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**LIE-THEORETIC GENERATING RELATIONS
OF TWO VARIABLE LAGUERRE POLYNOMIALS**

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Abstract

Generating relations involving two variable Laguerre polynomials $L_n(x, y)$ are derived. The process involves the construction of a three dimensional Lie algebra isomorphic to special linear algebra $sl(2)$ with the help of Weisner's method by giving suitable interpretations to the index n of the polynomials $L_n(x, y)$.

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1 INTRODUCTION

Recently Dattoli and Torre [4,5] introduced and discussed a theory of two variable Laguerre polynomials (TVLP). The reason of interest for this family of Laguerre polynomials is due to their intrinsic mathematical importance and to the fact that these polynomials are shown to be natural solutions of a particular set of partial differential equations, which often appears in the treatment of radiation physics problems such as the electromagnetic wave propagation and quantum beam life-time in storage rings, see Wrülich[13].

The TVLP $L_n(x, y)$ are specified by series ([2]; p.121(69))

$$L_n(x, y) = \sum_{r=0}^n \frac{(-1)^r n! x^r y^{n-r}}{(r!)^2 (n-r)!} , \quad (1.1)$$

and the generating function for $L_n(x, y)$ is given by ([4]; p.3(10)).

$$\sum_{n=0}^{\infty} L_n(x, y) t^n = \frac{1}{(1-yt)} \exp\left(-\frac{xt}{(1-yt)}\right) , \quad |yt| < 1. \quad (1.2)$$

These polynomials satisfy the following differential and pure recurrence relations:

$$\begin{aligned} \frac{\partial}{\partial x} L_n(x, y) &= \frac{n}{x} L_n(x, y) - \frac{ny}{x} L_{n-1}(x, y) , \\ \frac{\partial}{\partial y} L_n(x, y) &= n L_{n-1}(x, y) , \\ L_{n+1}(x, y) &= \frac{1}{(n+1)} \left(((2n+1)y - x) L_n(x, y) - ny^2 L_{n-1}(x, y) \right) . \end{aligned} \quad (1.3)$$

The TVLP $L_n(x, y)$ are linked to the Laguerre polynomials $L_n(x)$ [1] by the relation ([5];p.22(10b))

$$L_n(x, y) = y^n L_n\left(\frac{x}{y}\right) , \quad L_n(x, 0) = (-1)^n \frac{x^n}{n!} . \quad (1.4)$$

The differential equation satisfied by $L_n(x, y)$ is

$$\left(x \frac{d^2}{dx^2} + \left(1 - \frac{x}{y} \right) \frac{d}{dx} + \frac{n}{y} \right) L_n(x, y) = 0 . \quad (1.5)$$

Very recently Pathan *et al.* [9,10] obtained generating relations involving two and three variables generalized Hermite polynomials $H_n(x, y)$ and $H_n(x, y, z)$ [3] using the Lie-theoretic method. In the present paper we utilize Weisner's [12] group-theoretic method for obtaining generating relations in the case of TVLP $L_n(x, y)$ by giving suitable interpretations to the index n .

It is remarked that another, more general, approach to obtain Laguerre polynomials of many variables (and other polynomials as well, such as Hermite, Charlier) using a chaos decomposition

on spaces of Fock type can be found in [6] and the references therein. However, the approach in this paper is more direct and appropriate for obtaining generating relations.

The process involves the construction of a three dimensional Lie algebra isomorphic to special linear algebra $sl(2)$, the Lie algebra of $SL(2)$ ([8]; p.7). The 2×2 complex special linear group $SL(2)$ is the abstract matrix group of all 2×2 nonsingular matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C},$$

such that $\det g = 1$. $SL(2)$ is a three dimensional local Lie group. The elements

$$\mathcal{J}^+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{J}^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{J}^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},$$

satisfying the commutation relations

$$[\mathcal{J}^3, \mathcal{J}^\pm] = \pm \mathcal{J}^\pm, \quad [\mathcal{J}^+, \mathcal{J}^-] = 2\mathcal{J}^3$$

form a basis for $sl(2)$.

