Homogeneous Plane-wave Spacetimes and their Stability

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Abstract

We consider the stability of spatially homogeneous plane-wave spacetimes. We carry out a full analysis for plane-wave spacetimes in (4 + 1) dimensions, and find there are two cases to consider; what we call non-exceptional and exceptional. In the non-exceptional case the plane waves are stable to (spatially homogeneous) vacuum perturbations as well as a restricted set of matter perturbations. In the exceptional case we always find an instability. Also we consider the Milne universe in arbitrary dimensions and find it is also stable provided the strong energy condition is satisfied. This implies that there exists an open set of stable plane-wave solutions in arbitrary dimensions.

1 Introduction

Finding solutions to Einstein’s equations is a problem of great mathematical and physical interest. However, due to the fairly complicated nature of the Einstein equations, we usually impose symmetries in order to make the field equations more tractable. Some of the most successful schemes of symmetry reduction are the Bianchi models in (3+1)-dimensional cosmology [1, 3, 4]. For these models there exists a simply transitive group acting on the spatial hypersurfaces; hence, they are spatially homogeneous. For the spatially homogeneous models, the equations of motion reduce to a simple set of ordinary differential equations which makes it possible to study the evolution of these models in great detail. In particular, the asymptotic behaviour of all non-tilted Bianchi models with a perfect fluid have been determined [4–8]. Motivated by the success of the (3 + 1)-dimensional Bianchi models, here, we shall investigate some of their higher-dimensional counterparts.
In (3+1)-dimensional cosmology, algebraically special solutions usually play a particular role; they are equilibrium points, or sometimes even attractors, for more general classes of solutions. In particular the vacuum plane-wave solutions of dimension 3+1 are attractors for non-tilted non-inflationary models of Bianchi class B \[4,6,5\]. Hence, general solutions of Einstein’s equations of Bianchi class B containing non-tilted non-inflationary perfect fluids can be approximated by vacuum plane-wave solutions at late times. For higher-dimensional models the situation is still unsettled as no stability analysis has been done to date.

In this paper we intend to take the first step towards this and analyse the (4+1)-dimensional case. More precisely we look at spatially homogeneous plane waves, and find that they are stable to both perturbations in the geometry and also a restricted set of matter perturbations (which exclude inflationary fluids), provided a particular condition is satisfied; we call this the non-exceptional case. In the exceptional case, there exists an instability. The exceptional cases allows – due to the exact vanishing of one of the Einstein constraint equations – for an additional shear degree of freedom; it is this extra shear mode which turns out to be unstable. Also, as a precursor to future work, we show that the Milne universe is stable to perturbations in arbitrary dimensions provided the strong energy condition is satisfied. In particular this implies that in an arbitrary dimension, there exists an open set of stable plane-waves.

Recently, of course, plane waves (and pp-waves) have featured heavily in the String Theory literature \[9–11\]. Field theories in curved backgrounds usually assume that the back-reactions on the geometry can be neglected. However, the existence of an unstable mode would ultimately lead to the approximation being invalid. Therefore knowledge of the stability of such spacetimes in higher dimensions is crucial to such investigations, for studying quantum theories in such backgrounds inevitably leads to issues of back-reaction.

The outline of the paper is as follows. In section 2 we write down the relevant equations of motion for spatially homogeneous models. In section 3 we specialise to plane waves. Section 4 concerns the general stability analysis in 4+1 dimensions. In section 5 we consider matter perturbations, and in section 6 we analyse the stability of the (\(n+1\))-dimensional Milne universe. Finally we provide a brief discussion.

## 2 Equations of motion and constraints in \(n+1\) dimensions for spatially homogeneous space-times

In this section we recall some well known \[3,12\] results on spatially homogeneous spacetimes in \(n+1\) dimensions. Such spacetimes are defined as ones possessing a time-like vector \(n^\mu\) hypersurface orthogonal, such that there exists a Lie group \(G\) acting simply transitively on the hypersurfaces. Note that this implies that the spacetime is of the form \(\mathbb{R} \times \Sigma\) with metric \(g = -dt^2 + h(t)\), where \(n^\mu = (dt)^\mu\) and \(h(t)\) is a left-invariant metric on \(\Sigma\). For such spaces, \(n^\mu\) is geodesic and the extrinsic curvature on the hypersurfaces reduces to \(K_{\mu\nu} = \nabla_\mu n_\nu \equiv \theta_{\mu\nu}\). Now, we can introduce an orthonormal frame on our spacetime, so\(^1\)

\(^1\)Here, Greek indices \((\mu, \nu, \ldots)\) run over the full spacetime manifold, 0, 1, ..., \(n\); lower case Latin indices \((a, b, \ldots)\) run over the spatial hypersurfaces, 1, ..., \(n\); and upper case Latin indices
\[ g = \eta_{\mu\nu} \omega^\mu \omega^\nu = -dt^2 + \delta_{ab} \omega^a \omega^b, \]  

where we have picked \( \omega^0 = dt \). The dual basis to \( \{ \omega^\mu \} \) will be denoted by \( \{ e^\mu \} \), and we have

\[
[e^\mu, e^\nu] = \gamma^0_{\mu\nu} e_\rho, \tag{2}
\]
or equivalently \( d\omega^\mu = \Gamma^\mu_{\nu\rho} \omega^\nu \wedge \omega^\rho \), where \( \Gamma^\mu_{\nu\rho} = -\frac{1}{2} (\gamma^\rho_{\mu\nu} + \gamma^\mu_{\nu\rho} - \gamma^\nu_{\rho\mu}) \). Note that orthogonality implies that

\[
\gamma^0_{\mu\nu} = 0. \tag{3}
\]

