A More Improved Lattice Action for Heavy Quarks

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\textsuperscript{a}We extend the Fermilab formalism for heavy quarks to develop a more improved action. We give results of matching calculations of the improvement couplings at tree level. Finally, we estimate the discretization errors associated with the new action.

1. INTRODUCTION

One source of uncertainty in lattice QCD comes from the discretization effects in numerical simulations. Here we extend the Fermilab formalism to higher dimensional operators to obtain a more improved action for heavy quarks, aiming to reduce this uncertainty to 1–2%.

The method starts with Symanzik \cite{1}, who introduced a local effective Lagrangian

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD}} + \sum_j a^{\dim} \mathcal{L}_j - 4K_j \mathcal{L}_j \]  

of a continuum field theory. The $\mathcal{L}_j$ are higher-dimensional operators. The lattice action is adjusted so that the short-distance $K_j$ coefficients vanish, at least approximately.

The Fermilab formalism \cite{4} takes Wilson fermions \cite{2} with the clover action \cite{3} and introduces different couplings for space-like and time-like interactions. Ref. \cite{4} considers operators up to dimension five. Ref. \cite{5} adds dimension-six operators to this action. In this paper, we give a more complete action which has dimension six and some dimension seven operators and determine the couplings by matching at tree level.

2. QUARK BILINEARS

In this section, we consider the quark bilinears. The dimension-five (-six) interactions are listed in Ref. \cite{4} (Ref. \cite{5}). Redundant directions are identified by field transformations. They are used to eliminate the higher-order time derivatives, which would give large errors for large masses. We do not repeat this analysis here.

The corrections in Eq. (1) should be classified by a power-counting scheme. For light quark physics the only relevant ratio is $\Lambda$. For heavy quark physics, however, the power counting should follow HQET for heavy-light systems or NRQCD for quarkonium. In HQET, operators are assigned a power of $\lambda = \Lambda/m_Q$; in NRQCD, a power of relative velocity $v$. In these power-counting schemes, some dimension-seven interactions are commensurate with dimension-six interactions. As seen in the list in Table 1, they scale as $\lambda^3$ or $v^6$. Further dimension-seven operators carry power $\lambda^4$ (or higher) in HQET and $v^8$ (or higher) in NRQCD.

Including the needed dimension-seven operators, our lattice action takes the form

\[ S_F = S_0 + S_B + S_E \]

\[ + a^2c_1 \int \bar{\psi} \left( \gamma \cdot D, D^2 \right) \psi + a^2c_2 \int \bar{\psi} \gamma_1 D^2 \psi \]

\[ + a^3r_3 \int \bar{\psi} \left( D^2 \right)^2 \psi + a^3r_4 \int \bar{\psi} D^4 \psi \]

\[ + a^2c_5 \int \bar{\psi} \left( \gamma \cdot D, i \Sigma \cdot B \right) \psi \]

\[ + a^2c_6 \int \bar{\psi} \left( D \cdot E - E \cdot D \right) \psi \]

\[ + a^2c_7 \int \bar{\psi} \gamma \cdot \left( D \times B + B \times D \right) \psi \]

\[ + a^2c_8 \int \bar{\psi} \left( \gamma \cdot D, \alpha \cdot E \right) \psi \]
Table 1
Dimension six and seven bilinear interactions in $\mathcal{L}_{\text{eff}}$ with HQET and NRQCD power counting.

