i-j)}) \text{, for integer } n \tag{15} \]

for odd \( N \), and

\[ \sum_{n \in J} \Delta_{m,n} = 0, \text{ for integer } m \text{ (ghost) } \tag{16} \]

\[ \sum_{n \in J} \Delta_{m,n} = 2N \delta_{i,m} \delta_{j,n}, \text{ for half integer } m \tag{17} \]

\[ \sum_{m \in J} \Delta_{m,n} = 0, \text{ for integer } n \text{ (ghost) } \tag{18} \]

\[ \sum_{m \in J} \Delta_{m,n} = 2(\omega^{n(i-j)}) \text{, for half integer } n \tag{19} \]

for even \( N \).

### 3.3. Time evolution of the Wigner function

We define the Wigner function by

\[ \mathcal{W}(m,n) = \frac{1}{N} \text{tr}(\rho \Delta_{m,n}), \tag{20} \]

which is real valued and bounded by \( 1/N \),

\[ -\frac{1}{N} \leq \mathcal{W}(m,n) \leq \frac{1}{N}. \tag{21} \]

The average value of an observable \( \mathcal{O} \) is given by

\[ \langle \mathcal{O} \rangle = N \sum_{m,n \in J_+} \mathcal{O}(m,n) \mathcal{W}(m,n), \tag{22} \]

where \( \mathcal{O}(m,n) = \frac{1}{N} \text{tr}(O \Delta_{m,n}) \). We rewrite the time evolution equation of the density matrix \( \rho \),

\[ i \frac{\partial \rho}{\partial t} = [H, \rho], \tag{23} \]

in terms of the Wigner function. We expand the Hamiltonian \( H \) by using the Weyl operators in the basic region, and the density matrix by the Fano operators in the same region. Equating the coefficient of \( \Delta_{m,n} \), the time evolution equation of the Wigner function is given by

\[ \frac{d}{dt} W_{mn} = -2 \sum_{m''n'' \in J_+} \hat{H}^+(m'', n'') \sin \left\{ \frac{2\pi i}{N} (2(m'n' - n''m') - \alpha(m'', n'')) \right\} \]
\[ \theta(m'' + m', n'' + n') W_{m'n'}, \]  
where \( m'' = \text{mod}(m - m', \frac{2\pi}{N}) \), \( n'' = \text{mod}(n - n', \frac{2\pi}{N}) \) and \( \tilde{\mathcal{H}}^+(m,n) \) and \( \alpha(m,n) \) are determined by the polar decomposition \( \mathcal{H}(m, n) = \tilde{\mathcal{H}}^+(m, n) \omega^\alpha(m, n) \).

4. Construction of a Markov process

We extend the basic region with the dichotomic variable \( \sigma \in \{\pm 1\} \equiv B \), i.e., \( \tilde{J}_+ \times \tilde{J}_+ \rightarrow \tilde{J}_+ \times \tilde{J}_+ \times B \), and consider a new real valued function \( G(m, n, \sigma) \) on the extended space, 
\[
G(m, n, \sigma) = \frac{1}{4N} \left\{ \frac{2}{N} + \sigma W(m, n) \right\}. 
\]
It can be seen that \( G(m, n, \sigma) \) is positive and satisfies the inequality,
\[
\frac{1}{4N^2} \leq G(m, n, \sigma) \leq \frac{3}{4N^2}, 
\]
and normalization condition,
\[
\sum_{(m, n) \in \tilde{J}_+ \times \tilde{J}_+} G(m, n, \sigma) = 1. 
\]
The positivity of \( G(m, n, \sigma) \) follows from the boundedness of the Wigner function Eq.(21). The average value of an observable \( O \) is given by
\[
\langle O \rangle = 2N^2 \sum_{m, n \in \tilde{J}_+, \sigma \in B} \sigma O(m, n) G(m, n, \sigma). 
\]
It is natural to call \( G(m, n, \sigma) \) as a distribution function on the extended space \( \tilde{J}_+ \times \tilde{J}_+ \times B \). The time evolution of \( G(m, n, \sigma) \) is given by
\[
\frac{d}{dt} G(m, n, \sigma) = \sum_{m', n', \sigma' \in \{\pm 1\}} -\text{sgn}(\sigma \sigma') 
\]
\[
\tilde{\mathcal{H}}^+(m'', n'') \sin \left\{ \frac{2\pi i}{N} \left( 2(m''n' - n''m') - \alpha(m'', n'') \right) \right\} 
\]
\[
\times \theta(m'' + m', n'' + n') G(m', n', \sigma'). 
\]
Introducing a generating operator as 
\[
\mathcal{A}_t(m, n, \sigma; m', n', \sigma') = \tilde{\mathcal{H}}^+(m'', n'') \left[ \frac{1}{G(m', n', \sigma')} \right] 
\]
\[-\text{sgn}(\sigma \sigma') \times \sin \left\{ \frac{2\pi i}{N} \left( 2(m''n' - n''m') - \alpha(m'', n'') \right) \right\} \]
\[
\theta(m'' + m', n'' + n') \]  
for \( (m, n) \neq (m', n') \), 
\[
\mathcal{A}_t(m, n, \sigma; m, n, \sigma') = 0, 
\]
for \( (m, n) = (m', n') \), \( \sigma \neq \sigma' \), 
\[
\mathcal{A}_t(m, n, \sigma; m, n, \sigma) = -\frac{2}{G(m, n, \sigma)} 
\]
\[
\sum_{(m', n') \in \tilde{J}_+ \times \tilde{J}_+ \setminus \{0, 0\}} \tilde{\mathcal{H}}^+(m', n') 
\]
for \( (m, n, \sigma) = (m', n', \sigma') \), the equation for \( G(m, n, \sigma) \) can be written briefly as 
\[
\frac{d}{dt} G(m, n, \sigma) = \sum_{m', n' \in \tilde{J}_+, \sigma \in \{\pm 1\}} \mathcal{A}_t(m, n, \sigma; m', n', \sigma') G(m', n', \sigma'). 
\]
It can be checked that the generator \( \mathcal{A}_t(m, n, \sigma; m', n', \sigma') \) satisfies the Markov condition,
\[
\sum_{(m, n, \sigma) \in \tilde{J}_+ \times \tilde{J}_+ \times B} \mathcal{A}(m, n, \sigma; m', n', \sigma') = 0, 
\]
\[
\mathcal{A}(m, n, \sigma; m', n', \sigma') \geq 0, \quad \text{if} \ (m, n, \sigma) \neq (m', n', \sigma'), 
\]
which assures the existence of a background quantum Markov process on the extended lattice quantum phase space.

REFERENCES