The remaining structure constants are

\[
\gamma^a_{\mu b} = -\theta_a^b + \Omega^a_{b}, \tag{4}
\]
and \( \gamma^a_{bc} \equiv C^a_{bc} \); note \( \Omega_{ab} \) is the local angular velocity of a set of Fermi-propagated axes with respect to \( \{ e_a \} \), in the rest frame of an observer with velocity \( n^\mu \). The shear, \( \sigma_{\mu\nu} \), is defined to be the traceless part of \( \theta_{\mu\nu} \), so

\[
\theta_{\mu
u} = \sigma_{\mu\nu} + \frac{1}{n} \theta h_{\mu\nu}. \]

\[
\sigma^\mu_{\nu} = 0, \textbf{as we shall also} \]  

the dynamical equations,

\[
\frac{d}{dt} a_b + \frac{1}{n} \theta a_b - (\Omega\_b - \sigma\_b^d) a_d = 0, \tag{6}
\]

\[
\frac{d}{dt} D^a_{bc} + \frac{1}{n} \theta D^a_{bc} + 2(\Omega^{d}_b - \sigma^{d}_b) D^a_{cd} + (\Omega^a_{d} - \sigma^a_d)D^d_{bd} = 0. \tag{7}
\]

So far the only assumption is spatial homogeneity. Now we impose the field equations. Einstein’s equations, in a vacuum, are of course \( \nabla_{\mu} \nabla^\mu = 0 \). These can be used in conjunction with standard results on hypersurfaces to give us our equations of motion. Firstly we have [12],

\[
(n+1) R_{\mu\nu} n^\mu n^\nu = K^2 - K_{\mu\nu} K^{\mu\nu} - \nabla_{\mu}(n^\nu \nabla_{\nu} n^\mu) + \nabla_{\mu}(n^\nu \nabla_{\nu} n^\mu), \tag{8}
\]
can be used to derive an \( (n+1) \)-dimensional version of Raychaudhuri’s equation, namely,

\[
\dot{\theta} + \frac{1}{n} \theta^2 + \sigma_{\mu\nu} \sigma^{\mu\nu} = 0, \tag{9}
\]

\( (A, B, ...) \) run over \( 1, ..., n - 1 \).
where \( \equiv n^\mu \nabla_\mu \). Another nice fact is \([12]\),

\[
(n+1) G_{\mu\nu} n^\mu n^\nu = \frac{1}{2} (n R + K^2 - K_{\mu\nu} K^{\mu\nu}),
\]

which allows us to deduce the Friedmann constraint,

\[
\left(1 - \frac{1}{n}\right) \theta^2 = \sigma_{\mu\nu} \sigma^{\mu\nu} - (n) R.
\]

The \((n+1) R_{ab} = h^a_c h^b_c (n+1) R_{\mu\nu}\) components of the vacuum equations lead to the shear equations of motion,\(^2\)

\[
\dot{\sigma}_{ab} + \theta \sigma_{ab} + (n) R_{(ab)} = 0,
\]

where \(A_{(ab)}\) of a tensor \(A_{ab}\) means the trace-free part. One can calculate the curvature tensors on the hypersurface in terms of the structure constants. We get,

\[
(n) R_{ab} = b_{ab} - \frac{(n-1)}{2} (D_{abd} + D_{bad}) a^d - (n-1) a^2 h_{ab},
\]

\[
(n) R = b^a_a - n(n-1) a^2,
\]

\[
b_{ab} \equiv -\frac{1}{4} (2 D^c_{ad} D_{eb}^d - D_{acd} D_b^c + 2 D^c_{ad} D^d_{bc}).
\]

Finally the Gauss-Codazzi equation,

\[
D_a K_b^a - D_b K = (n+1) R_{\mu\nu} n^\nu h_b^a
\]

where \(D_a\) is the metric connection of \(h\), leads to \(\theta_b^a C^b_{ca} - \theta^a_c C^b_{ab} = 0\), which in turn leads to the constraint,

\[
n \sigma_{ac} a^a - \sigma^a_b D^b_{ac} = 0.
\]

Now we will recast these equations in terms of expansion-normalised variables following the \((3+1)\)-dimensional analysis \([4]\). So we define \(\Sigma_{ab} \equiv \sigma_{ab}/\theta\), \(\Omega_{ab} \equiv \Omega_{ab}/\theta\), \(A_b \equiv a_b/\theta\), \((n) R_{(a)} \equiv (n) R_{ab}/\theta^2\) and introduce a new time variable, \(\tau\), by \(d\tau/dt \equiv \theta\). Then using Raychaudhuri\’s equation, the shear equations, the Friedmann constraint and the constraint implied by the Gauss-Codazzi equations, we find,

\[
\Sigma_{ab}' = \left(\Sigma^2 - \frac{n-1}{n}\right) \Sigma_{ab} - 2 \Sigma_{(a} \Omega_{b)c} - (n) R_{(ab)},
\]

\[
\left(1 - \frac{1}{n}\right) = \Sigma^2 - (n) R,
\]

\[
n \Sigma_{ac} A^a - \sigma_{ac} D^b_{ac} = 0,
\]

\[
\sigma_{ac} a^a - \sigma^a_b D^b_{ac} = 0.
\]

