Abstract

The field equations associated with the Born-Infeld-Einstein action are derived using the Palatini variational technique. In this approach the metric and connection are varied independently and the Ricci tensor is generally not symmetric. For sufficiently small curvatures the resulting field equations can be divided into two sets. One set, involving the antisymmetric part of the Ricci tensor $R^\mu_\nu$, consists of the field equation for a massive vector field. The other set consists of the Einstein field equations with an energy momentum tensor for the vector field plus additional corrections. In a vacuum with $R^\mu_\nu = 0$ the field equations are shown to be the usual Einstein vacuum equations. This extends the universality of the vacuum Einstein equations, discussed by Ferraris et al. [6, 7], to the Born-Infeld-Einstein action. In the simplest version of the theory there is a single coupling constant and by requiring that the Einstein field equations hold to a good approximation in neutron stars it is shown that mass of the vector field exceeds the lower bound on the mass of the photon. Thus, in this case the vector field cannot represent the electromagnetic field and would describe a new geometrical field. In a more general version in which the symmetric and antisymmetric parts of the Ricci tensor have different coupling constants it is possible to satisfy all of the observational constraints if the antisymmetric coupling is much larger than the symmetric coupling. In this case the antisymmetric part of the Ricci tensor can describe the electromagnetic field.
**Introduction**

In the 1930’s Born and Infeld [3] attempted to eliminate the divergent self energy of the electron by modifying Maxwell’s theory. Born-Infeld electrodynamics follows from the Lagrangian

\[
L = -\frac{1}{4\pi b} \left\{ \sqrt{-\det(g_{\mu\nu} + bF_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})} \right\},
\]  

(1)

where \(g_{\mu\nu}\) is the metric tensor and \(F_{\mu\nu}\) is the electromagnetic field tensor. In the weak field limit this Lagrangian reduces to the Maxwell Lagrangian plus small corrections. For strong fields the field equations deviate significantly from Maxwell’s theory and the self energy of the electron can be shown to be finite. The Born-Infeld action also appears in string theory. The action for a D-brane is of the Born-Infeld form with two fields, a gauge field on the brane and the projection of the Neveu-Schwarz B-field onto the brane [4].

In the same spirit one can attempt to modify the Lagrangian of general relativity from \(L = -(1/2\kappa)R\), where \(R\) is the Ricci scalar, to

\[
L = -\frac{1}{\kappa b} \left\{ \sqrt{-\det(g_{\mu\nu} + bR_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})} \right\} + L_M
\]  

(2)

where \(R_{\mu\nu}\) is the Ricci tensor, \(L_M\) is the matter Lagrangian and \(\kappa = 8\pi G\). Deser and Gibbons [5] considered this type of Lagrangian and derived the equations of motion by using a purely metric variation, i.e. they took the connection to be the Christoffel symbol. This leads to fourth order equations with ghosts. To eliminate the ghosts they considered modifying the action so that the quadratic terms in the Taylor series expansion of the Lagrangian vanished. Feigenbaum, Freund and Pigli [6] and Feigenbaum [7] have also examined Born-Infeld like gravitational actions in two and four dimensions using a purely metric variation. They found that the form of the action produces a curvature limiting effect and that the spacetime inside a black hole is nonsingular.

Here I take a different approach and derive the field equations using a Palatini variation. In the Palatini approach the connection and metric are treated as independent variables. If the action is taken to be \(L = -(1/2\kappa)R + L_M\) the variation with respect to the connection fixes the connection to be the Christoffel symbol. The variation with respect to the metric then gives the Einstein field equations. However, the purely metric variation and the Palatini variation give different field equations for Lagrangians that are general functions of the Riemann tensor and its contractions. For example if \(L = f(R) + L_M\) it can be shown [8] that the field equations derived from the Palatini approach are second order in contrast to the fourth order equation that follow from a purely metric variation. If the curvature is is much less than \(1/b\) the field equations that follow from (2) describe a massive vector field, with \(R_{\mu\nu}\) being proportional to the electromagnetic field tensor, and the Einstein equations plus small corrections. The energy momentum tensor for the massive vector field automatically appears in the Einstein equations.
In a vacuum with $R_{\mu\nu} = 0$ the equations reduce to the usual vacuum Einstein field equations. This extends the universality of the vacuum Einstein field equations discovered by Ferraris et al. [6, 7] to the Born-Infeld-Einstein action. This implies that there is no curvature limiting effect and that black holes will contain a spacetime singularity. There are however, corrections to Einstein’s equations in interior regions. If these corrections are taken to be small in neutron stars the mass of the vector field exceeds the lower limit on the mass of the photon. Thus, if this theory is realized in nature it describes a new massive field.

