General covariance of the non-abelian DBI-action: Checks and Balances

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Abstract

We perform three tests on our proposal to implement diffeomorphism invariance in the non-abelian D0-brane DBI action as a basepoint independence constraint between matrix Riemann normal coordinate systems. First we show that T-duality along an isometry correctly interchanges the potential and kinetic terms in the action. Second, we show that the method to impose basepoint independence using an auxiliary $dN^2$-dimensional non-linear sigma model also works for metrics which are curved along the brane, provided a physical gauge choice is made at the end. Third, we show that without alteration this method is applicable to higher order in velocities. Testing specifically to order four, we elucidate the range of validity of the symmetrized trace approximation to the non-abelian DBI action.
1 Introduction

The nature of spacetime at the smallest scales is still an open question, many recent advances in non-perturbative string theory notwithstanding. Yet, the fundamental principle that the laws of physics should be observer independent, leads us to expect that even at the smallest scales general coordinate invariance manifests itself in some form or other. The one explicit proposal for non-perturbative string theory we currently possess, M(atrix) theory, contains explicit couplings to the graviton. M(atrix) theory is formulated in the light-cone gauge, and not background independent, but these couplings ought to reflect the freedom to choose different (transverse) coordinates. This expectation is based on the connection between M(atrix) theory and the low-energy-effective-action for $N$ superposed D0-branes. The latter is derivable from string theory and constitutes a generalization of the invariant length point particle action $S = \int ds = \int \sqrt{g_{ij}\dot{x}^i\dot{x}^j}$ where the coordinates $x^\mu$ have been promoted to $U(N)$-valued non-abelian matrices $X^{n,ab}$, together with a potential term proportional to commutators. Such matrix-valued coordinates no longer commute, and it begs the question how to implement the nonlinear coupling to gravity. Gravity is the gauge theory which imposes general coordinate invariance, and the problem is therefore equivalent to finding the invariant action for non-abelian D0-branes in a curved background.

Guided by these symmetry considerations we attempted in [1] to construct the full coupling of $N$ non-abelian D0-branes to gravity by imposing that the action be invariant under general coordinate transformations. Naturally, the matrix-valued nature of the coordinates $X^\mu$ goes beyond Riemannian geometry and general coordinate transformations and the diffeomorphism group will take a wholly new form. Moreover, the non-abelian nature of the coordinates does make the problem a very difficult one. Considering maps between (matrix) Riemann normal coordinate systems centered on different base points $\mathcal{P}_i$, we put forth that diffeomorphism invariance can be implemented as a new symmetry principle: base-point independence. The advantage of this method is that (i) the definition of matrix normal coordinates — that $X^i(\tau) = \tau Y^i$ with $Y^i$ constant is a solution to the non-abelian geodesic equation, which is the field equation of the action $S$ — yields an additional set of constraints on the action, and (ii) in normal coordinates all nonlinear terms in the action are tensors at the basepoint $\mathcal{P}$ and transform covariantly under change of basepoint. These two points plus the constraint that the action itself should be invariant under a basepoint transformation allowed us to construct an algorithm to determine the action order by order in the matrices $X^\mu$. Solving the algorithm explicitly to first nontrivial order $\mathcal{O}(X^6)$, we found a number of surprises:

(i) In contrast to the abelian point particle action, the more stringent constraints imposed by base-point independence do not determine the curved space action for non-abelian D0-branes uniquely. Signs that this would be the case had been found earlier: at the linearized level in diffeomorphisms, two different stress tensors are compatible with current conservation [2]. One arises in the low-energy effective action (LEEA) for non-abelian D0-branes in the bosonic string; the second appears in the LEEA for non-abelian D0-branes in type II superstring theory.
(ii) Despite the remaining arbitrariness in the action, one can show that (a) a fully symmetrized trace structure at all orders is incompatible with base-point independence and (b) compared to the abelian case, there are always new vertices, i.e. couplings to the gravitational background proportional to commutators.

A consequence of (ii) is that the gravitational potential felt by non-abelian D0-branes is fundamentally different than that felt by abelian particles. Exemplifying this point is the evidence that in hyperbolic spacetimes such non-abelian D0-branes behave collectively rather than independently: there are signs that a gravitational analogue to the Myers effect exists [3, 4]. The order $\mathcal{O}(X^6)$ base-point independent action furthermore obeys all Douglas’s axioms of D-geometry: necessary properties of the action for non-abelian D0-branes in curved space [5]. In particular, with a specific ansatz for a generalization of the flat-space potential to one built out of manifestly base-point independent objects (see section 3) the masses of fluctuations around a diagonal configuration are given by the geodesic distances between the diagonal entries.

Complementary to these results is recent work of van Raamsdonk on gauge-theories on “spaces” with $U(N)$-valued coordinates [6], which confirmed a number of qualitative aspects. In string theory, these theories arise as the LEEA of D0-branes embedded in the worldvolume of a higher-dimensional Dp-brane. String theory similarly predicts that a LEEA of non-abelian D0-branes coupled to gravity should exist. Yet with the strong constraints imposed by base-point independence for matrix-valued coordinates, it is a wonder we were able to find a solution at all, let alone a family of solutions. To support the answer we found in [1] and confirm the consistency of the solutions, we perform here three tests on our answer and the underlying idea of base-point independence. In section 3 we will show that T-dualizing along an isometry transforms the ansatz made for the potential term in [1] into the base-point independent kinetic term. This confirms our ansatz for the potential. A shortcoming of the base-point independence method as put forth in [1] we correct in section 4. In [1] we only addressed metric spaces with curvature strictly transverse to the the D0-brane worldvolume. Here we show that our method extends to spaces with curvature in all directions. In doing so, we solve the paradox between coordinate invariance and the distinct nature between tangential (commuting) and transverse (non-abelian) coordinates with respect to the worldvolume. The resolution lies in an extended definition of the ‘physical gauge’. Finally in section 5 we extend the method of base-point independence to higher derivative terms in LEEA of non-abelian D0-branes. This is a necessary condition for our philosophy to be consistent, and it elucidates the range of applicability of the symmetrized trace approximation for non-abelian D0-branes. We will find in particular that the symmetrized trace prescription is incorrect for terms linear in the graviton and of fourth order in velocities. We begin, however, with a brief review of diffeomorphism invariance for matrix-valued coordinates and base-point independence.
2 Diffeomorphisms, covariant background field formalism and base-point dependence