\(^2\)Note that \(\dot{\sigma}_{ab} = \frac{\theta}{\dot{\tau}} \sigma_{ab} + 2 \sigma_{(a} \Omega_{b)c} \).
where $' = d/d\tau$ and $\Sigma^2 \equiv \Sigma_{ab}\Sigma^{ab}$.

At this point it is important to note that our equations of motion look like $X' = F(X)$ where $X$ is a column vector containing each of the independent dynamical variables. This is an autonomous system of differential equations, which lends itself to a straightforward stability analysis. Suppose $X_0$ is an equilibrium point; i.e.

$$F(X_0) = 0.$$  

Consider perturbing about this, so $X = X_0 + \delta X$. Then one finds, that we can rewrite the system as

$$\delta X' = DF(X_0)\delta X + O(\delta X^2),$$

where $DF$ is the Jacobian matrix, i.e. the matrix of derivatives $\partial F_i/\partial X_j$. Hence, close to the equilibrium point, we can drop the quadratic terms in $\delta X$ so the stability depends only on the matrix $DF(X_0)$. More precisely, the stability criterion for this system is that if the eigenvalues all have negative real parts, then we have stability; zero eigenvalues need more analysis and the existence of a sole eigenvalue with positive real part implies instability.

In particular, if the equilibrium points are actually a space parameterised by $m$ variables (as is the case in the subsequent analysis), then perturbing inside this space, i.e. $F(X_0 + \delta X) = 0$, implies

$$DF(X_0)\delta X = 0;$$

namely, $\delta X$ is an eigenvector of $DF(X_0)$ with zero eigenvalue, and thus, $DF(X_0)$ will possess at least $m$ zero eigenvalues.

### 3 Homogeneous plane-waves

Plane-waves are solutions to Einstein’s equations possessing a covariantly constant null vector. Now, define the null directions $\eta^\pm = 1/\sqrt{2}(\omega^\pm + \omega^0)$, and similarly $e^\pm = 1/\sqrt{2}(e^\pm + e_0)$, so

$$g = 2\eta^+\eta^- + \delta_{AB}\omega^A\omega^B,$$  

(22)

where $A = 1, 2, \ldots, n-1$.

The existence of a null covariantly constant vector implies two things: firstly that it is a Killing vector and secondly that its metric dual $g(\xi, \cdot)$ is closed. We can choose $\xi$ to be parallel to $e_-$ so $\xi = fe_-$ where $f$ is a function. Choosing a left-invariant frame (which we are free to do) implies that $[\xi, e_A] = 0$ which gives,

$$\gamma^a_{A0} = \gamma^a_{An},$$  

(23)

$$e_A(f) = 0.$$  

(24)

Secondly, the metric dual to $\xi$, i.e. $f\eta^+$ is closed so $d(f\eta^+) = 0$ which gives,

$$\gamma^0_{ab} + \gamma^n_{ab} = 0,$$  

(25)

$$\Rightarrow C^a_{ab} = 0 \text{ due to orthogonality},$$  

(26)
together with a condition on \( f \) which we will not deal with. The wave front is required to be flat which tells us that there is an \((n - 1)\)-dimensional abelian subalgebra and this forces,

\[
C_{BC}^A = 0. \tag{27}
\]

This implies that \( a_B = \frac{1}{n-1} C_a^a C_{aB} = \frac{1}{n-1} C^n_{nB} = 0 \), which means \( a_b = a \delta_{bn} \) and upon using the evolution equation for \( a_b \), we get,

\[
\sigma^n_A = \Omega^n_A. \tag{28}
\]

Thus we are left with only the following non-zero components,

\[
C^A_{Bn} \equiv \Theta^A_{Bn}, \tag{29}
\]

\[
\gamma^A_{B0} = C^A_{Bn}. \tag{30}
\]

For every element \( \Theta \in \text{Mat}(n - 1, \mathbb{R}) \) we have a possible plane-wave solution. However there exists some redundancy; namely the group of rotations \( O(n - 1) \) will simply redefine our orthonormal frame \( \{ \omega^A \} \). Also, we are allowed an overall dilation, since the Einstein equations can be used to determine \( \Theta^A_{\text{\{B\}} = (n - 1)a} \). Thus the space of plane waves is isomorphic to \( \text{Mat}(n - 1, \mathbb{R})/[O(n - 1) \times \mathbb{R}] \); note the dimension of this space is \( \frac{1}{2} n(n - 1) - 1 \), see Appendix B. Now, the structure constants \( \gamma^a_{\text{\{b\}} = -\theta^a_b + \Omega^a_{\text{\{b\}}} \) imply the following constraint,

\[
\theta^A_B - \Omega^A_B = \Theta^A_B. \tag{31}
\]

We will define \( T_{AB} = \Theta_{\text{\{AB\}}} \), i.e the traceless part of \( \Theta \). Thus we have,

\[
\sigma_{AB} = T_{\text{\{AB\}}}. \tag{32}
\]

Finally, the expansion-normalised \( T_{AB} \) will be denoted by \( \tilde{T}_{AB} \).