On a manifold with a general connection there are two nonzero contractions of the Riemann tensor. One can consider including both contractions in the Born-Infeld-Einstein Lagrangian and this theory will then have two coupling constants. One coupling is associated with the symmetric part of the Riemann tensor and the other is associated with the antisymmetric part. If the antisymmetric coupling is much larger than the symmetric coupling all the constraints can be satisfied and the antisymmetric part of the Ricci tensor can describe the electromagnetic field.

## The Field Equations I

The field equations for the theory follow from the Born-Infeld-Einstein action

$$ L = -\frac{1}{\kappa b} \left\{ \sqrt{-\det(g_{\mu\nu} + bR_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})} \right\} + L_M $$

(3)

where $R_{\mu\nu}$ is the Ricci tensor, $\kappa = 8\pi G$ and $L_M$ is the matter Lagrangian. The Ricci tensor is given by

$$ R_{\mu\nu} = \partial_\nu \Gamma^\alpha_{\mu\alpha} - \partial_\alpha \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\beta\nu} \Gamma^\beta_{\mu\alpha} + \Gamma^\alpha_{\beta\mu} \Gamma^\beta_{\alpha\nu} $$

(4)

and the connection is taken to be symmetric. Note that $R_{\mu\nu}$ is not symmetric in general. If the curvature is much smaller than $b^{-1}$ an expansion of (3) gives the Einstein Lagrangian to lowest order in $b$. Thus, for sufficiently weak fields Einstein’s equations will hold to a good approximation.

Varying the action with respect to $g_{\mu\nu}$ gives

$$ \sqrt{P} \left( P^{-1} \right)^{\mu \nu} - \sqrt{g} g^{\mu \nu} = \sqrt{g} \kappa b T^{\mu \nu} $$

(5)

where $P_{\mu\nu} = g_{\mu\nu} + bR_{\mu\nu}$, $P^{-1}$ is the inverse of $P$, $(P^{-1})^{\mu \nu}$ is the symmetric part of $P^{-1}$, $P = -\det(P_{\mu\nu})$ and $g = -\det(g_{\mu\nu})$. Varying with respect to $\Gamma^\alpha_{\mu\nu}$ gives

$$ \nabla_\alpha \left[ \sqrt{P} \left( P^{-1} \right)^{\mu \nu} \right] - \frac{1}{2} \nabla_\beta \left\{ \sqrt{P} \left[ \delta^\mu_\alpha \left( P^{-1} \right)^{\beta \nu} + \delta^\nu_\alpha \left( P^{-1} \right)^{\beta \mu} \right] \right\} = 0 $$

(6)

and contracting over $\alpha$ and $\nu$ gives

$$ \nabla_\alpha \left[ \sqrt{P} \left( P^{-1} \right)^{\mu \nu} \right] = -\frac{3}{5} \nabla_\alpha \left[ \sqrt{P} \left( P^{-1} \right)^{\mu \nu} \right] $$

(7)
where \((P^{-1})^{\mu\nu}\) is the antisymmetric part of \(P^{-1}\).

