ON CERTAIN PROPERTIES OF SOME GENERALIZED SPECIAL FUNCTIONS

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Abstract

In this paper, we derive a result concerning eigenvector for the product of two operators defined on a Lie algebra of endomorphisms of a vector space. The results given by Radulescu, Mandal and authors follow as special cases of this result. Further using these results, we deduce certain properties of generalized Hermite polynomials and Hermite Tricomi functions.

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1. Introduction

The interplay between differential equations, generalized special functions and Lie theory is particularly useful in applications. The theory of generalized special functions has witnessed a rather significant evolution during the last years (see Dattoli et al. [1,2,3,4]). In applicative fields, we note that for some physical problems the use of new classes of special functions provides solutions hardly achievable with conventional analytical and numerical means.


In this paper, we establish a result concerning eigenvector for the product of two operators defined on a Lie algebra of endomorphisms of the vector space \( V \). The results given by Radulescu [8], Mandal [6] and Pathan and Khan [7] follow as special cases of this result.

Further, we extend the approach of Radulescu and Mandal [8,6] to deduce some properties of two variable Hermite-Kampé de Fériét polynomials (TVHKdFP) and Hermite-Tricomi functions (HTF). The analytic methodology developed in this paper can easily be adopted to the study of some other special functions of mathematical physics.

The TVHKdFP \( H_n(x, y) \) [2] are specified by the series

\[
H_n(x, y) = \sum_{r=0}^{[n/2]} \frac{n!y^r x^{n-2r}}{r!(n-2r)!},
\]

and the generating function for \( H_n(x, y) \) is given by

\[
\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = \exp(xt + yt^2).
\]

The polynomials \( h_n(x, y; \xi) \) [2] generated by

\[
\sum_{n=0}^{\infty} h_n(x, y; \xi) \frac{t^n}{n!} = \exp(2xt - t^2 + 2y\xi t - \xi^2 t^2),
\]

can be written in terms of \( H_n(x, y) \) as follows:

\[
h_n(x, y; \xi) = H_n(2(x + y\xi), -(1 + \xi^2)).
\]
\[
\sum_{n=0}^{\infty} \frac{v^n H_{jn+m}(x)}{(pn)!}.
\]

Further we note that the HTF \( HC_n(x, y) \) are defined by the series ([4]; p.24 (36 b))

\[
HC_n(x, y) = \sum_{s=0}^{\infty} \frac{(-1)^s H_s(x, y)}{s!(n+s)!},
\]

and the generating function for \( HC_n(x, y) \) is given as ([4]; p.25 (47))

\[
\sum_{n=-\infty}^{\infty} HC_n(x, y)t^n = \exp\left(t - \frac{x}{t} + \frac{y}{t^2}\right).
\]

For \( y = 0 \), the HTF \( HC_n(x, y) \) reduces to the Tricomi functions \( C_n(x) \), which are linked to the ordinary Bessel functions by the relation [1]

\[
C_n(x) = x^{-n/2} J_n(2\sqrt{x}).
\]

2. Main Theorem

Let \( \text{End} \, V \) be the Lie algebra of endomorphisms of the vector space \( V \), endowed with the Lie bracket \([\cdot, \cdot]\) defined by \([A, B] = AB - BA\), for every \( A, B \in \text{End} \, V \). We denote by \( I \) the identity operator of \( V \).

The main theorem of the paper is as follows.

**Theorem - 1:** Let \( A, B \in \text{End} \, V \) be such that \([A, B]y_n = (a(2n+1)+b)y_n\), where the sequence \((y_n)_n \subset V \) is defined as follows: \( Ay_1 = y_0 \) and \( By_n = \left(\frac{(a(n^2+2n+2)+bn+1)}{(an+bn+1)}\right)y_{n+1} \), for every \( n \geq 0 \). Then \( Ay_{n+1} = (an+bn+1)y_n \), and \( y_n \) is an eigenvector of eigenvalue \((a(n^2-1)+b(n-1)+1)\) for \( BA \), for every \( n \geq 1 \).

**Proof.** First we show that

\[
Ay_{n+1} = (an+bn+1)y_n, \text{ for every } n \geq 1.
\]

We have

\[
[A, B]y_1 = (3a+b)y_1,
\]

\[
A(By_1) - B(Ay_1) = (3a+b)y_1.
\]

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Also, \(Ay_1 = y_0\), \(By_0 = y_1\) and \(B y_1 = \frac{(3a + b + 1)}{(a + b + 1)} y_2\), and therefore \(A y_2 = (a + b + 1) y_1\).