### 4 General analysis in \( 4 + 1 \) dimensions

In this section we will describe the main results of the paper. In Appendix A the equations of motion can be found for the \( 4 + 1 \) case. We provide the corresponding plane-wave solutions and find that these spacetimes are indeed stable to perturbations in the geometry in the non-exceptional case, whereas unstable in the exceptional case. We should make a remark on exactly what sort of perturbations we are dealing with: they correspond to perturbations in a subspace of spatially homogeneous models, namely ones for which the structure constants are \( C^A_{B4} = \Theta^A_{B4} \) with the rest vanishing.\(^3\) Homogeneous plane-waves certainly belong to this class as we showed in the previous section, although it is not clear that all perturbations should lie in this space too. For now we assume so, however we will discuss this issue later.

It should also be emphasised that when we refer to stability we mean in the technical sense; i.e. the perturbations are bounded, and not in the asymptotic sense; i.e. all perturbations tend to zero at late times.

\(^3\) All spatially homogeneous cosmological models of dimension \( 4+1 \) are classified in [13]. We refer to this work for the other possibilities for the structure constants.
4.1 Parameterisation of independent variables

First of all we note that the constraint (20) leads to $\Sigma_{44} = 0$ in what we call the non-exceptional case (see Appendix A). We will use our freedom to rotate the orthonormal frame to put $T_{AB}$ in upper triangular form; note this can always be done if it has real eigenvalues using an orthogonal matrix. The complex eigenvalue case can also be done, but this time one needs a unitary matrix (see Appendix B). We parameterise the variables by,

$$
\Sigma_{AB} = \begin{pmatrix}
S - 2\Sigma_1 & \Sigma_{12} & \Sigma_{13} \\
\Sigma_{12} & S + \Sigma_1 + \sqrt{3}\Sigma_2 & \Sigma_{23} \\
\Sigma_{13} & \Sigma_{23} & S + \Sigma_1 - \sqrt{3}\Sigma_2
\end{pmatrix}
$$

(33)

$$
\tau_{AB} = \begin{pmatrix}
-2\tau_1 & \tau_{12} & \tau_{13} \\
0 & \tau_1 + \sqrt{3}\tau_2 & \tau_{23} \\
0 & 0 & \tau_1 - \sqrt{3}\tau_2
\end{pmatrix},
$$

(34)

where $\tau_{AB} \equiv \tilde{T}_{AB}/A$. Note that

$$
\Sigma^2 = 12S^2 + 6(\Sigma_1^2 + \Sigma_2^2) + 2(\Sigma_{12}^2 + \Sigma_{13}^2 + \Sigma_{23}^2).
$$

(35)

The independent variables left after elimination of the constraints (see Appendix A) are $\Sigma_{AB}$ (for $A \neq B$), $\Sigma_1, \Sigma_2$ and $\tau_{AB}$ (for $A < B$), $\tau_1, \tau_2$; the constraints are used to eliminate $A$ and $S$. Thus we have 10 independent variables. Note that the definition of $\tau_{AB}$ simplifies the equations of motion, in particular the diagonal components, which correspond to the group parameters (see Appendix B), are constants of motion. Thus we see a further simplification and hence we are left with effectively only 8 independent variables for each group type.

4.2 Exact solutions

Here we write down the general plane-wave solution to the equations of motion. These are equilibrium points to the equations of motion. We find,

$$
\tau_1 = p, \quad \tau_2 = q, \quad \tau_{12} = Q_1, \quad \tau_{13} = Q_2, \quad \tau_{23} = Q_3,
$$

$$
A = \frac{1}{2(p + 2)}, \quad S = -\frac{A}{2} \rho, \quad \Sigma_1 = \frac{1}{2} Ap, \quad \Sigma_2 = \frac{1}{2} Aq,
$$

$$
\Sigma_{12} = \frac{1}{2} A Q_1, \quad \Sigma_{13} = \frac{1}{2} A Q_2, \quad \Sigma_{23} = \frac{1}{2} A Q_3,
$$

where

$$
\rho = p^2 + q^2 + \frac{1}{12}(Q_1^2 + Q_2^2 + Q_3^2),
$$

(36)

and $p, q, Q_1, Q_2, Q_3$ are constants. Their explicit metrics are given in [14], however, these are not needed in this analysis. Our family of plane waves is thus parameterised by five constants. Note that this means we expect the Jacobian matrix to have three zero eigenvalues, since we have already discarded the variables $\tau_1$ and $\tau_2$ on account of them being constants of motion. These three eigenvalues correspond to eigenvectors tangential to the set of exact plane-wave solutions.
4.3 Stability