It will be instructive to consider the vacuum equations when \(R_{\mu\nu}\) is symmetric. Since the antisymmetric part of \(R_{\mu\nu}\) is to be proportional to the vector field this will describe a spacetime free of all matter, including the vector field. In this case the left hand side of (7) vanishes implying that

\[
\nabla_\alpha \left[ \sqrt{P} (P^{-1})^{\mu\nu} \right] = 0. \tag{8}
\]

Substituting equation (5) into equation (8) gives

\[
\nabla_\alpha (\sqrt{g} g^{\mu\nu}) = 0 \tag{9}
\]

and this tells us that the connection is the Christoffel symbol. Taking the determinant of (5) gives \(p = g\) and we see that \(R_{\mu\nu} = 0\). \(\tag{10}\)

Thus, in a vacuum the resulting field equations are the Einstein field equations, independent of the value of \(b\). This extends the universality of the Einstein vacuum equations discovered by Ferraris et al. [6, 7] to Born-Infeld like actions. In their papers Ferraris et al. showed that Lagrangians of the form \(L = F(R)\) and \(L = f(R^{\mu\nu} R_{\mu\nu})\) always give the Einstein vacuum equations under a Palatini variation.

Now go back to the general case where \(R_{\mu\nu}\) is not symmetric. From equation (7) we see that

\[
\nabla_\alpha \left[ \sqrt{P} (P^{-1})^{\alpha\mu} \right] = \frac{2}{5} \nabla_\alpha \left[ \sqrt{P} (P^{-1})^{\alpha\mu} \right]. \tag{11}
\]

Substituting equations (5) and (11) into (6) gives

\[
\nabla_\alpha \left[ \sqrt{g} (g^{\mu\nu} + \kappa b T^{\mu\nu}) \right] - \frac{1}{5} \nabla_\beta \left\{ \sqrt{g} \left[ (g^{\mu\beta} + \kappa b T^{\mu\beta}) \delta^\nu_\alpha + (g^{\nu\beta} + \kappa b T^{\nu\beta}) \delta^\mu_\alpha \right] \right\} = 0. \tag{12}
\]

Since the trace of the above system of equations vanishes there are four too few equations and the system is underdetermined. Thus, we expect four arbitrary functions in the solution. Such a solution is given by

\[
\nabla_\alpha \left[ \sqrt{g} (g^{\mu\nu} + \kappa b T^{\mu\nu}) \right] = \sqrt{g} [\delta^\mu_\alpha V^\nu + \delta^\nu_\alpha V^\mu] \tag{13}
\]

where \(V^\mu\) is an arbitrary vector.

Now consider vacuum solutions with \(T^{\mu\nu} = 0\). Contracting equation (13) with \(g_{\mu\nu}\) and using

\[
\sqrt{g} g_{\mu\nu} \nabla_\alpha g^{\mu\nu} = \sqrt{g} \left[ g_{\mu\nu} \partial_\alpha g^{\mu\nu} + 2 \Gamma^\mu_{\alpha\mu} \right] = -2 \nabla_\alpha \sqrt{g} \tag{14}
\]

gives

\[
\nabla_\alpha \sqrt{g} = \sqrt{g} V_\alpha. \tag{15}
\]

Thus, we have

\[
\nabla_\alpha g_{\mu\nu} = g_{\mu\nu} V_\alpha - g_{\mu\alpha} V_\nu - g_{\nu\alpha} V_\mu. \tag{16}
\]
The same procedure that is used in general relativity to find the connection gives

\[ \Gamma^\alpha_{\mu\nu} = \frac{1}{2} \left[ 3g_{\mu\nu} V^\alpha - \delta^\alpha_\mu V_\nu - \delta^\alpha_\nu V_\mu \right], \quad (17) \]

where the first term on the right hand side is the Christoffel symbol. Now go back to equation (7) and substitute in equations (5) and (13) to get (recall that \( T_{\mu\nu} = 0 \))

\[ \nabla_\alpha \left[ \sqrt{P} \left( P^{-1} \right)^{\nu}_\mu \right] = -3\sqrt{g} V^\mu. \quad (18) \]

Now consider the “Einstein limit” where \( |bR_{\mu\nu}| << |g_{\mu\nu}| \). To second order in \( b \)

\[ \left( P^{-1} \right)^{\mu\nu} = g^{\mu\nu} - bR^{\mu\nu} + b^2 R^\mu_\alpha R^\alpha_\nu \quad (19) \]

and

\[ \sqrt{P} = \sqrt{g} \left[ 1 + \frac{1}{2} bR + \frac{1}{8} b^2 R^2 - \frac{1}{4} \frac{b^2 R_{\alpha\beta} R^{\alpha\beta}}{g} \right]. \quad (20) \]