2 GROUP-THEORETIC METHOD

On replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$ and n by $z\frac{\partial}{\partial z}$ in differential equation (1.5), we construct a partial differential equation

$$\left(x \frac{\partial^2}{\partial x^2} + \left(1 - \frac{x}{y} \right) \frac{\partial}{\partial x} + \frac{z}{y} \frac{\partial}{\partial z} \right) f(x, y, z) = 0. \quad (2.1)$$

Thus $f(x, y, z) = L_n(x, y)z^n$ is a solution of (2.1) since $L_n(x, y)$ is a solution of (1.5).

First, we consider the following first order linear partial differential operators

$$\begin{aligned} J^3 &= z \frac{\partial}{\partial z} + \frac{1}{2}, \\ J^+ &= xyz \frac{\partial}{\partial x} + yz^2 \frac{\partial}{\partial z} + (y - x)z, \\ J^- &= \frac{x}{yz} \frac{\partial}{\partial x} - \frac{1}{y} \frac{\partial}{\partial z}, \end{aligned} \quad (2.2)$$

such that

$$\begin{aligned}
J^3[L_n(x, y)z^n] &= \left(n + \frac{1}{2}\right) L_n(x, y)z^n , \\
J^+[L_n(x, y)z^n] &= (n + 1)L_{n+1}(x, y)z^{n+1} , \\
J^-[L_n(x, y)z^n] &= -nL_{n-1}(x, y)z^{n-1} .
\end{aligned}$$

We note that these operators satisfy the commutation relations

$$[J^3, J^\pm] = \pm J^\pm , \quad [J^+, J^-] = 2J^3 , \quad (2.3)$$

where $[A, B] = AB - BA$.

The above commutation relations show that the set of J -operators $\{J^3, J^+, J^-\}$ generate a three dimensional Lie algebra isomorphic to $sl(2)$, the Lie algebra of $SL(2)$.

In terms of the J -operators, we introduce the Casimir operator ([8]; p.32)

$$\begin{aligned}
\mathcal{C} &= J^+ J^- + J^3 J^3 - J^3 \\
&= x \left(x \frac{\partial^2}{\partial x^2} + \left(1 - \frac{x}{y}\right) \frac{\partial}{\partial x} + \frac{z}{y} \frac{\partial}{\partial z} \right) - \frac{1}{4}.
\end{aligned} \quad (2.4)$$

It is easy to verify that $\mathcal{C}$ commutes with J^3, J^+ and J^- , that is

$$[\mathcal{C}, J^3] = [\mathcal{C}, J^\pm] = 0. \quad (2.5)$$

Equation (2.4) enables us to write (2.1) as

$$\mathcal{C}f(x, y; z) = -\frac{1}{4}f(x, y; z). \quad (2.6)$$

Now we proceed to compute the multiplier representation $[T(g)f](x, y; z)$, $g \in SL(2)$, induced by the J -operators (2.2). A simple computation using ([11];p.320 (Theorem 7)) and (2.2) gives

$$\begin{aligned}
[T(\exp(b' \mathcal{J}^+))f](x, y; z) &= (1 - b'yz)^{-1} \exp\left(\frac{-b'xz}{(1 - b'yz)}\right) \\
&\quad \cdot f\left(\frac{x}{(1 - b'yz)}, y; \frac{z}{(1 - b'yz)}\right) , \quad |b'yz| < 1, \\
[T(\exp(c' \mathcal{J}^-))f](x, y; z) &= f\left(\frac{x}{(1 - \frac{c'}{yz})}, y; z(1 - \frac{c'}{yz})\right) , \quad \left|\frac{c'}{yz}\right| < 1, \\
[T(\exp(\tau' \mathcal{J}^3))f](x, y; z) &= \exp\left(\frac{\tau'}{2}\right) f(x, y; z \exp(\tau')) ,
\end{aligned} \quad (2.7)$$

defined for $|b'|, |c'|$ and $|\tau'|$ sufficiently small.