The equations of motion form an autonomous system of first order differential equations, and as previously explained, one simply needs the eigenvalues of the Jacobian matrix evaluated at the point we are perturbing about. Upon linearising our system we find the following eigenvalues:

\[
\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0
\]
\[
\lambda_4 = -\frac{3}{2(\rho + 2)}, \quad \begin{bmatrix}
\lambda_5 \\
\lambda_6
\end{bmatrix} = -\frac{3 \pm 2i\sqrt{Q_1^2 + Q_2^2 + Q_3^2}}{2(\rho + 2)},
\]
\[
\begin{bmatrix}
\lambda_7 \\
\lambda_8
\end{bmatrix} = -\frac{3 \pm i\sqrt{Q_1^2 + Q_2^2 + Q_3^2}}{2(\rho + 2)}.
\]

(37)

Note that, ignoring the zero eigenvalues, we have 5 eigenvalues with manifestly negative real parts (which, curiously are also all equal). Thus in the non-exceptional case we have stability. The zero eigenvalues were expected and correspond to moving in the space of plane-wave solutions.

4.4 Exceptional case

This occurs when \( T^B_A + 4a\delta^B_A \) is not invertible, and \( a \neq 0 \). Therefore the constraint leads to \( \sigma_{4B}(\tau^B_A + 4a\delta^B_A)A = 0 \) which implies \( 2\sigma_{41}A(2 - \tau_1) = 0 \) since \( \tau \) is upper triangular. Therefore we see the exceptional case corresponds to the particular group types where \( \tau_1 = 2 \), which allows for \( \sigma_{41} = \Omega_{41} \neq 0 \) in general. Let us restrict to \( \sigma_{24} = \sigma_{34} = 0 \). The equation of motion for \( \Sigma_{41} \) is,

\[
\Sigma'_{41} = \left( \Sigma^2 - \frac{3}{4} - 4S + 2\Sigma_1 \right) \Sigma_{41}.
\]

(38)

Note that the exact solution is unchanged, since \( \Sigma_{41} = 0 \) together with the exact solution in the non-exceptional case solves all equations of motion. If we evaluate \( \Sigma^2 - \frac{3}{4} - 4S + 2\Sigma_1 \) at the exact plane wave solution, it is easy to show that it is always greater that \( 3/(\rho + 2)^2 \). Now, linearising the equation for \( \Sigma_{41} \) we get,

\[
\delta\Sigma'_{41} = \lambda\delta\Sigma_{41},
\]

(39)

where \( \lambda > \frac{3}{(\rho + 2)^2} \).

Therefore we have found an instability.

5 Matter perturbations

Having shown that the homogeneous plane waves are stable to geometric perturbations in \((4 + 1)\) dimensions, it is desirable to test their sensitivity to the insertion of matter. It proves convenient to do so in \((n + 1)\) dimensions. We begin with a stress-energy tensor for a perfect fluid,

\[
T_{\mu\nu} = \rho n_{\mu} n_{\nu} + P h_{\mu\nu}.
\]

(41)
From the Einstein equations, one finds

\[ R_{\mu\nu} n^\mu n^\nu = \kappa \left( \frac{n-1}{n-2} \rho + \frac{n}{n-1} P \right). \tag{42} \]

We will consider matter satisfying the barotropic equation of state \( P = (\gamma - 1) \rho \). In this case the conservation of energy implies

\[ \dot{\rho} + \gamma \rho \theta = 0. \tag{43} \]

Raychaudhuri's equation now becomes

\[ \dot{\theta} + \frac{1}{n} \theta^2 + \sigma^2 + \kappa \rho (n \gamma - 2 n - 1) = 0. \tag{44} \]

We now define an expansion-normalised density, \( \Omega \equiv \rho \theta^2 \). We can then compute its derivative with respect to \( \tau \), and using (44) we obtain

\[ \Omega' = \Omega \left( -\gamma + 2n + 2 \Sigma^2 + 2 \kappa \Omega (n \gamma - 2) n - 1 \right). \tag{45} \]

To linear order, then, recalling that we are perturbing about the vacuum, we arrive at

\[ \delta \Omega' = \delta \Omega \left( -\gamma + 2n + 2 \Sigma^2_0 \right). \tag{46} \]

Here \( \Sigma^2_0 \) corresponds to the vacuum plane-wave solution we are perturbing about. We immediately deduce that these spacetimes are stable against such matter perturbations provided that

\[ 2n + 2 \Sigma^2_0 < \gamma, \tag{47} \]

for \( n = 4 \). This follows from the fact that the linearised equation for \( \delta \Omega \) only depends on \( \delta \Omega \) and thus the eigenvalues of the Jacobian matrix are unaltered. Given that \( \Sigma^2 \) is a positive-definite quantity, it becomes clear demanding stability in this model is slightly more restrictive than simply requiring the Strong Energy Condition, namely \( \gamma > 2n \), to hold. Note that the case \( \gamma = 0 \), which is equivalent to having a cosmological constant, is always unstable. On the other hand, if \( \gamma = 1 \) as it does for dust, we can always find a stable plane wave.