Equation (5) becomes

\[ G_{\mu\nu} = -\kappa \left( S_{\mu\nu} + \hat{S}_{\mu\nu} \right) \quad (21) \]

where

\[ S_{\mu\nu} = \frac{b}{2\kappa} \left[ RR_{\mu\nu} - \frac{1}{4} R^2 g_{\mu\nu} - 2g^{\alpha\beta} R_{\mu\alpha\nu\beta} + \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} \right], \quad (22) \]

\[ \hat{S}_{\mu\nu} = \frac{b}{\kappa} \left[ g_{\mu\nu} R^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} \right]. \quad (23) \]

\( R = g^{\mu\nu} R_{\mu\nu} \) and \( G_{\mu\nu} \) is the Einstein tensor with respect to the connection \( \Gamma^\alpha_{\mu\nu} \). Equation (18) becomes

\[ \nabla_\alpha \left\{ \sqrt{g} \left[ \left( 1 + \frac{1}{2} bR \right) R_\alpha^{\mu}\nu + bR^{\beta[\alpha} R^{\beta]\nu] \right] \right\} = \frac{3}{b} \sqrt{g} V^\mu \quad (24) \]

where the square brackets around \( \alpha \) and \( \mu \) denote antisymmetrization on these indices. The antisymmetric part of the Ricci tensor is proportional to the curl of the vector field and is given by

\[ R_\alpha^{\mu\nu} = \frac{1}{2} \left[ \nabla_\mu V_\nu - \nabla_\nu V_\mu \right]. \quad (25) \]

Note that to lowest order in \( b \) equations (24) and (25) describe a massive vector field with mass \( \sqrt{6/b} \). This implies that we must take \( b \) to be positive. Equations (21) to (25) plus (17) are the field equations of the theory. To compare to the Einstein field equations we need to write these equations in terms of the Einstein tensor \( \tilde{G}_{\mu\nu} \), which is defined in terms of the Christoffel symbol. The relationship between the Ricci tensors is given by

\[ R_{\mu\nu} = \tilde{R}_{\mu\nu} - \bar{\nabla}_\alpha H^\alpha_{\mu\nu} + \tilde{\nabla}_\nu H^\alpha_{\alpha\mu} - H^\alpha_{\alpha\beta} H^\beta_{\mu\nu} + H^\alpha_{\mu\beta} H^\beta_{\alpha\nu} \quad (26) \]
where $H^\alpha_{\mu\nu}$ is the tensor field

$$H^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\}$$  \hspace{1cm} (27)$$

and $\tilde{\nabla}$ is the metric compatible covariant derivative. A simple calculation shows that

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} + \frac{3}{2} \left[ V_\mu V_\nu + \frac{1}{2} g_{\mu\nu} \left( 2 \tilde{\nabla}_\alpha V^\alpha - V_\alpha V^\alpha \right) \right].$$  \hspace{1cm} (28)$$

It is easy to simplify the right hand side by showing that $\tilde{\nabla}_\alpha V^\alpha = 0$. To begin take the divergence of equation (24) and use (13) to get

$$\nabla_\mu \nabla_\alpha R^{\alpha\mu} + \nabla_\mu \left( V_\alpha R^{\alpha\mu} \right) + \nabla_\mu U^\mu = \frac{3}{b} \nabla_\mu V^\mu ,$$  \hspace{1cm} (29)$$

where $U^\mu$ contains the terms in (24) proportional to $b$. A short calculation shows that

$$\nabla_\mu \nabla_\alpha R^{\alpha\mu} = R^{\alpha\mu}$$  \hspace{1cm} (30)$$

and that $\nabla_\mu U^\mu$ is third order in the curvature so that it can be neglected. Substituting (30) and (24) into (29) and neglecting third order terms in the curvature gives