If $g \in SL(2)$ and $d \neq 0$, it is a straightforward computation to show that

$$g = \exp(b' \mathcal{J}^+) \exp(c' \mathcal{J}^-) \exp(\tau' \mathcal{J}^3),$$

where $b' = -\frac{b}{d}$, $c' = -cd$, $\exp(\frac{\tau'}{2}) = \frac{1}{d}$ and $ad - bc = 1$.

Hence the operator $T(g)$ is given by

$$\begin{aligned} [T(g)f](x, y; z) &= (d + byz)^{-1} \exp\left(\frac{bxz}{(d + byz)}\right) \\ &\cdot f\left(\frac{xyz}{(d + byz)(c + ayz)}, y; \frac{(c + ayz)}{y(d + byz)}\right), \quad (2.8) \\ \max\left(\left|\frac{byz}{d}\right|, \left|\frac{c}{ayz}\right|\right) &< 1, \quad |\arg(d)| < \pi. \end{aligned}$$

3 GENERATING RELATIONS

To accomplish our job of obtaining generating relations, we search for the function $f(x, y; z)$ which satisfies (2.6). Consider the case when $f(x, y; z)$ is a common eigenfunction of $\mathcal{C}$ and J^3 , that is, let $f(x, y; z)$ be a solution of the simultaneous equations

$$\begin{aligned} \mathcal{C}f(x, y; z) &= -\frac{1}{4}f(x, y; z), \\ J^3f(x, y; z) &= \left(n + \frac{1}{2}\right)f(x, y; z), \end{aligned} \quad (3.1)$$

which may be rewritten as

$$\begin{aligned} \left(x \frac{\partial^2}{\partial x^2} + \left(1 - \frac{x}{y}\right) \frac{\partial}{\partial x} + \frac{z}{y} \frac{\partial}{\partial z}\right) f(x, y; z) &= 0, \\ \left(z \frac{\partial}{\partial z} - n\right) f(x, y; z) &= 0. \end{aligned} \quad (3.2)$$

Equations (3.2) yield

$$f(x, y; z) = L_n(x, y) z^n,$$

so that, we have

$$\begin{aligned} [T(g)f](x, y; z) &= (d + byz)^{-n-1} \left(\frac{(c + ayz)}{y}\right)^n \exp\left(\frac{bxz}{(d + byz)}\right) \\ &\cdot L_n\left(\frac{xyz}{(d + byz)(c + ayz)}, y\right), \end{aligned} \quad (3.3)$$

satisfying the relation

$$\mathcal{C}[T(g)f](x, y; z) = -\frac{1}{4}[T(g)f](x, y; z).$$

If n is a non-negative integer, (3.3) has an expansion of the form

$$[T(g)f](x, y; z) = \sum_{k=0}^{\infty} A_{kn}(g) L_k(x, y) z^k ,$$

which simplifies to the identity

$$\begin{aligned} (d + byz)^{-n-1} \left(\frac{(c + ayz)}{y} \right)^n \exp \left(\frac{bxz}{(d + byz)} \right) L_n \left(\frac{xyz}{(d + byz)(c + ayz)}, y \right) \\ = \sum_{k=0}^{\infty} A_{kn}(g) L_k(x, y) z^k \end{aligned} \quad (3.4)$$