2.1 The base-point transformation

A guiding rule in theoretical physics is to keep as many symmetries manifest as possible. In an action describing fluctuations, this rule is automatic for a linearly acting symmetry: in that case both the background and the fluctuation transform in the same way. When a symmetry acts nonlinearly on fields, this rule is a priori difficult to keep. Consider general coordinate invariance for a first quantized particle action,

$$S = \int \sqrt{g} \frac{d^4x}{\sqrt{-g}}.$$  

The explicit reason why the symmetry is difficult to maintain, is that the quantum fluctuation $\delta x^i(\tau) = x^i(\tau) - x^i_{bg}(\tau)$ is not a covariant object under background coordinate transformations $\delta x^i_{bg} = \epsilon^i_{\alpha}(x^\alpha_{bg})$. The resolution is to expand the fluctuations nonlinearly as well, in such a way that they become covariant. Let $\xi^i(\tau)$ be the tangent vector $\xi^i(\tau)$ at $x^i_{bg}$ along the geodesic towards $x^i(\tau)$, and solve for the geodesic between $x^i(\tau)$ and $x^i_{bg}(\tau)$ in terms of $\xi^i$:

$$x^i = x^i_{bg} + \xi^i - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^i_{j_1...j_n}(x_{bg}) \xi^{j_1} \cdots \xi^{j_n}. \tag{2.1}$$

Here $\Gamma^i_{j_1...j_n} \equiv \nabla^\text{cov}_{j_1} \Gamma^i_{j_2...j_n}$ are generalized connection symbols, where $\nabla^\text{cov}$ only acts on the lower indices; see for example [7, 8]. As $\xi^i$ is a vector, it transforms co(ntra)variantly under background gauge transformations. Eq. (2.1) therefore constitutes a (nonlinear) expansion in fluctuations $\xi^i$ which is consistent with the symmetries.

We can also use the symmetry to our advantage. Eq. (2.1) also defines a coordinate transformation between coordinates $x^i$ and (Riemann normal) coordinates $\xi^i$. In this new coordinate system $x^i_{bg}$ is the origin and it is not difficult to show that all geodesics through $x^i_{bg}$ are straight lines. In normal coordinates $\xi^i$, the geodesic connecting $\xi^i$ and the origin $x^i_{bg}$ reads

$$\xi^i(\tau) = \tau \xi^i. \tag{2.2}$$

As eq. (2.1) holds for all coordinate systems, this in turn implies that in Riemann normal coordinates (RNC) the generalized connection coefficients vanish at the origin

$$\Gamma^i_{j_1...j_n}(x^i_{bg}) \big|_{\text{RNC around } x^i_{bg} = 0} = 0. \tag{2.3}$$

The action for the fluctuations, built out of objects evaluated at $x^i_{bg}$ therefore contains no connection terms. It consists only of true tensors at $x^i_{bg}$ and the coordinate $\xi^i$, originally a tangent vector at $x^i_{bg}$ and will be manifestly covariant under background coordinate transformations.

$$S^{\text{exp}}_{\text{RNC}} = S[g(x_{bg}), R(x_{bg}), \nabla R(x_{bg}), \ldots, \xi].$$

In fact, a moment’s thought reveals that because the fluctuation $\xi$ transforms co(ntra)variantly, the action after the expansion (2.1) in any coordinate system is built from covariant quantities. We will see below, though, that the specific choice of normal coordinates is very helpful.
This non-linear expansion is familiar as the non-linear sigma model version of the covariant background field expansion [7, 8]. To discuss what base-point independence means, it is useful to recall the covariant BG field method for Yang-Mills theories. The covariant BG field method entails a split of the gauge connection $A_\mu = A^{bg}_\mu + Q_\mu$ into a background part $A^{bg}_\mu$ and a quantum fluctuation $Q_\mu$. The gauge transformation

$$\delta A_\mu = \delta(A^{bg}_\mu + Q_\mu) = D_\mu(A^{bg} + Q)\Lambda$$

(2.4)
decomposes into a standard gauge transformation of the background field,

$$\delta^{bg}A^{bg}_\mu = D_\mu(A^{bg})\Lambda,$$

(2.5)

plus a covariant gauge rotation of the quantum field $Q_\mu$,

$$\delta^{bg}Q_\mu = [Q_\mu, \Lambda].$$

(2.6)

Essential for proving equivalence with the standard approach to correlation functions, is that the background expanded action only depends on the combination $A^{bg}_\mu + Q_\mu$. The background expanded action has an additional shift symmetry $A^{bg}_\mu \rightarrow A^{bg}_\mu + \epsilon_\mu$; $Q_\mu \rightarrow Q_\mu - \epsilon_\mu$ [9]. Vice versa, suppose an action is invariant under both this shift symmetry and the background gauge transformations (2.5) and (2.6). The true symmetry of the action is then eq. (2.4), and we recover the standard YM action.

Compare this with the background field method for the non-linear sigma model. Analogous to eqs. (2.4)-(2.6) the now non-linear expansion in fluctuations (2.1) guarantees that the original general coordinate invariance

$$\delta x^i = \delta(x^{bg}_i + \xi^i -'' \Gamma'') = \epsilon^i(x = x^{bg} + \xi -'' \Gamma'')$$

(2.7)
decomposes into a standard coordinate transformation for the background field,

$$\delta^{bg}x^{bg}_i = \epsilon^i(x^{bg})$$

(2.8)

plus a co(ntra)variant transformation of the quantum fluctuation, $\xi^i$

$$\delta^{bg}\xi = -\epsilon^j(x^{bg})\partial_j\xi^i + \xi^k\partial^i\epsilon_k(x^{bg})$$

(2.9)

In addition the background expanded action should have a “shift” symmetry which guarantees that it only depends on the particular combination of $x^{bg}_i$ and $\xi^i$ in (2.1). Due to the nonlinear nature of the expansion, this “shift” symmetry will be nonlinear as well. To see what this “shift” symmetry exactly is, we hark back the geometric principles underlying the background field expansion. In particular, recall that the normal coordinate system is defined by its properties with respect to the origin, i.e. the point $x^{bg}$. This suggest that we can compensate for a shift symmetry $x^{bg}_i \rightarrow x^{bg}_i + \epsilon^i(x^{bg})$ by choosing a new RNC coordinate system around a new
'basepoint' \( \tilde{x}_{bg} \equiv x^i_{bg} + \epsilon^i(x_{bg}) \). By construction this compensating coordinate transformation to a new set of RNC is given by considering the geodesic from \( \tilde{x}_{bg} \) to \( \xi^i \)  

\[
x_{bg} + \xi^i = \tilde{x}_{bg} + \chi - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^i_{j_1...j_n}(\tilde{x}_{bg}) \chi^{j_1} \cdots \chi^{j_n}.
\]

(2.10)  

\[
\Rightarrow \quad \xi^i = \epsilon^i + \chi^i + \epsilon^k \sum_{n=2}^{\infty} \frac{1}{(n+1)!} \nabla_{j_n} \cdots \nabla_{j_3} R_{j_1(k)i}{}_{j_2}(0) \chi^{j_1} \cdots \chi^{j_n}.
\]