Next, suppose that \(A y_{n+1} = (a n + b n + 1) y_n\). Now since we may write

\[
[A, B] y_{n+1} = (a (2n + 3) + b) y_{n+1},
\]

\[
A(B y_{n+1}) - B(A y_{n+1}) = (a (2n + 3) + b) y_{n+1},
\]

\[
\frac{(a ((n + 1)^2 + 2(n + 1)) + b(n + 1) + 1)}{(a(n + 1) + b(n + 1) + 1)} A y_{n+2} - (a n + b n + 1) B y_n = (a (2n + 3) + b) y_{n+1},
\]

\[
\frac{(a(n^2 + 4n + 3) + b(n + 1) + 1)}{(a(n + 1) + b(n + 1) + 1)} A y_{n+2} = (a(n^2 + 4n + 3) + b(n + 1) + 1) y_{n+1},
\]

\[
A y_{n+2} = (a(n + 1) + b(n + 1) + 1) y_{n+1}.
\]

Hence by mathematical induction \(A y_{n+1} = (a n + b n + 1) y_n\), for every \(n \geq 1\). It follows that

\[
(BA) y_n = (a(n - 1) + b(n - 1) + 1) B y_{n-1},
\]

i.e.

\[
(BA) y_n = (a(n^2 - 1) + b(n - 1) + 1) y_n.
\]

Hence \(y_n\) is an eigenvector of eigenvalue \((a(n^2 - 1) + b(n - 1) + 1)\) for \(BA\), for every \(n \geq 1\).

3. Deductions and Applications of Theorem 1.

We note that, for \(a = 0, b = 1\), Theorem-1 yields the main result of Radulescu ([8]; p.67 (Theorem-1)). Again for \(a = 0, b = 0\), it yields the main result of Mandal ([6]; p. 273 (Theorem-1)). And finally for \(a = 1\) and \(b = 0\), it gives the main result of Pathan and Khan [7].

First, we consider the case, when \(a = 0\) and \(b = 1\), that is, we recall the following main theorem of Redulescu [8]:

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Theorem-2: Let $C, D \in \text{End} \ V$ be such that $[C, D] = I$. We define the sequence $(Z_n)_n \subset V$ as follows: $CZ_0 = 0$ and $Z_n = DZ_{n-1}$, for every $n \geq 1$. Then $CZ_n = nZ_{n-1}$ and $Z_n$ is an eigenvector of eigenvalue $n$ for $DC$, for every $n \geq 1$.

Let $V = C^\infty (\mathbb{R} \times \mathbb{R})$. We define the operators $C, D \in \text{End} \ V$ by

\[
Cf(x, y) = f'(x, y),
\]
\[
Df(x, y) = 2yf'(x, y) + xf(x, y),
\]
for every $(x, y) \in \mathbb{R} \times \mathbb{R}$, and where $f'$ denotes $\frac{\partial f}{\partial x}$. It can easily be seen that these operators satisfy the commutation relation $[C, D] = I$.

Next, we prove that

\[
D^n f(x, y) = (-1)^n(-y)^{n/2} \left[ e^{t^2} (f(x, y)e^{-t^2})^{(n)} \right] \bigg| _{t = \frac{x}{2\sqrt{-y}}}.
\]

(3.2)

From the definition of $D$, the above equality holds for $n = 1$. Inductively, taking into account $D^{n+1} f(x, y) = D(D^n f(x, y))$, it follows that

\[
D^{n+1} f(x, y) = (-1)^{n+1}(-y)^{(n+1)/2} \left[ e^{t^2} (f(x, y)e^{-t^2})^{(n+1)} \right] \bigg| _{t = \frac{x}{2\sqrt{-y}}},
\]

which ends our proof.

The TVHKdFP, $H_n(x, y)$ satisfies the following differential equation

\[
2yZ''(x, y) + xZ'(x, y) - nZ(x, y) = 0,
\]

(3.3)

where $n$ is a positive integer.

Equation (3.3) may be written as

\[
2yZ''(x, y) + xZ'(x, y) = nZ(x, y),
\]

which by using (3.1) reduces to

\[
DZ'(x, y) = nZ(x, y).
\]

Therefore

\[
DCZ(x, y) = nZ(x, y).
\]

By Theorem-2, it follows that $Z_n(x, y)$ is a solution of the differential equation (3.3). Setting $Z_0(x, y) = 1$, we obtain $Z_n(x, y) = D^n(1)$. Therefore defining $H_n(x, y) = Z_n(x, y)$, we deduce the Rodrigues-type formula

\[
H_n(x, y) = (-1)^n(-y)^{n/2} \left[ e^{t^2} \left( \frac{d}{dt} \right)^n e^{-t^2} \right] \bigg| _{t = \frac{x}{2\sqrt{-y}}}.
\]
Proceeding in this manner we are able to deduce other properties of TVHKdFP $H_n(x, y)$.