$$\nabla_\mu V^\mu = -V^\alpha V_\alpha .$$  \hspace{1cm} (31)$$

Using (17) it is easy to show that

$$\tilde{\nabla}_\mu V^\mu = 0 .$$  \hspace{1cm} (32)$$

Now define the vector potential $A_\mu$ by

$$A_\mu = \frac{1}{2\alpha} V_\mu .$$  \hspace{1cm} (33)$$

The antisymmetric part of the Ricci tensor is then given by

$$R^{\alpha\mu} = \alpha F^{\alpha\mu}$$  \hspace{1cm} (34)$$

where $F_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu$. Note that the $\nabla$ operator in (25) can be replaced by $\tilde{\nabla}$ or by $\partial$.

To summarize field equations are given by

$$\tilde{G}_{\mu\nu} = -\kappa \left[ T^{R}_{\mu\nu} + T^{A}_{\mu\nu} \right]$$  \hspace{1cm} (35)$$

and by

$$\tilde{\nabla}_\alpha F^{\alpha\mu} - \frac{6}{b} A^\mu = S^\mu$$  \hspace{1cm} (36)$$
where
\[
T^R_{\mu\nu} = \frac{b}{2\kappa} \left[ RR_{\mu\nu} - \frac{1}{4} R^2 g_{\mu\nu} - 2g^{\alpha\beta} R_{\mu\alpha} R_{\nu\beta} + \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} \right],
\tag{37}
\]
and
\[
T^A_{\mu\nu} = \frac{\alpha^2 b}{\kappa} \left[ F_{\mu\alpha} F_{\nu}{}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} + \frac{6}{b} \left( A_{\mu} A_{\nu} - \frac{1}{2} g_{\mu\nu} A^\alpha A_\alpha \right) \right],
\tag{38}
\]
and $S^\mu$ contains the terms in (24) that are proportional to $b$. Note that equation (36) describes a vector field with mass $\sqrt{6/b}$ and $T^A_{\mu\nu}$ is its corresponding energy momentum tensor.

As discussed below equation (10) the vacuum field equations with $F_{\mu\nu} = 0$ are the Einstein equations for all values of $b$. Thus, vacuum tests of general relativity do not constrain the value of $b$. We will see below however that the value of $b$ is severely constrained by taking general relativity to be valid in the interior of neutron stars.

Now consider the field equations in the presence of matter and sources for the vector field. To the Lagrangian given in (3) we need to add an interaction term that couples the vector field to its sources. The simplest coupling is given by
\[
L_c = \frac{\alpha b}{2\kappa} \sqrt{g} \Gamma_{\alpha\mu}^\alpha J^\mu
\tag{39}
\]
where $J^\mu$ is the conserved current associated with the source. At first sight it may appear that there is a problem with this choice as $\Gamma_{\alpha\mu}^\alpha$ is not a tensor. An analogous situation occurs in electrodynamics where the interaction Lagrangian $\sqrt{g} A_\mu J^\mu$ appears not to be gauge invariant. However, if $\nabla_\mu J^\mu = 0$ the Lagrangian only changes by a total derivative under a gauge transformation. Now, under a coordinate transformation
\[
\tilde{\Gamma}_{\alpha\mu}^\alpha = \partial x^\nu \Gamma_{\nu\mu}^\alpha - \frac{\partial}{\partial \bar{x}^\mu} \ln \frac{\partial x}{\partial \bar{x}}
\tag{40}
\]
where $|\partial x/\partial \bar{x}|$ is the Jacobian of the transformation. This is analogous to a gauge transformation and it is easy to show that the Lagrangian $L_c$ only changes by a total derivative if $\nabla_\mu J^\mu = 0$. Another way of looking at it is to use (17) to write
\[
\sqrt{g} \Gamma_{\alpha\mu}^\alpha J^\mu = \sqrt{g} [\partial_\mu \ln \sqrt{g} - V_\mu] J^\mu.
\tag{41}
\]
The first term can be rewritten as a total divergence and can therefore be neglected. The second term is a scalar density and so the Lagrangian $L_c$ has the correct transformation properties. A simple example of a conserved current is a point source with charge $e$. The current is given by
\[
J^\mu(x^\alpha) = \frac{e}{\sqrt{g}} \int U^\mu(\tau) \delta(x^\alpha - x^\alpha(\tau)) d\tau
\tag{42}
\]
where $U^\mu$ is the four velocity of the particle and $\tau$ is the proper length along its world line. Of course the fields that produce $J^\mu$ must appear in $L_M$. One interesting property of
The choice of Lagrangian is that the current $J^\mu$ does not enter into the equation defining the connection, so that (17) is still valid. In the Einstein limit where $b|R_{\mu\nu}| << |g_{\mu\nu}|$ and $\kappa b|T^{\mu\nu}| << |g^{\mu\nu}|$ the field equations are given by