(2.11)  

The tangent vector \( \chi^i \) at \( \tilde{x}_{bg} \) from \( \tilde{x}^i_{bg} \) to \( \xi^i \) is the new normal coordinate around \( \tilde{x}_{bg} \). Furthermore, due to the special properties of RNC, all tensors — the other building blocks of the action — transform covariantly under this “shift”. Schematically

\[
R(x_{bg}) \rightarrow R(\tilde{x}_{bg}) = R(x_{bg}) + \epsilon \partial R(x_{bg}) \overset{RNC}{=} R(x_{bg}) + \epsilon \nabla R(x_{bg}).
\]

(2.12)  

Thus if the action is invariant under this “shift” of basepoint, i.e.

\[
S = S[g(x_{bg}), R(x_{bg}), \ldots, \epsilon + \chi + \epsilon R(x_{bg}) \chi \ldots \chi] \\
= S[g(x_{bg}) + \epsilon \nabla g(x_{bg}), R(x_{bg}) + \epsilon \nabla R(x_{bg}), \ldots, \chi] \\
= S[g(\tilde{x}_{bg}), R(\tilde{x}_{bg}), \ldots, \chi],
\]

(2.13)  

it in fact only depends on the nonlinear combination (2.1).\(^2\) In combination with the manifest invariance under background coordinate transformations, this establishes that the action \( S \) is in fact diffeomorphism invariant. Geometrically the meaning of the “shift” symmetry is clear. A point \( x \) on a manifold \( \mathcal{M} \) can be reached either by successive infinitesimal translations along a vector \( \xi \) from the basepoint \( x_{bg} \) or by translations along \( \chi \) from \( \tilde{x}_{bg} \). General coordinate invariance means that formal local expressions, e.g. the line element, constructed as a function of \( \xi \) in relation to \( x_{bg} \) or by translations along \( \chi \) from \( \tilde{x}_{bg} \). General coordinate invariance means that formal local expressions, e.g. the line element, constructed as a function of \( \xi \) in relation to \( x_{bg} \) do not depend on which basepoint one picks. This is the manifestation of diffeomorphism invariance as “base-point independence”.

### 2.2 Generalization to matrix geometry

The above summary establishes why general coordinate invariance is equivalent to a base-point independent action for covariant fluctuations with tensorial couplings. Importantly, string theory tells us that it is precisely the vector-like fluctuations of the action for non-abelian D0-branes which become matrix-valued. It is therefore more natural to impose general coordinate

\(^1\)In the last step we have used that in the old RNC the connection coefficients vanish at the origin, viz.

\[
\Gamma^i_{jk}(\tilde{x}_{bg}) \simeq \epsilon^i \partial \Gamma^i_{jk}(x_{bg}) \overset{RNC}{=} \frac{1}{3} \epsilon^i R_{i(jk)} (x_{bg}).
\]

This also sets our convention for the Riemann tensor.

\(^2\)From the technical perspective on the covariant background field expansion, the choice of RNC corresponds to a choice of background field which obeys the equations of motion. For abelian geometry, this is not necessary but it does make the geometrical picture clearer.
invariance in the guise of base-point independence rather than a matrix generalization of Riemannian geometry. We should caution that, though certainly a necessary condition, base-point independence may not be truly equivalent to a “diffeomorphism” invariance for matrix-valued coordinates. We simply do not know enough about the latter, and we will proceed on the assumption that it is so.

With this input from string theory that it is the vectorlike fluctuations which are promoted to matrices, and that we should therefore impose diffeomorphism invariance as base-point independence, the way to construct a constraints on the LEEA for non-abelian D0-branes are clear.

(i) Write the most general two-derivative action in $U(N)$-valued fluctuations $X^{i,ab}$, invariant under $U(N)$ rotations (it should be a single trace, see below), and with tensorial couplings evaluated at an abelian basepoint $x_{bg}$; the origin of the normal coordinate system.

(ii) Enforce that the action is indeed in matrix normal coordinates: i.e. tune the couplings such that

$$X^{i,ab}(\tau) = \tau Y^{i,ab},$$

(2.14)

with $Y^{i,ab}$ a constant matrix, is a solution to the field equations. In matrix geometry this step is crucial, for it also fixes novel matrix-type diffeomorphisms of the form (see e.g. [10])

$$\delta X \sim [X, X].$$

(2.15)

(iii) Require that the action is (abelian) basepoint independent: i.e. solve the field equation for matrix-geodesics between an abelian point $\hat{X}^{i,ab}_{bg} = \epsilon^{i} \delta_{ab}$ and a matrix point $X^{j,cd}$ in terms of the tangent vector $Z^{i,ab} = \dot{X}^{i,ab}$. Substitute this coordinate change in the action, together with a shift of the background tensors and demand that the action be invariant.

We should note that string theory only predicts that coordinates transverse to the worldline of the non-abelian D0-brane are promoted to matrices. Strictly speaking, we therefore consider only spacetimes which are curved orthogonal to the brane. One of the purposes of this article is to remedy this situation; we will do so in section 4.

Predictions from string theory place two more constraints on the base-point independent action of non-abelian D0-branes:

(iv) Tseytlin observed that the action must consist of only a single trace over $U(N)$-indices [11].

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3 For instance, it is unclear if the base-point independence constraint also correctly accounts for “coordinate transformations” strictly proportional to commutators $\delta X \sim [X, X]$. At the same time, from the string theory point of view, it is unclear to what extent these are really geometrical (see e.g. [10]).

4 We have also chosen the gauge $A_0 = 0$ on the worldline. There is a corresponding Gauss’s law constraint on the matrix-valued coordinates. For the purposes of this paper, it will play no role, and we will ignore its consequences.
(v) Explicit computations in type II superstring theory have revealed that in the linearized weak field approximation (i.e. linear in the small fluctuation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; in normal coordinates linear in the Riemann tensor and its symmetrized derivatives) the ordering is completely symmetrized [12, 2]. Surprisingly the symmetrized ordering does not arise for D0-branes in the bosonic string. We must therefore choose which representation of matrix-valued diffeomorphisms we are interested in, and insist that at the linearized level our answer reproduces this. This will partly but by no means completely fix the freedom in the action that remains after we impose the requirement of diffeomorphism invariance.

The remaining axioms of D-geometry [5] can be shown to follow from these two constraints plus base-point independence [1].