To determine $A_{kn}(g)$, we set $x = 0$ and $y = 1$ in (3.4), and thus we have

$$A_{kn}(g) = \sum_{r=0}^n \frac{(-n)_r (n - r + 1)_{k-n+r}}{(k - n + r)! r!} (-b)^{k-n+r} (-c)^r d^{-n-k-1} , \quad (3.5)$$

$$k, n \geq 0$$

Substituting (3.5) into (3.4) and simplifying, we obtain the generating relation

$$\begin{aligned} (ad)^n \left(1 + \frac{byz}{d} \right)^{-n-1} \left(1 + \frac{c}{ayz} \right)^n \exp \left(\frac{bxz}{(d + byz)} \right) L_n \left(\frac{xyz}{(d + byz)(c + ayz)}, y \right) \\ = \sum_{k=0}^{\infty} \sum_{r=0}^n \frac{(-n)_r (n - r + 1)_k}{k! r!} \left(\frac{-bz}{d} \right)^k \left(\frac{-cd}{z} \right)^r L_{n+k-r}(x, y). \end{aligned} \quad (3.6)$$

We consider some special cases of (3.6).

I. Taking $a = c = d = 1, b = 0$ and replacing z by $-\frac{1}{t}$, in (3.6), we obtain

$$\left(1 - \frac{t}{y} \right)^n L_n \left(\frac{x}{(1 - \frac{t}{y})}, y \right) = \sum_{r=0}^n \frac{(-n)_r}{r!} L_{n-r}(x, y) t^r , \quad (3.7)$$

which for $y = 1$ reduces to a special case of ([11]; p.331(31)) (with $\alpha = 0$)

$$(1 - t)^n L_n \left(\frac{x}{(1 - t)} \right) = \sum_{r=0}^n \frac{(-n)_r}{r!} L_{n-r}(x) t^r .$$

II. Taking $a = b = d = 1, c = 0$ and replacing z by $-t$, in (3.6), we obtain

$$(1 - yt)^{-n-1} \exp \left(\frac{-xt}{(1 - yt)} \right) L_n \left(\frac{x}{(1 - yt)}, y \right) = \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!} L_{n+k}(x, y) t^k , \quad (3.8)$$

which for $y = 1$ reduces to ([7]; p.20(12))

$$(1 - t)^{-n-1} \exp \left(\frac{-xt}{(1 - t)} \right) L_n \left(\frac{x}{(1 - t)} \right) = \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!} L_{n+k}(x) t^k .$$

III. Taking $a = d = 1$ and $bc \neq 0$, without any loss of generality we can choose $bz = -t_1$ and $\frac{c}{z} = -t_2$ in (3.6) and we obtain

$$\begin{aligned} & (1 - yt_1)^{-n-1} \left(1 - \frac{t_2}{y}\right)^n \exp\left(\frac{-xt_1}{(1 - yt_1)}\right) L_n\left(\frac{x}{(1 - yt_1)(1 - \frac{t_2}{y})}, y\right) \\ &= \sum_{k=0}^{\infty} \sum_{r=0}^n \frac{(-n)_r (n - r + 1)_k}{k!r!} t_1^k t_2^r L_{n+k-r}(x, y), \end{aligned} \quad (3.9)$$

which for $y = 1$ yields

$$\begin{aligned} & (1 - t_1)^{-n-1} (1 - t_2)^n \exp\left(\frac{-xt_1}{(1 - t_1)}\right) L_n\left(\frac{x}{(1 - t_1)(1 - t_2)}\right) \\ &= \sum_{k=0}^{\infty} \sum_{r=0}^n \frac{(-n)_r (n - r + 1)_k}{k!r!} t_1^k t_2^r L_{n+k-r}(x). \end{aligned} \quad (3.10)$$

Further taking $t_1 = t_2 = t$ in (3.10), we get

$$\begin{aligned} & (1 - t)^{-1} \exp\left(\frac{-xt}{(1 - t)}\right) L_n\left(\frac{x}{(1 - t)^2}\right) \\ &= \sum_{k=0}^{\infty} \sum_{r=0}^n \frac{(-n)_r (n - r + 1)_k}{k!r!} t^{k+r} L_{n+k-r}(x). \end{aligned} \quad (3.11)$$

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