Since $H_n(x, y) = Z_n(x, y) = D^n(1)$, we can use the equality $CZ_n = nZ_{n-1}$. Now from the definition of the operator $C$, it follows that

$$Z'_n(x, y) = nZ_{n-1}(x, y),$$

(3.4)
i.e.

$$\frac{\partial}{\partial x} H_n(x, y) = nH_{n-1}(x, y),$$

Again, by the definition of the operator $D$, we have

$$DZ_n(x, y) - 2yZ'_n(x, y) - xZ_n(x, y) = 0,$$

which by using (3.4) can be written as

$$Z_{n+1}(x, y) - 2yZ_{n-1}(x, y) - xZ_n(x, y) = 0,$$

(3.5)
or equivalently

$$H_{n+1}(x, y) + xH_n(x, y) - 2yH_{n-1}(x, y) = 0.$$

Further, we may derive corresponding properties for $h_n(x, y; \xi)$ using (1.4).

Next, we consider the case when $a = 0$ and $b = 0$, that is, we consider the following theorem of Mandal [6]:

**Theorem-3:** Let $E, F \in End V$ be such that $[E, F] = 0$. We define the sequence $(W_n)_n \subset V$ as follows: $EW_1 = W_0$ and $W_{n+1} = FW_n$ for every $n \geq 0$. Then $EW_{n+1} = W_n$ for every $n \geq 1$ and $W_n$ is an eigenvector of eigenvalue 1 for $FE$ for every $n \geq 1$.

Let $V = C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$. We define the operators $E, F \in End V$ by

$$Eu(x, y, t) = \frac{\partial}{\partial x} u_x + \frac{2y}{t} u_y + u_t,$$

$$Fu(x, y, t) = -t u_x,$$

(3.6)

for every $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, where $u_x$, $u_y$ and $u_t$ denote $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t}$ respectively. It can be easily seen that these operators satisfy the commutation relation $[E, F] = 0$.

Now if $u(x, y, t)$ assumes the form $W_n(x, y, t) = V_n(x, y)t^n \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, then $FW_n = W_{n+1} \implies EW_{n+1} = W_n$ by virtue of Theorem-3. Now the relation $FW_n = W_{n+1}$ yields
\[ \left(-t \frac{\partial}{\partial x}\right) (V_n(x,y) t^n) = V_{n+1}(x,y) t^{n+1}, \]
i.e.
\[ \frac{\partial}{\partial x} V_n(x,y) = -V_{n+1}(x,y). \]  
(3.7)

Again the relation \( EW_{n+1} = W_n \) yields
\[ \left( x \frac{\partial}{\partial x} + \frac{2y}{t} \frac{\partial}{\partial y} + \frac{\partial}{\partial t} \right) (V_n(x,y) t^n) = V_{n-1}(x,y) t^{n-1}, \]
i.e.
\[ \left( x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + n \right) V_n(x,y) = V_{n-1}(x,y), \]  
(3.8)

which by using (3.7) can be written as
\[ xV_{n+1}(x,y) - (2y \frac{\partial}{\partial y} + n)V_n(x,y) + V_{n-1}(x,y) = 0, \]  
(3.9)

Again by virtue of Theorem-3, we can write

\[ FEW_n(x,y,t) = W_n(x,y,t), \]
i.e.
\[ \left(-t \frac{\partial}{\partial x}\right) \left( x \frac{\partial}{\partial x} + \frac{2y}{t} \frac{\partial}{\partial y} + \frac{\partial}{\partial t} \right) (V_n(x,y) t^n) = V_n(x,y) t^n, \]
i.e.
\[ \left( x \frac{\partial^2}{\partial x^2} + 2y \frac{\partial^2}{\partial x \partial y} + (n+1) \frac{\partial}{\partial x} + 1 \right) V_n(x,y) = 0. \]  
(3.10)

Now it is evident from the differential equation (3.10) that HTF \( H_n(x,y) \) (1.5)-(1.6) is a solution of the above differential equation. It is interesting to note that (3.7), (3.8) and (3.9) are differential recurrence relations satisfied by \( H_n(x,y) \).

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