$$\tilde{G}_{\mu\nu} = -\kappa \left[ T_{\mu\nu} + T^R_{\mu\nu} + T^A_{\mu\nu} + T^\Gamma_{\mu\nu} \right]$$

(43)

and

$$\tilde{\nabla}_a F^{\alpha\mu} - \frac{6}{b} A^\mu = J^\mu + S^\mu$$

(44)

where $T^R_{\mu\nu}$ is given by (37), $T^A_{\mu\nu}$ is given by (38), $S^\mu$ contains the terms proportional to $b$ in (24) and $T^\Gamma_{\mu\nu}$ contains the additional terms that enter through the connection from the $\kappa b T^{\mu\nu}$ terms in (12).

For the Einstein equations to be approximately valid it is necessary that $\kappa b|T^{\mu\nu}| << |g^{\mu\nu}|$. If $g_{\mu\nu} \sim \eta_{\mu\nu}$ and if $T^{\mu\nu}$ describes matter with a density $\rho$ then the constraint can be written as

$$b << \frac{1}{\kappa \rho}.$$  

(45)

For a neutron star $\rho \approx 10^{18} \text{ kg/m}^3$ and the constraint becomes

$$b << 10^9 \text{ m}^2.$$  

(46)

This corresponds to the mass constraint

$$m >> 10^{-47} \text{ kg}.$$  

(47)

Since the mass of the photon is constrained to be less than $10^{-52} \text{ kg}$ [9, 10, 11] this vector field cannot be the electromagnetic field.

The field equations (43) can be derived from the Lagrangian

$$L = -\frac{1}{2 \kappa} \sqrt{g} \left[ R + \frac{1}{4} b R^2 - \frac{1}{2} b R_{\alpha\beta} R^{\alpha\beta} \right] + L_c + L_M$$

(48)

which is the Born-Infeld-Einstein Lagrangian (3) expanded to order $b$. Equation (36) also follows, but without the $S^\mu$ term which is of order $b^2$ in the expansion of the Lagrangian. Thus, this term can be consistently neglected to this order.

**Field Equations II**

On a manifold with a general connection there are two nonzero contractions of the Riemann tensor: $R_{\mu\nu}$ given in (4) and

$$S_{\mu\nu} = R_{\alpha\mu\nu} = \partial_\mu \Gamma^\alpha_{\alpha\nu} - \partial_\nu \Gamma^\alpha_{\alpha\mu}.$$  

(49)
The simplest Born-Infeld Lagrangian that includes both $R_{\mu\nu}$ and $S_{\mu\nu}$ is given by replacing $bR_{\mu\nu}$ in (3) by the linear combination $bR_{\mu\nu} + dS_{\mu\nu}$. Since $S_{\mu\nu} = -2R_{\mu\nu} \vee$ this Lagrangian can be written as

$$L = -\frac{1}{\kappa b} \left\{ \sqrt{-\det(g_{\mu\nu} + bR_{\mu\nu} + aR_{\nu\vee})} - \sqrt{-\det(g_{\mu\nu})} \right\} + L_M \quad (50)$$