6 The Milne universe

In this section we will use the equations set up in the first section to analyse the stability of the Milne universe in an arbitrary dimension. This spacetime can be defined as one with \( \tilde{T}^A_B = \Sigma_{AB} = 0 \); note that this implies \( A^2 = 1/n^2 \). Explicitly the metric, in horospherical coordinates, looks like,

\[ ds^2 = -dt^2 + t^2 \left( dz^2 + e^{2z} \sum_{i=1}^{n-1} (dx_i^2) \right). \tag{48} \]

If we perturb about this solution we get the linearised equations,

\[ \delta \Sigma'_{AB} = -\frac{n-1}{n} \delta \Sigma_{AB} + (n-1) A \delta \tilde{T}_{(AB)}, \tag{49} \]

\[ \delta \tilde{T}'_{AB} = 0, \tag{50} \]
where the $' = d/d\tau$ since we are working at linearised level. We see that this system of autonomous equations gives rise to a block diagonal Jacobian matrix whose eigenvalues are zero or negative. The zeros actually correspond to moving towards another plane-wave solution. Therefore the Milne universe is stable in the sense that the perturbations remain bounded. To see the origin of the zero eigenvalues, write,

$$
\begin{pmatrix}
\frac{\delta \Sigma_{AB}}{\delta T_{(AB)}} \\
\frac{\delta \Sigma_{AB}}{\delta \tilde{T}_{(AB)}}
\end{pmatrix}' = \begin{pmatrix}
-\frac{a-1}{n} & \frac{a-1}{n} \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\delta \Sigma_{AB} \\
\delta \tilde{T}_{(AB)}
\end{pmatrix}
$$

(51)

where we have substituted $A = 1/n$ which is true for the Milne universe ($A = -1/n$ can be excluded since we can pick its orientation). Therefore, the zero eigenvalue corresponds to $\delta \Sigma_{AB} = \delta \tilde{T}_{(AB)}$, and thus we see that we do not move off the surface $\Sigma_{AB} = \tilde{T}_{(AB)}$ which contains our plane waves as we saw earlier.

Interestingly, a simple argument allows us to say something about the stability of plane-waves in arbitrary dimensions from the above analysis. The Milne universe is the $\Sigma^2 \rightarrow 0$ limit of the set of plane-wave solutions. Observing that the Jacobian matrix is a continuous function of the state space variables (which implies that its eigenvalues have to be so too) we can deduce that: \textit{provided that eq. (47) holds, there will exist an open set of plane-wave solutions which are stable to spatially homogeneous perturbations.}

## 7 Discussion

We have found that the stability analysis of vacuum spatially homogeneous plane-waves in $4+1$ dimensions separate into two cases. For the non-exceptional case they are stable to (spatially homogeneous) perturbations in the geometry. They are also stable to matter perturbations provided $\gamma > 1/2 + 2\Sigma^2_0$. This, of course, excludes inflationary fluids.\footnote{The cosmological no-hair theorem tells us that in dimension $4+1$ for $\gamma < 1/2$ (e.g. inflationary type fluids) the late-time asymptotes are the flat FRW universe, which is consistent with our matter perturbation results.} The exceptional case, however, is always unstable. This result is completely analogous to the $(3 + 1)$-dimensional case where a similar instability for the plane-wave solution for the exceptional Bianchi type VI$^*$ exists.

It should be noted, however, if one allows for perturbations with respect to group types then two subtleties may occur. For some group types, even though they cannot be written as $C^A_{B4} = \Theta^A_{B}$, there exist Lie algebra contractions for which the limit is an algebra of this type \cite{16}. Also, for some special solutions an extra Killing vector may appear which may allow for simply transitive symmetry groups other than those considered here \cite{17}. In any case, these possibilities only arise for a set of measure zero in the set of plane-wave solutions.

Also, we found that the Milne universe in arbitrary dimensions is stable provided $\gamma > \frac{2}{n}$, which is the strong energy condition for a perfect fluid. The implications of this are that in arbitrary dimensions, there always exists a neighbourhood of stable solutions (around the Milne solution).

Of course, one can consider more general types of fluids than those considered here; namely tilted (non-comoving) fluids, anisotropic stresses, dissipative...
fluids, $p$-form fluxes, etc. For example, if one takes a tilted perfect fluid, then generalising the arguments in [18] indicates that non-comoving velocities may have a significant effect on the late-time evolution of cosmological models.\(^5\) An interesting extension of our work would be to include a systematic study of these more general fluids.

Based on the above analysis, one is tempted to believe that all higher-dimensional plane-wave spacetimes, except for the exceptional models, within the class considered here, are stable for all spatially homogeneous vacuum perturbations. We have already shown that some stable plane-wave spacetimes exist in any dimension, and the stability has explicitly been shown for the (3+1)- and (4+1)-dimensional cases. However, whether this is true in all dimensions awaits to be seen.