### 2.3 Applying base-point independence

As formulated these conditions prescribe a consistent algorithm to find a base-point independent action for matrix-valued normal coordinates. This direct approach is cumbersome, however, and in [1] we used a more convenient and intuitive method. The basis of this “matrix-geometry” is the realization that the final action must be a constrained form of a $dN^2$-dimensional non-linear sigma model (NL$\sigma$M).

$$ S = \int d\tau G_{IJ}(X) \dot{X}^I \dot{X}^J. \quad (2.1) $$

Each index $I$ describes a triplet $I = \{i; ab\}$ built from a $d$-dimensional space-time index $i$ and two $U(N)$-indices $a, b$. The $dN^2$-dimensional metric $G_{IJ}(g_{ij}, R_{ijkl}, \ldots, X)$ is a functional of the $d$-dimensional metric $g_{ij}$ and its derivatives. Using a $dN^2$-dimensional expansion in normal coordinates, one must impose the following conditions to obtain the $d$-dimensional base-point independent action for matrix normal coordinates:

(a) When functionally expressed in terms of the $d$-dimensional constituents the $dN^2$ dimensional metric, Riemann tensor, and covariant derivatives thereof must obey all the usual identities of symmetry/antisymmetry, Bianchi identity, commutation relations of covariant derivatives, etc.

(b) The $U(N)$ indices must be such that the action is a single trace; i.e. no traces may occur within the functional expressions.

(c) It should have the right $U(1)$ limit for diagonal matrices.

(d) At linearized order the symmetrized ordering should emerge.

(e) Most importantly, base-point independence follows from the requirement that the “trace” of the $dN^2$-dimensional covariant derivative, acts as the $d$-dimensional covariant derivative:

$$ \delta^{ab} \nabla_{i;ab}(\text{anything}) = \nabla_i(\text{anything}). \quad (2.2) $$
For instance at order 4 and 5 in matrix normal coordinates $X^{iab}$ the two relevant $dN^2$-dimensional tensors are the Riemann tensor and its covariant derivative. Imposing the above matrix-geometry constraints one finds that in terms of $d$-dimensional curvature tensors, they are

$$ R_{IJKL} = R_{ijkl} \Sigma_{a_i b_i a_j b_j a_k b_k} , $$

$$ \nabla_M R_{IJKL} = \nabla_m R_{ijkl} \Sigma_{a_m b_m a_i b_i a_j b_j a_k b_k} , $$

(2.3)

where $\Sigma_{a_1 b_1 \ldots a_n b_n}$ is the object that when contracted with $n$-matrices returns the symmetrized trace

$$ \Sigma_{a_1 b_1 \ldots a_n b_n} O^{i_1 a_1 b_1} \ldots O^{i_n a_n b_n} = \text{Str}(O^{i_1} \ldots O^{i_n}) . $$

At order six, the fully symmetrized ordering is no longer consistent with the identity

$$ [\nabla_N, \nabla_M] R_{IJKL} = R_{NMI}^P R_{PJKL} + \ldots $$

This illustrates why the symmetrized approximation corresponds to the linearized approximation.

Finally, it will also turn out to be convenient to introduce a $dN^2$-vielbein $E^A_I$, which we define below, and which will be used to define the potential and discuss T-duality in section 3.

2.3.1 Second order in $\dot{X}$

To explicitly show how the matrix-geometry generates a base-point independent action, we review here the application for the kinetic term — order two in derivatives— to order $O(X^4)$, i.e. we show that the action

$$ L_2 = -\frac{1}{2} (\delta_{ij} \text{tr}(\dot{X}^i \dot{X}^j) + \frac{1}{3} R_{ijkl} \text{Str}((\dot{X}^i \dot{X}^j X^k X^l))) + O(X^5) $$

(2.4)

is base-point independent.

Writing the action in terms of a $dN^2 NL\sigma M$, we find

$$ L = -\frac{1}{2} \eta_{AB} \Pi^A \Pi^B , $$

(2.5)

with

$$ \Pi^A = E^A_I \dot{X}^I . $$

Matching with the flatspace result, the tangent space metric acts as a trace on the matrix indices:

$$ \eta_{AB} = \eta_{iab,jcd} = \delta_{ij} \delta_{ad} \delta_{bc} . $$

(2.6)
The metric $\eta_{AB}$ is a twisted version of the $SO(dN^2)$ invariant metric; ‘twisted’ means that it equals a Wick-rotated $SO(dN^2)$ metric up to a similarity transformation.\(^5\) Note that this is slightly different from the convention used in [1].

Expanding in RNC the vielbein equals:

$$E_I^A = \delta_I^A + \frac{1}{12} R^{A}_{(PQ)I} X^P X^Q + \frac{1}{24} \nabla_P R^{A}_{(QRI)} X^P X^Q X^R + \ldots$$  \hspace{1cm} (2.7)

Substituting this into equation 2.5, and using (2.3), we recover equation (2.4). By virtue of the fact that eq. (2.3) is the solution to the matrix-geometry constraints, this action is basepoint independent. For this simple case, one can check it explicitly [1].

An instructive illustration of the power of the matrix-geometry method, is the following exercise. Although we know from the flat space limit that the one-form $\Pi^A$ should be contracted with the tangent space metric $\eta_{AB}$ of eq. (2.6), we can consider a more general case.

$$L_2 = -\frac{1}{2} M_{AB}(X) \Pi^A \Pi^B, \hspace{1cm} (2.8)$$

Expanding in RNC as prescribed one obtains the action (with the base-point $\bar{X}$, see [8] for a convenient algorithm):

$$\begin{align*}
-2L_2 &= \left\{ M_{AB|\bar{X}} + \nabla_C M_{AB|\bar{X}} X^C + \frac{1}{2} \left( \frac{2}{3} M_{QBR}^{\quad CDA} + \nabla_D \nabla_C M_{AB} \right) \right\} X^A \hat{X}^B. \\
&= \left\{ \frac{2}{3} (\nabla^C M_{AB} + \nabla_D \nabla_C M_{AB}) \right\} X^A \hat{X}^B.
\end{align*}$$  \hspace{1cm} (2.9)

Comparing with the flat space case we read off: $M_{AB|\bar{X}} = \eta_{AB}$. Assume that it is possible to set $\nabla \ldots \nabla M|_{\bar{X}} = 0$, then we get $M(X)_{AB} = \eta_{AB}$ (remember: in RNC partial and covariant derivates are the same). This results in the action:

$$-2L_2 = (\eta_{AB} + \frac{1}{6} R_{B(CD)} X^C X^D) \hat{X}^A \hat{X}^B = \eta_{AB} \Pi^A \Pi^B. \hspace{1cm} (2.10)$$

Given that the vielbein is constructed from tensors obeying the matrix-geometry constraints, we can check the properties the action needs to have, without explicit calculations. We only need to verify that $M_{AB}$ also satisfies the matrix-geometry constraints. Single traceness of the action follows from the fact that $M_{AB} = \eta_{AB}$ has no internal $U(N)$ contractions. The correct $U(1)$ limit follows from the vanishing of all the covariant derivatives $\nabla \ldots \nabla M = \nabla \ldots \nabla \eta = 0.$