<table>
<thead>
<tr>
<th>dim</th>
<th>Operator</th>
<th>HQET</th>
<th>NRQCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$\bar{Q}iD_j^i\gamma Q$</td>
<td>$\lambda^3$</td>
<td>$v^4$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}\gamma \cdot DQ$</td>
<td>$\lambda^3$</td>
<td>$v^4$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}{\gamma \cdot D, \alpha \cdot E}Q$</td>
<td>$\lambda^2$</td>
<td>$v^4$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}\gamma \cdot iD_1\Sigma \cdot BQ$</td>
<td>$\lambda^3$</td>
<td>$v^6$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}[D_4, \gamma \cdot E]Q$</td>
<td>$\lambda^4$</td>
<td>$v^6$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}_{\gamma 4}(D \cdot E - E \cdot D)Q$</td>
<td>$\lambda^2$</td>
<td>$v^4$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}\gamma \cdot (D \times B + B \times D)Q$</td>
<td>$\lambda^3$</td>
<td>$v^6$</td>
</tr>
<tr>
<td>7</td>
<td>$\bar{Q}(D^2)^2Q$</td>
<td>$\lambda^3$</td>
<td>$v^4$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}D_i^iQ$</td>
<td>$\lambda^3$</td>
<td>$v^4$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}{D^2, i\Sigma \cdot B}Q$</td>
<td>$\lambda^3$</td>
<td>$v^6$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}\gamma \cdot iD_1\Sigma \cdot B \cdot DQ$</td>
<td>$\lambda^3$</td>
<td>$v^6$</td>
</tr>
<tr>
<td></td>
<td>$\sum_{i\neq j}\bar{Q}{D_j^i, i\Sigma_n B_i}Q$</td>
<td>$\lambda^3$</td>
<td>$v^6$</td>
</tr>
<tr>
<td></td>
<td>$\sum_{i\neq j}\bar{Q}i\Sigma_n D_iB_iD_jQ$</td>
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<td>$v^6$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}D_1\Sigma \cdot BD_1Q$</td>
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<td>$v^6$</td>
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<tr>
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<td>$\bar{Q}D_1\Sigma \cdot (B \times D)Q$</td>
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<td>$v^6$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}E \cdot EQ$</td>
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<tr>
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<td>$\bar{Q}\Sigma \cdot (E \times E)Q$</td>
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<td>$v^6$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}B \cdot BQ$</td>
<td>$\lambda^3$</td>
<td>$v^8$</td>
</tr>
<tr>
<td></td>
<td>$\bar{Q}\Sigma \cdot (B \times B)Q$</td>
<td>$\lambda^3$</td>
<td>$v^8$</td>
</tr>
</tbody>
</table>

are redundant. Matching of the last four rows in Table 1 requires a calculation of Compton scattering, which is is still in progress.

3. MATCHING

We derive matching conditions for the improvement couplings in Eq. (2) by matching on-shell quantities for small momentum but any $m_Qa$. We do this by calculating the energy momentum relation and the scattering of a heavy quark off a background field. The energy momentum relation has been studied [4] and summarized [5] before. It gives us conditions on $c_1$, $c_2$, $c_3$ and $c_4$. Results for the chromoelectric terms $c_E$, $c_6$ and $c_8$ are given in Ref. [5], determined from the temporal component of the background field. Scattering from the spatial component of the background field yields results for the chromomagnetic terms. Our results yield the following matching conditions. The energy momentum relation gives:

$$16\zeta c_1 = \frac{4\zeta^4(1 - \zeta^2)}{[m_0(2 + m_0)]^2} - \frac{3r_s^2\zeta^4}{(1 + m_0)^2}$$

$$+ \frac{\zeta^3[2\zeta + 4r_s(1 + m_0) - 6r_s\zeta^2/(1 + m_0)]}{m_0(2 + m_0)}$$

$$+ \frac{m_0(2 + m_0)}{2(1 + m_0)} \frac{r_s^2\zeta^2}{(1 + m_0)^2} - \frac{r_s^3\zeta^3}{(1 + m_0)^2} - 8r_3].$$

(3)

where $r_3$ can be chosen to be zero. Also

$$0 = \zeta^2 + 6\zeta c_2 + (r_s\zeta - 24r_4)m_0(2 + m_0)$$

$$8(1 + m_0).$$

(4)

We do not set $r_4 = 0$ or $r_s\zeta/24$ but choose it below so that $r_{11} = 0$. For the chromoelectric interactions one finds

$$\zeta^2 c_E^2 + c_8 \frac{2m_0(2 + m_0)}{(1 + m_0)} = \frac{\zeta^2(\zeta^2 - 1)}{m_0(2 + m_0)} + \frac{r_s^2}{(1 + m_0)}$$

$$+ \frac{r_s^2m_0(2 + m_0)}{4(1 + m_0)^2}.$$

(5)

We fix $c_0 = 0$.