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**A Equations of motion in 4 + 1 dimensions**

Here follows the differential equations and constraints used in the analysis of this paper. We will concern ourselves mainly with the expansion-normalised variables:

\[
\Sigma_{AB} \equiv \sigma_{AB}\theta \\
\tilde{T}_{AB} \equiv T_{AB}\theta \\
S \equiv \sigma^A_A 3\theta \\
A \equiv a\theta
\]

Latin capital indices run from 1...3, the coordinates of the wave front. Further we have the following equations

\[
4\sigma_A a + \sigma^B_B T^B_A = 0 \\
\sigma^B_B (3a\delta^A_B + \Theta^B_A) = 0 \\
4\sigma_{44} a - \sigma^B_B T^B_A = 0.
\]

In the non-exceptional case (defined as $\Theta^A_B + 3a\delta^A_B$ being invertible) we infer from the second line above $\sigma^A_A = 0 = \Omega^A_A$. Recall that under our parameterisation, $\sigma_{44} = -3s$. The last of the above equations then reduces to, using (52)

\[
-12SA - \Sigma_{AB}\tilde{T}^{BA} = 0.
\]

Under the useful field redefinition, $\tau_{AB} \equiv \tilde{T}_{AB} A$, we can solve for $S$

\[
S = -\Sigma_{AB} \tau^{BA} 12.
\]

\(^5\)A quick calculation shows that conservation of angular momentum in universes close to isotropy implies that the peculiar velocities are growing at late times if $\gamma > 1 + 1/n$. 

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Although the function $S$ appears in our parameterisation of the $\Sigma_A^B$, it will not appear on the right hand side on account of the tracelessness of $\tau_A^B$.

The introduction of the $\tau$ variable also simplifies the (first order) equations of motion, which we now produce for convenience:

\[
A' = (3\mathcal{S} + \Sigma^2)A \\
\tau_1' = \tau_2' = 0 \\
\tau_2' = \left(-3\Sigma_1 - \sqrt{3}\Sigma_2\right)\tau_2 + 2\Sigma_{12} \left(3\tau_1 + \sqrt{3}\tau_2\right) \\
\tau_3' = -\left(3\Sigma_1 - \sqrt{3}\Sigma_2\right)\tau_3 + 2\Sigma_{13} \left(3\tau_1 - \sqrt{3}\tau_2\right) - 2\Sigma_{23}\tau_1 + 2\Sigma_{12}\tau_3 \\
\Sigma_{AB}' = (\Sigma^2 - 34)\Sigma_{AB} - \chi_{AB}
\]  

(56)

where the symmetric matrix $\chi_{AB}$ is defined to be

\[
\begin{align*}
\chi_{11} &= 6A^2\tau_1 - 2\Sigma_{12}^2 - 2\Sigma_{13}^2 + 58A^2(\tau_{12}^2 + \tau_{13}^2) + 32A^2(\tau_1^2 + \tau_2^2) + A^2\tau_{23}^2 \\
\chi_{12} &= -3A^2\tau_2 - 3\Sigma_1\Sigma_{12} - \sqrt{3}\Sigma_2\Sigma_{12} - 2\Sigma_{13}\Sigma_{23} + 32A^2\tau_1\tau_2 + A^2\tau_{13}\tau_{23} \\
\chi_{13} &= -3A^2\tau_3 - 3\Sigma_1\Sigma_{13} + \sqrt{3}\Sigma_2\Sigma_{13} + 32A^2\tau_1\tau_3 - \sqrt{3}A^2\tau_{13}\tau_2 \\
\chi_{22} &= -3A^2(\tau_1 + \sqrt{3}\tau_2) + 2\Sigma_{12}^2 - 2\Sigma_{23}^2 - 38A^2\tau_{12}^2 + 58A^2\tau_{23}^2 + 32A^2(\tau_1^2 + \tau_2^2) + 18A^2\tau_{13}^2 \\
\chi_{23} &= -32A^2\tau_{23} + 2\Sigma_{12}\Sigma_{13} + 2\sqrt{3}\Sigma_{23}\Sigma_2 - 12A^2\tau_{12}\tau_{13} - \sqrt{3}A^2\tau_{23}\tau_2 \\
\chi_{33} &= -3A^2\tau_1 + 3\sqrt{3}A^2\tau_2 + 2\Sigma_{13}^2 + 2\Sigma_{23}^2 - 38A^2(\tau_{13}^2 + \tau_{23}^2) + 32A^2(\tau_1^2 + \tau_2^2) + 18A^2\tau_{12}^2.
\end{align*}
\]  

(57)

In addition we have the Friedmann constraint, which allows us to eliminate $A$ from the equations:

\[
38 = 12\Sigma^2 + A^2(\tau^2 + 6),
\]

(58)

where $\tau^2 \equiv \tau^{(AB)}\tau_{(AB)}$. This leads unambiguously to

\[
A^2 = 34 - \Sigma^2\tau^2 + 12.
\]

(59)

Note that the first equation of (56) was used to derive the subsequent equations for the $\tau_A^B$ so it is included automatically. Thus, $S$ and $A$ have been removed and given $\tau_1$ and $\tau_2$ (the group type) we are left with eight dynamic variables.