If $a = b$ we obtain the theory discussed in the previous section. Varying the action with respect to $g_{\mu\nu}$ gives

$$\nabla_\alpha \left[ \sqrt{P} \left( P^{-1} \right)^{\mu\nu} \right] - \frac{1}{2} \nabla_\beta \left\{ \sqrt{P} \left[ \left( P^{-1} \right)^{\mu\beta} + \frac{a}{b} \left( P^{-1} \right)^{\beta\nu} \right] \delta^\alpha_\nu + \sqrt{P} \left[ \left( P^{-1} \right)^{\nu\beta} + \frac{a}{b} \left( P^{-1} \right)^{\beta\mu} \right] \delta^\mu_\alpha \right\} = 0 \quad (52)$$

and contracting over $\alpha$ and $\nu$ gives

$$\nabla_\alpha \left[ \sqrt{P} \left( P^{-1} \right)^{\alpha\nu} \right] = -\frac{3b}{5a} \nabla_\alpha \left[ \sqrt{P} \left( P^{-1} \right)^{\alpha\mu} \right] \quad (53)$$

As before, the field equations reduce to the Einstein field equations in a vacuum if $R_{\nu\vee} = 0$. Substituting equations (51) and (53) into (52) gives (12). Once again set

$$\nabla_\alpha \left[ \sqrt{g} \left( g^{\mu\nu} + \kappa bT^{\mu\nu} \right) \right] = \sqrt{g} \left[ \delta^\mu_\alpha V^\nu + \delta^\nu_\alpha V^\mu \right] \quad (54)$$

where $V^\mu$ is an arbitrary vector and the connection is still given by (17). Now substitute equations (51) and (54) into (53) to get

$$\nabla_\alpha \left[ \sqrt{P} \left( P^{-1} \right)^{\alpha\nu} \right] = -\frac{3b}{a} \sqrt{g} V^\mu \quad (55)$$

Now consider the “Einstein limit” where $b|R_{\mu\nu}| << |g_{\mu\nu}|$. To second order in $b$ and $a$

$$\left( P^{-1} \right)^{\mu\nu} = g^{\mu\nu} - bR^{\mu\nu} + b^2 g_{\alpha\beta} R^{\mu\alpha} R^{\nu\beta} + a^2 g_{\alpha\beta} R^{\mu\alpha} R^{\nu\beta}, \quad (56)$$

$$\left( P^{-1} \right)^{\mu\nu \vee} = -a \left[ R^{\mu\nu} - b \left( R^{\mu\alpha} R^{\nu\beta} + R^{\mu\beta} R^{\nu\alpha} \right) \right] \quad (57)$$

and

$$\sqrt{P} = \sqrt{g} \left[ 1 + \frac{1}{2} bR + \frac{1}{8} b^2 R^2 - \frac{1}{4} b^2 R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{4} a^2 R_{\alpha\beta\vee} R^{\alpha\beta} \right], \quad (58)$$

where $R = g^{\mu\nu} R_{\mu\nu}$. Sources can be coupled to the vector field by using the interaction Lagrangian $L_c$ given in (39) with $b$ replaced by $a$. Using equation (28) and the above gives

$$\tilde{G}_{\mu\nu} = -\kappa \left[ T_{\mu\nu} + T^{R}_{\mu\nu} + T^{A}_{\mu\nu} + T^{\Gamma}_{\mu\nu} \right] \quad (59)$$
and
\[ \tilde{\nabla}_\alpha F^{\alpha \mu} - \frac{6b}{a^2} A^\mu = J^\mu + S^\mu \] (60)

where
\[ T^{R}_{\mu \nu} = \frac{b}{2\kappa} \left[ RR_{\mu \nu} - \frac{1}{4} R^2 g_{\mu \nu} - 2g^{\alpha \beta} R_{\mu \alpha} R_{\nu \beta} + \frac{1}{2} g_{\mu \nu} R_{\alpha \beta} R^{\alpha \beta} \right], \] (61)
\[ T^A_{\mu \nu} = \frac{\alpha^2 c^2}{b\kappa} \left[ F_{\mu \alpha} F^\alpha_{\nu} - \frac{1}{4} g_{\mu \nu} F^{\alpha \beta} F_{\alpha \beta} + \frac{6b}{a^2} \left( A_{\mu} A_{\nu} - \frac{1}{2} g_{\mu \nu} A^{\alpha} A_{\alpha} \right) \right], \] (62)

and \( S^\mu \) contains the terms in (24) that are proportional to \( b \). It is easy to show that the constraint \( \tilde{\nabla}_\mu A^\mu = 0 \) is still satisfied. Note that equation (60) describes a vector field with mass \( \sqrt{6b/a^2} \) and \( T^A_{\mu \nu} \) is its corresponding energy momentum tensor.