\(^5\)E.g. for $N = 2$, one finds

$$\eta_{abc, jcd} = \delta_{ij} \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}$$

with rows (columns) labeled by $ab = \{11, 12, 21, 22\}$. Hence in the $N = 2$ case the twisted metric is that of $SO(3d, d)$. 
The crucial property to check is base-point independence. Note that, since the expansion of \( M_{AB} \) is \( SO(dN^2) \) covariant, it is manifest that the action is invariant under any matrix valued diffeomorphism. However, the bi-tensor \( M_{AB} \) should also be a functional of the \( d \)-dimensional metric, and its derivatives. This functional will be consistent with base-point independence, if under a shift in base-point, it is parallel transported in the \( d \)-dimensional sense. As before, this is guaranteed if

\[
\epsilon^k \delta^{ab} \nabla_{kab} = \epsilon^k \nabla_k ,
\] (2.11)
on the tensor \( M_{AB} \). For \( M_{AB} = \eta_{AB} \) this is obviously so, so that the result is indeed base-point independent. Note that this is a truly non-trivial constraint that does not follow from the fact that the Lagrangian \( L_2 \) is a scalar quantity under matrix valued diffeomorphisms.

Focussing on \( M_{AB} \) alone, these results can be extended to arbitrary order in \( X \); at all orders \( \nabla \ldots \nabla M \) can be set to vanish consistent with the matrix-geometry constraints. In particular, the non-trivial identity

\[
\nabla [C \nabla_D] M_{AB} \big|_{\bar{X}} = R_{CDA}^Q M_{QB} \big|_{\bar{X}} + R_{CDB}^Q M_{QA} \big|_{\bar{X}} = R_{CDAB} + R_{CDBA} = 0 .
\] (2.12)
and higher order analogues are satisfied. For these, it is crucial that \( M_{AB} \) evaluated at the base-point is equal to the tangent space metric. In section 5, where we discuss consistency of the base-point independence approach for higher derivative terms, we will see that these non-trivial identities do impose constraints. We thus recover the two-derivative action (2.5). The lesson is that base-point independence and the other constraints of matrix geometry are automatically satisfied when we can put \( \nabla \ldots \nabla M \) to zero. The power of the above argument by generalization is that the expansion of the action in RNC allows us to, almost, read off whether the action is a candidate for the non-Abelian generalization of the DBI-action.

We now proceed with a number of checks and extensions of the action for non-abelian D0-branes. In section 5 we show how the method just discussed extends to actions fourth order in derivatives. But before that, we perform two checks on our method. In the next section we prove that the base-point independent potential term, conjectured in [1], turns into the kinetic term of eq. (2.5) after T-duality. And in section 4, we show how the matrix-geometry approach has a straightforward extension to spacetimes with curvature along the brane (recall that the above approach is only valid for transverse curvature).

### 3 Checks: T-duality

One of the consequences of the non-abelian nature of the coordinates \( X \) is that new terms can be present in the action, proportional to commutators, which have no \( U(1) \) equivalent. Indeed for non-abelian D0-branes in flat space, string theory tells us that at lowest order in derivatives there is a potential equal to

\[
V = -\frac{T \lambda^2}{2} \text{Tr}([X^i, X^j][X_i, X_j]) .
\] (3.1)
The form of the potential is dictated by consistency with T-duality. Under this stringy symmetry the potential and kinetic term are exchanged. T-duality holds for any spacetime with isometries, and the curved space analogues of both the potential and the kinetic term should be consistent with the duality. In addition to constructing a base-point independent kinetic term, in [1] we also put forward a potential for D0-branes in a curved background. This conjectured potential passed a strong consistency test. It satisfied the non-trivial D-geometry constraint that fluctuations around a diagonal background have masses proportional to geodesic lengths. We will now show that the conjectured form of the potential reproduces the kinetic term after a T-duality transformation. This is strong confirmation that our guess is correct.

To generalize the expression (3.1) to curved space, we need an analogue of the vector \( \dot{X}^I \), which can be contracted with the vielbein \( E^A_I \). Define the “commutation” operator

\[
D(X)^{ich;ad} \equiv \delta^{ab} X^{icd} - \delta^{cd} X^{iab}.
\]

(3.2)

Acting on a matrix \( M_{da} \) it returns the commutator

\[
\mathcal{D}(X)^{ich;ad} M_{da} = [X^i, M]^{cb}.
\]

(3.3)

Commutators obey the Leibniz rule and act as a derivation on the space of matrices. Analogous to the standard time derivative \( \dot{X}^i \), we expect any \( X^{iab} \) appearing inside a commutator to transform as a vector under matrix coordinate (i.e. base-point) transformations. The matrix-valued “commutation operator” \( \mathcal{D}(X)^{I,ad} \) can therefore be pushed forward to the tangent space with the \( SO(dN^2) \) vielbein. Supporting and consistent with the notion that the “commutation” operator is the covariant building block of the potential, is the expression for flat space potential (3.1) in terms of \( \mathcal{D}(X)^{I,ab} \). A small calculation shows that it is equivalent to four building blocks contracted with exactly twice the \( SO(dN^2) \) metric [1].

\[
V_{\text{flat}} = -\frac{T \lambda^2}{4} \eta_{IK} \eta_{JL} \mathcal{D}(X)^{I,ab} \mathcal{D}(X)^{J,bc} \mathcal{D}(X)^{K,cd} \mathcal{D}(X)^{L,da}
\]

\[
= -\frac{T \lambda^2}{4} \eta_{IK} \eta_{JL} \text{Tr} \mathcal{D}(X)^I \mathcal{D}(X)^J \mathcal{D}(X)^K \mathcal{D}(X)^L.
\]

(3.4)

In the last line the trace is only over the explicit \( U(N) \) indices of the matrix valued \( SO(dN^2) \) vector \( \mathcal{D}(X)^{I,ab} \).

The generalization to curved space is now straightforward. We simply insert the appropriate number of vielbeins into the flat space potential:

\[
V_{\text{curved}} = -\frac{T \lambda^2}{4} \eta_{AC} \eta_{BD} E^A_i E^B_j E^C_K E^D_L \text{Tr} \mathcal{D}(X)^I \mathcal{D}(X)^J \mathcal{D}(X)^K \mathcal{D}(X)^L
\]

\[
= -\frac{T \lambda^2}{4} (\eta_{AC} \eta_{BD} + \frac{4}{12} R_{C(PQ)A} X^P X^Q \eta_{BD}) \text{Tr} \mathcal{D}(X)^A \mathcal{D}(X)^B \mathcal{D}(X)^C \mathcal{D}(X)^D + \mathcal{O}(\nabla^2)
\]

\[
= -\frac{T \lambda^2}{4} (\text{Tr}([X^i, X^k][X^j, X^l]) \delta_{ij} \delta_{kl} + \frac{1}{3} R_{ijkl} \delta_{mn} \text{Str}(X^k X^l [X^i, X^m][X^j, X^n])) + \ldots
\]

(3.5)
The last two steps show that in the linearized approximation it correctly reproduces the symmetrized result from [12], as is expected.