(6)

For the chromomagnetic interactions one finds

$$c_B = r_s, \quad c_7 = 0, \quad c_5 = c_1, \quad r_9 = r_3,$$

(7)

$$r_{10} = -\frac{1}{48}(r_s\zeta - 24r_4) + \frac{1}{12}c_B\zeta,$$

(8)
\[ r_{11} = -\frac{1}{48}(r_3 \zeta - 24r_4) - \frac{1}{6}c_B \zeta, \tag{9} \]
\[ r_{12} = 0. \tag{10} \]

In summary, we may set \( c_6 = c_7 = c_8 = r_3 = r_9 = r_{11} = r_{12} = 0 \), leaving only five non-zero couplings: \( c_1, c_2, r_4, c_5 \) and \( r_{10} \).

4. ERROR ESTIMATES

We now estimate the discretization effects. It is advantageous to use a heavy-quark description of cutoff effects [6], because then contributions from, say, \( \bar{Q} \alpha \cdot E Q \) and \( \bar{Q} \{\gamma \cdot D, \alpha \cdot E\} Q \) are automatically combined. Thus, we express the uncertainties as

\[ \text{Error}_i = (C_{\text{lat}}^i - C_{\text{cont}}^i) \langle O_i \rangle \tag{11} \]

where \( C_i \) are HQET/NRQCD short-distance coefficients for lattice gauge theory and continuum QCD [6]. The mismatched coefficients can be obtained from the results of the previous section.

The matrix elements \( \langle O_i \rangle \) are estimated by using HQET (NRQCD) power counting for heavy-light systems (quarkonia). Since higher-order calculations are not yet available we multiply the tree-level mismatch with \( a_s \) to estimate the \( l \)-loop mismatch. This is a conservative estimate because the most pessimistic asymptotic behavior for the coefficients of the improvement operators is the same at higher orders as in the tree level formulas of previous section. To obtain the numbers shown in Figure 1, we take \( a_s = 0.25 \).

Fig. 1 shows results for the chromomagnetic energy (labelled \( 1/2m_B \)), chromoelectric energy (labelled \( 1/4m_E^2 \)), and the relativistic correction term (labelled \( 1/8m_3^2 \)). See Refs. [4,6] for details. The vertical lines represent \( a = 1/8 \) fm and \( a = 1/11 \) fm, corresponding to the unquenched MILC lattices [7]. The black (gray) curves represent the charmed (bottom) quark. Note that, in most cases, bottom quarks have smaller discretization effects than charmed quarks. Thus, Fig. 1 shows concretely that the Fermilab formalism avoids errors of order \((m_Qa)^n\).

Figure 1. Truncation errors for the new lattice action. The dotted curves correspond to the unimproved case. The dashed and solid curves show the errors from tree-level and one-loop matching, respectively.

5. CONCLUSIONS

We have presented a more improved action for heavy quarks with five new operators. Our error analysis shows that, to reduce errors below 1%, one-loop matching is essential for the coupling \( c_B \) of the chromomagnetic clover term. Tree-level matching for the coupling of the chromoelectric term, \( c_E \), is usually enough, but one-loop may be necessary for charmonium. Tree-level matching suffices for the other couplings in Eq. (2).

M.B.O. and A.X.K. are supported in part by the U.S. Department of Energy under contract DE-FG02-91ER40677. Fermilab is operated by the Universities Research Association Inc., under contract with the U.S. Department of Energy.

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