The value of \( b \) is constrained by
\[ b << 10^8 \text{ m}^2 \] (63)

if we require that general relativity holds to a good approximation inside neutron stars. The mass of the photon is constrained to be less than \( 10^{-52} \text{ kg} \) [9, 10, 11] and this gives the constraint
\[ a > 10^{10} \sqrt{b}. \] (64)

Thus, the coupling constant associated with the antisymmetric part of the Ricci tensor must be much larger than the coupling constant associated with the symmetric part.

The field equations without the \( S^\mu \) term can be derived from the Lagrangian
\[ L = -\frac{1}{2\kappa} \sqrt{g} \left[ R + \frac{1}{4} b R^2 - \frac{1}{2} b R_{\alpha \beta} R^{\alpha \beta} + \frac{c^2}{2b} R_{\alpha \beta} R_{\alpha \beta} \right] + L_c + L_M \] (65)

which is the expansion of (50) to first order in \( b \) and \( c^2/b \).

### Curvature squared Lagrangian

In the previous two sections it was shown that Born-Infeld-Einstein Lagrangians give gravity coupled to a massive vector field to first order in the parameters \( b \) and \( c^2/b \). In this section a Lagrangian which exactly produces the Einstein field equations coupled to a massive vector field will be given.

The Lagrangian for the theory in the absence of sources is
\[ L = -\frac{1}{2\kappa} \sqrt{g} \left( R + \frac{1}{2} b R_{\alpha \beta} R^{\alpha \beta} \right). \] (66)

This is (65) with the quadratic terms in \( R_{\mu \nu} \) dropped.

Varying with respect to \( g_{\mu \nu} \) gives
\[ G_{\mu \nu} = -\alpha^2 b \left[ F_{\mu \alpha} F^\alpha_{\nu} - \frac{1}{4} g_{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} \right]. \] (67)
Varying with respect to $\Gamma^\alpha_{\mu\nu}$ one finds that the connection is again given by (17) and the field equation for $F^\mu\nu$ is given by

$$\tilde{\nabla}_\alpha F^\alpha\mu = \frac{6}{b} A^\mu$$

(68)

Converting $G^\mu\nu$ to $\tilde{G}^\mu\nu$ in (67) gives the Einstein field equations with the energy momentum tensor (38). Thus, the field equations describe a massive vector field coupled to gravity.

**Conclusion**

The field equations for the Born-Infeld-Einstein action were derived using a Palatini variation. The vacuum field equations with $R^\nu_{\nu\nu} = 0$ were shown to be the Einstein vacuum equations, independent of the value of $b$. This extends the universality property of the vacuum Einstein field equations discovered by Ferraris et al. to the Born-Infeld-Einstein action. For sufficiently small curvatures the field equations describe a massive vector field with $R^\nu_{\nu\nu}$ being proportional to the field tensor and the Einstein equations plus small corrections. The simplest version of the theory uses only the Ricci tensor $R^\mu\nu$. By requiring that the Einstein field equations hold to a good approximation in neutron stars it was shown that the mass of the vector field exceeds the limit on the photon mass. Thus, in this case the vector field would necessarily describe a new field.

On a manifold with a general connection there are two nonzero contractions of the Riemann tensor: $R^\mu\nu$ and $S^\mu\nu$. If both $R^\mu\nu$ and $S^\mu\nu$ are used it is possible to have the Einstein equations hold in neutron stars and for the mass of the field to satisfy the constraints on the photon mass. For this to work the coupling constant associated with the antisymmetric part of the Ricci tensor must be much larger than the coupling constant associated with the symmetric part.

**References**