Under T-duality the “parallel” part of the curved space potential (3.5) must transform into the kinetic term (2.5). To check this, assume that the \(d\)-dimensional geometry is a product of a \((d-1)\)-dimensional piece times a circle along the direction \(i = 9\). This is not sufficient to test the full nature of T-duality, but in this situation everything is tractable and certainly should work. We thus have the following expression for the \(dN^2\) metric:

\[
G_{\alpha\beta,j\gamma\delta}(X^i) = \begin{pmatrix} G_{\mu\alpha\beta,\nu\gamma\delta}(X^\rho) & 0 \\ 0 & \delta_{\alpha\delta} \delta_{\beta\gamma} \end{pmatrix} \quad \mu, \nu, \rho = 1, \ldots, d - 1.
\] (3.6)

The expression in the lower right corner is simply a consequence of the non-trivial form of the flat-space metric:

\[
G_{\alpha\beta,j\gamma\delta}^{flat} = \eta_{\alpha\beta,j\gamma\delta} = \delta_{ij} \delta_{\alpha\delta} \delta_{\beta\gamma}.
\] (3.7)

Because we have chosen a direct product form for the space time, the tangent space decomposes trivially:

\[
\eta_{\alpha\beta,I\gamma\delta}^{d-dim} = \begin{pmatrix} \eta_{\alpha\beta,\nu\gamma\delta}^{(d-1)-dim} & 0 \\ 0 & \delta_{\alpha\delta} \delta_{\beta\gamma} \end{pmatrix}.
\] (3.8)

In particular, the component of the vielbein \(E^A_I\) along the circle is

\[
E_{9a\beta}^{\alpha\zeta} = \delta_9^{a\alpha} \delta_\beta^{\zeta}.
\] (3.9)

In this background the potential splits into three parts. One has no tangent-vectors lying along the isometry direction; this will become the potential in the T-dual case. A second term has all components along the circle: since we know the flat limit corresponds to the commutator squared, this term will vanish. The crossterm with half the components along the circle is the interesting part. Using that the vielbein is trivial in the ninth direction, it equals

\[
V_{cross}^{curv} = -\frac{T \lambda^2}{2} \eta_{ij,9cd} \eta_{BD} E_B^D E_L^D \text{Tr} (X^9) \eta_{ij,9cd} \eta_{BD} E_B^D \text{Tr} (X^9) \eta_{ij,9cd} \eta_{BD} E_B^D\left[ \text{Tr} (X^9) \text{Tr} (X^9) \right].
\] (3.10)

The triviality of the vielbein allows the use of the following contraction identity,

\[
\eta_{ij,9cd} \text{Tr} (X^9) \text{Tr} (X^9) = 2(X^k)^{a\beta} (X_k)^{ed} - \delta^{a\beta} (X^2)^{ed} - \delta^{ed} (X^2)^{a\beta},
\] (3.11)

in the direction of the circle. We get

\[
V_{cross}^{curv} = -T \lambda^2 \eta_{BD} E_B^D E_L^D \left[ \text{Tr} (X^9) \text{Tr} (X^9) \right].
\]

Due to the defining property (3.3) of the commutation operator, the last term vanishes as \(\text{Tr}(D(X)^I) = [X^i, \mathbb{I} ] = 0\). The remaining term yields

\[
V_{cross}^{curv} = -T \lambda^2 \eta_{BD} E_B^D E_L^D \left[ [X^i, X^9]^{a\beta} [X^\ell, X^9]^{\gamma\delta} \right].
\] (3.12)
Upon using the standard T-duality rule which replaces commutators with derivatives,

\[ i\lambda [X^g, \mathcal{F}(X)] \rightarrow \partial_g \mathcal{F}, \]

we recover exactly the kinetic term (2.5). Notice that the vielbeins have basically just gone along for the ride. The proof of T-duality in a flat background would be identical. This shows the power of the matrix-geometry approach.

4 Checks: Non-transverse metrics

A second crucial test which the base-point independent action for non-abelian D0-branes must pass, is that it must be able to account for gravitational polarization along the worldvolume in addition to purely transverse curvature. So far we have only dealt with the latter situation, both for simplicity as well as the string theory indication that only the transverse coordinates get promoted to \( U(N) \)-valued matrices. This is not a very satisfactory situation. Most interesting metrics, e.g. Schwarzschild, (A)dS, have curvature in the timelike tangential direction. Moreover, from a diffeomorphic perspective it is very strange. It appears to conflict with the idea of general coordinate invariance, since some directions are said to be more special than others, based on data not intrinsic to the space-time. Introducing an extension to a physical gauge choice, we will see that this conflict is spurious. This important conclusion has been instrumental in providing evidence that there is a gravitational analogue to the Myers effect [4].

There is one obvious answer to deal with more general metrics, that ensures an action that is fully diffeomorphism invariant. That answer is to start with the \( dN^2 + p + 1 \) dimensional NL\( \sigma \)M,

\[
S = \int d^{p+1} \xi \sqrt{\text{det}(G_{MN}(X) \frac{\partial X^M}{\partial \xi^a} \frac{\partial X^N}{\partial \xi^a})},
\]

where \( M \) is now the multi-index \( \hat{M} = m\alpha\beta, a; \) demand that \( G_{MN}(X) \) is a functional of the metric \( g_{mn} \) and its derivatives, and solves the set of constraints

(a) The \( dN^2 + p + 1 \) dim Riemann tensor \( R_{IJKL} \), its covariant derivatives, etc. have all the usual properties.

(b) The action is a single trace over the \( U(N) \) indices.

(c) The action has the correct linearized form and \( U(1) \) limit.

(d) The action is base-point independent: i.e. \( \delta^{\alpha\beta} \nabla_{m\alpha\beta} = \nabla_m \).

6The results in this section were obtained together with Eric Gimon.
Of course to set up the system of constraints algebraically one needs to be careful whether the index $M$ is orthogonal to the brane (in the $\hat{M}$ direction) or parallel (in the $a$ direction).