\section{B Some results on $n$-dimensional Lie algebras}

Two Lie algebras, given by the structure constants $C^k_{ij}$ and $\tilde{C}^k_{ij}$, are isomorphic (over $\mathbb{R}$) if and only if there exist a matrix $A \in GL(n, \mathbb{R})$ such that

\[
\tilde{C}^k_{ij} = (A^{-1})^k_l C^l_{pq}A^p_i A^q_j.
\]

(60)
We are interested in the Lie algebras where
\[ C^A_{BC} = 0, \quad C^A_{Bn} = \Theta^A_B. \] (61)
For this class of algebras, the transformation (60) can be written in terms of \( \Theta^A_B \):
\[ \tilde{\Theta}^A_B = (A^{-1})^A_B \Theta^A_B A^B_A \phi, \] (62)
in some suitable new orthogonal coordinates (marked with tildes), and \( \phi \) is a non-zero constant. Hence, the Lie algebras are determined in this case by equivalence classes of the map (in matrix form)
\[ \Theta \mapsto \phi A^{-1} \Theta, \quad A \in GL(n-1, \mathbb{R}), \quad \phi \in \mathbb{R} - \{0\}. \] (63)
The \( \phi \) allows us to do a global rescaling of \( \Theta \), while the conjugation leaves the characteristic equation,
\[ \det(\Theta - \lambda I) = 0, \] (64)
invariant. Hence, the Lie algebras can be given in terms of the eigenvalues of \( \Theta \) modulo a constant rescaling. If \( \lambda_i \) are the eigenvalues, and \( |\lambda_1| \neq 0 \), say, then the \((n-2)\)-tuple,
\[ \Lambda = \left( \frac{\lambda_2}{|\lambda_1|}, \ldots, \frac{\lambda_{n-1}}{|\lambda_1|} \right), \quad |\lambda_i| \geq |\lambda_{i+1}| \] (65)
will be invariant under the transformation (63). This means that different \( \Lambda \) yields non-isomorphic Lie algebras. If all eigenvalues are distinct, then \( \Lambda \) will uniquely determine the Lie algebra. However, in the degenerate cases, where two or more eigenvalues are equal, we have to use the Jordan canonical form to determine the Lie algebra. From this one sees that in general the space of distinct Lie algebras within the class given in eq.(61), is of dimension \((n-2)\).

As an example, consider the 4-dimensional real indecomposible Lie algebras [19–21]. Here, the general types are the two two-parameter families \( A^{pq}_{4,5} \) (three real eigenvalues) and \( A^{pq}_{4,6} \) (one real and two complex conjugate eigenvalues). The algebras \( A_{4,1-4,4} \) are the degenerate cases with non-trivial Jordan canonical form; \( A_{4,1} \) has three zero eigenvalues; \( A_{4,2} \) has two equal and non-zero eigenvalues; \( A_{4,3} \) has two zero eigenvalues; and \( A_{4,4} \) has three equal non-zero eigenvalues.

In our case, \( \Theta \) is a real matrix. This implies that the eigenvalues \( \lambda_i \) are either real, or come in complex conjugate pairs. Due to the gauge symmetry of determining the orientation of the frame \( \{\omega^A\} \), we are allowed to perform a \( O(n-1) \) rotation of the spatial frame to simplify the matrix \( \Theta \).

Let us first assume that all the eigenvalues \( \lambda_i \) are real. Then for any \( \Theta \in \text{Mat}(n-1, \mathbb{R}) \) there exist an \( R \in O(n-1) \) such that \( R^{-1} \Theta R \) is an upper triangular matrix. Hence, by a suitable choice of frame we can assume that \( \Theta \) is upper triangular with its eigenvalues along the diagonal:
\[ \Theta = \begin{pmatrix}
\lambda_1 & \Theta^1_2 & \cdots & \Theta^1_{n-1} \\
0 & \lambda_2 & \cdots & \Theta^2_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{n-1}
\end{pmatrix}. \] (66)
Note that, as explained earlier, the ratio of the diagonal elements determine the group type.

If some of the eigenvalues are complex, then it is clearly not possible to perform a real rotation so that the $\Theta$ is on upper triangular form. However, using a little trick, we can still put it into an upper triangular form. For any $\Theta \in \text{Mat}(n - 1, \mathbb{R})$ there exist a $U \in U(n - 1)$ such that $U^\dagger \Theta U$ is an upper triangular matrix. Using unitary matrices we can put it into the form (66) with the eigenvalues along the diagonal. However, using a unitary “rotation” we have to be careful about interpreting the dynamical variables, $\sigma^A_B$, $T^A_B$ and $\Omega^A_B$. In matrix form, the unitary transformation implies the following transformation on $\sigma^A_B$, $T^A_B$:

$$\sigma \mapsto U^\dagger \sigma U, \quad T \mapsto U^\dagger TU,$$

and $\Omega^A_B$ have to be reinterpreted as a $u(n - 1)$-connection:

$$\Omega \mapsto U^\dagger \frac{d}{dt} U + U^\dagger \Omega U.$$  

Furthermore, the equations of motion and the constraints have to be amended similarly:

$$\sigma^A_B T^B_A \mapsto \text{Tr}(\sigma^A_T), \quad \sigma^A_B \sigma^A_B \mapsto \text{Tr}(\sigma^A_T),$$

etc. The equations of motion can now be solved in a similar manner.

For the plane-wave spacetimes, we can use this to find the dimension of the space of plane-wave solutions. Basically, a plane-wave spacetime is given by specifying the upper-triangular trace-free matrix $T$. All other dynamical variables can be given, due to the Einstein equations and constraints, and the plane-wave criterion, in terms of $T$. Hence, there is an $(n-2)(n+1)/2$-parameter family of homogeneous plane-wave solutions of which $(n-2)$ parameters specify the Lie algebra type.

References


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