This procedure emphasizes the dichotomy which conflicts with diffeomorphism invariance that directions perpendicular to the brane are treated differently than those parallel. For a single D-brane, we know this is a fake problem. Introducing a worldline metric, we can "undo" the physical gauge choice $\tau = x^0$; the extra degree of freedom $X^0(\tau)$ is compensated by the additional worldsheet diffeomorphism symmetry. Up to two derivatives, the action is then exactly the same as before, except that the range of indices now also includes the tangential directions.

For non-abelian D0-branes the conflict due to this dichotomy is far more acute: from its origins in string theory, we expect the tangential coordinates to be commutative, while diffeomorphism invariance tells us that they should be of the same non-commutative nature as the transverse coordinates. Physically, however, we expect that a “physical gauge” solution should also exist for $N$ D-branes. Consider e.g. the example of $N$ static D1-branes wrapping the equator of a 2-sphere. Geometrically the normal and tangential directions are equivalent; yet the natural construction advocated above seems to say the opposite. This cannot be true. Simply rotating the system around a fixed point on the equator, should leave everything invariant. This argues that ”undoing” the physical gauge choice should also give tangential matrix valued coordinates. Thus we have a straightforward guess for the solution to our dilemma. Up to two derivatives the action is simply the same as before (2.5) but with the range of indices including the tangential directions. And the system has enough symmetry that we may choose a physical gauge

$$X^{\parallel\alpha\beta} = \xi^{\parallel\delta\alpha\beta}.$$  \hspace{1cm} (4.2)

It seems difficult to justify this procedure from a world-sheet point of view. Nevertheless geometrically it seems to be the most natural thing to do. We simply propose that this is an inherent part of matrix-diffeomorphisms. Support for this proposal is that the final action will obey all the correct symmetries.

Of course, so would the action constructed by the dichotomous approach proposed below eq. (4.1). This is as it should be. It is not so hard to show, that the actions resulting from either imposing a matrix-physical gauge or the ”natural” p-brane procedure above are the same. This is the evidence in support of the fact that in matrix-geometry enough symmetries exist, to fix the matrix-physical gauge. To show equivalence, we use the single trace property. Let us first look at the Riemann tensor. We use tilded indices to denote the $(d+p+1)^2$ directions, greek indices in the middle of the alphabet for the underlying $d+p+1$ dimensional spacetime, greek indices in the beginning of the alphabet for $U(N)$ indices, $a,b$’s and $m,n$’s for the tangential and normal directions respectively, capital indices for the $dN^2 + p + 1 \text{NLoM}$, splitting in $dN^2 \hat{M}$’s = $ma\beta$ and $p + 1 a$’s.

We start from the $(d+p+1)^2$ solution to the constraints for the Riemann tensor:

$$R_{MNPS} = R_{\rho\alpha_1\alpha_2,\rho\beta_1\beta_2,\rho\gamma_1\gamma_2,\sigma\delta_1\delta_2} = \sum_{n=1}^{6} T^{(n)}_{\mu\nu\rho\sigma} \Delta_{p_n}(\alpha_\beta\gamma\delta),$$ \hspace{1cm} (4.3)
where \( p_n(\alpha\beta\gamma\delta) \) is the \( n \)th permutation and
\[
\Delta_{\alpha\beta\gamma\delta} \equiv \delta_{\alpha_2\beta_1}\delta_{\beta_2\gamma_1}\delta_{\gamma_2\delta_1}\delta_{\delta_2\alpha_2}
\] (4.4)
is the cyclic contraction. Hence the total number of inequivalent permutations \( p_n(\alpha\beta\gamma\delta) \) is 6 for a 4-tensor; or \( (n-1)! \) for \( n \) indices. \( T_{\mu\nu\rho\sigma}^{(n)} \) is functional of the \( d+p+1 \)-dimensional Riemann tensor. Now we impose the physical gauge by contracting those indices in \( \Delta_{\alpha\beta\gamma\delta} \) with \( \delta_{\alpha_1\alpha_2} \) that correspond with tangential directions. The answer is obvious, the contraction of \( \Delta_{\alpha\beta\gamma\delta} \) with \( \delta_{\alpha_1\alpha_2} \) yields a \( \Delta \) tensor with one less index. Or specifically
\[
R_{\tilde{A}\tilde{N}\tilde{P}\tilde{S}}|_{\text{phys. gaug}} = \sum_{n=1}^{2} T^{(n)}_{\alpha\beta\gamma\delta} \Delta_{p_n(\beta\gamma\delta)}
\]
\[
R_{\tilde{A}\tilde{B}\tilde{P}\tilde{S}}|_{\text{phys. gaug}} = \sum_{n=1}^{1} T^{(n)}_{\alpha\beta\gamma\delta} \Delta_{p_n(\gamma\delta)}
\]
\[
R_{\tilde{A}\tilde{N}\tilde{B}\tilde{S}}|_{\text{phys. gaug}} = \sum_{n=1}^{1} T^{(n)}_{\alpha\beta\gamma\delta} \Delta_{p_n(\beta\delta)}
\]
\[
R_{\tilde{A}\tilde{B}\tilde{C}\tilde{S}}|_{\text{phys. gaug}} = T_{\alpha\beta\gamma\delta}
\] (4.5)
These are exactly the first set of constraints one would write down for the \( dN^2 + p + 1 \) dimensional dichotomous sigma model. And there is a beautiful corollary to this Riemann tensor equivalence of the two approaches: the base point independence constraint (in other words diffeomorphism invariance), \( \delta_{\alpha\beta} \nabla_{\mu \alpha \beta} = \nabla_\mu \), guarantees that this also holds for all derivatives of the Riemann tensor. Thus the physical expectation that diffeomorphism invariance should treat all coordinates equally is born out, once we "undo" the physical gauge choice \( X_\parallel = \xi_\parallel \delta^{ab} \).

5 Balances: Higher order corrections: Fourth order in \( \dot{X} \)

We have thus seen that the base-point independent action including the potential is consistent with T-duality and that the method itself captures diffeomorphism invariance fully in that it is extensible to non-transverse curvature for actions up to two derivatives. We conclude here with an application of the base-point independence approach to the next order in derivatives. This will illustrate the universality of our method and we will show at this order the symmetrized trace prescription is no longer consistent with base-point independence, even if we treat gravity at the linearized level. We conclude that the symmetrized trace approximation is that of linearized gravity up to two derivatives.

In section 2.3 we saw that the covariant expansion is a straightforward approach to determine the crucial properties of the action. We will therefore pursue this route also for the action \( L_4 \), the part of the DBI-action of fourth order in \( \dot{X} \). Before doing that, let us show explicitly why the symmetrized trace approximation starts to fail at this order. If the action would be the
symmetrized trace, then it could be written as:

\[
L_4 = -\frac{1}{8} \text{Str}(g_{ij}(X)g_{kl}(X)\dot{X}^i \dot{X}^j \dot{X}^k \dot{X}^l) \\
= -\frac{1}{8} (\delta_{ij}\delta_{kl} \text{Str}(\dot{X}^i \dot{X}^j \dot{X}^k \dot{X}^l) + \frac{2}{3} R_{mnjl}\delta_{kl} \text{Str}(X^m X^n \dot{X}^i \dot{X}^j \dot{X}^k \dot{X}^l)) + \ldots .
\]  

(5.1)

Substituting the base-point transformation (a constant shift \(\epsilon\)) (the explicit expression follows from the geodesic equation in matrix space, which follows from \(L_2\)):

\[
\Delta X_i = \epsilon_i + \frac{1}{6} \epsilon^k \text{Sym}(Z^p_1 Z^p_2) R^{p_1(ki)p_2},
\]

(5.2)

we get for the variation (schematically and up to first order in the Riemann tensor):

\[
\Delta L_4 \propto \epsilon \text{RStr}(\ddot{Z}^3 \text{Sym}(\ddot{Z} Z)) + \epsilon \text{RStr}(\ddot{Z}^4 Z).
\]

(5.3)

From this we see that the two combinatorial structures cannot be combined, hence the symmetrized trace prescription does not yield a base-point independent action.

To determine what is the correct ordering, we follow the same steps as before. Our starting point is an action of order four in one-forms \(\Pi^A\) contracted with an arbitrary symmetric four-tensor \(M_{ABCD}(X)\):

\[
L_4 = -\frac{1}{8} M_{ABCD}(X)\Pi^A\Pi^B\Pi^C\Pi^D.
\]

(5.4)

Expanding in RNC up to second order in \(X\), we find

\[
-8L_4 = \left\{ M_{ABCD} |_{\dot{X}} + \nabla_F M_{ABCD} |_{\dot{X}} X^F \\
+ \frac{1}{2} (\frac{4}{3} M_{QBCD} R^Q_{EFA} + \nabla_E \nabla_F M_{ABCD}) |_{\dot{X}} X^E X^F \right\} \dot{X}^A \dot{X}^B \dot{X}^C \dot{X}^D.
\]

(5.5)

The lesson from section 2.3 is whether we can put \(\nabla \ldots \nabla |_{\dot{X}} = 0\)? This would ensure the correct \(U(1)\) limit and base-point independence. Suppose we could. In that case the action would be:

\[
-8L_4 = \left\{ M_{ABCD} |_{\dot{X}} + \frac{2}{3} M_{QBCD} R^Q_{EFA} \right\} \dot{X}^A \dot{X}^B \dot{X}^C \dot{X}^D
\]

(5.6)

From the flat space limit,\(^7\)

\[
-8L_4^{\text{flat}} = M_{ABCD} |_{\ddot{X}} \dot{X}^A \dot{X}^B \dot{X}^C \dot{X}^D = \text{Str}(\ddot{X}^i \ddot{X}^j \ddot{X}^k \ddot{X}^l)\delta_{ij}\delta_{kl},
\]

(5.7)

we learn that:

\[
M_{ABCD} |_{\ddot{X}} \equiv \eta_{ABCD} = \delta_{ab}\delta_{cd} \sum_{\alpha_1 \beta_1 \ldots \alpha_4 \beta_4} \neq \eta_{AB}\eta_{CD}.
\]

(5.8)

\(^7\)The flat space limit follows indirectly from explicit string computations which show that the non-abelian DBI-action at order \(F^4\) is given by the symmetrized trace [13]. This breaks down at higher orders; see [14] for the latest status.
Note that the single trace requirement means that the tensor $M$ is not proportional to two tangent metrics. This will be important.

Are all the requirements (a)-(d) of matrix geometry satisfied? Since we’ve assumed all covariant derivatives on $M$ can be put to zero we have obtained the right U(1) limit. The single trace condition is met, by construction. What about the focus of this article: base-point independence? This is also guaranteed if we can truly put all covariant derivatives on $M$ in independence by hand. Recall that the base-point transformation is given by:

$$\Delta X_i = \epsilon_i + \frac{1}{6} \epsilon^k \text{Sym}(Z^{p_1} Z^{p_2}) R_{p_1 (k)p_2}.$$ (5.11)

Using this transformation we calculate the variation of $L_4$ with $M_{ABCD} = \eta_{ABCD}$:

$$-8L_4 = \eta_{ABCD} E_i^A X^I E_j^B X^J E_k^C X^K E_L^D X^L$$

$$= \delta_{ij} \delta_{kl} \text{Str}(\dot{X}^i \dot{X}^j \dot{X}^k \dot{X}^l) + \frac{1}{3} R_{i(kl)} \delta_{mn} \text{Str}(\dot{X}^i \dot{X}^m \dot{X}^n \text{Sym}(\dot{X}^j \dot{X}^k \dot{X}^l)),$$ (5.12)

$$-\Delta L_4 = \frac{142}{312} \epsilon^{k} R_{\beta (k)p_2} \text{Str}(\dot{Z}^\alpha \dot{Z}^2 \text{Sym}(\dot{Z}^\beta Z^{p_2})) + \frac{2}{38} R_{k(\alpha)p_2} \epsilon^k \text{Str}(\dot{Z}^\alpha \dot{Z}^2 \text{Sym}(\dot{Z}^\beta Z^{p_2}))$$

$$= \frac{1}{6} \epsilon^k \text{Str}(\dot{Z}^\alpha \dot{Z}^2 \text{Sym}(\dot{Z}^\beta Z^{p_2})) \left\{ R_{\beta (k)p_2} + \frac{1}{2} R_{k(\alpha)p_2} \right\}.$$ (5.13)

Note that \text{Sym}(\ldots) expressions are treated as one block within the symmetrized trace. Using (note $A_{(ab)} = A_{ab} + A_{ba}$),

$$R_{\alpha (p_1 \beta)p_2} = -\frac{1}{2} R_{p_1 (\alpha \beta)p_2} + \frac{3}{2} R_{\alpha \beta p_1 p_2},$$

8One might object that the identity (5.10) appears to be irrelevant as in the action $\nabla_E \nabla_F M_{ABCD}$ is contracted with the symmetric combination $X^E X^F$. The remaining discussion will show why this is not so.
this simplifies to (with the notation $A_{p_1\ldots p_n} = A_{1\ldots n}$):

$$\Delta L_4 = -\frac{1}{4} e^6 R_{3456}\delta_{12}\text{Str}(\dot{Z}^1 \dot{Z}^2 \dot{Z}^3 \text{Sym}(\dot{Z}^4 Z^5)) \neq 0. \quad (5.14)$$

So the proposed action is indeed not base-point independent. To see that the single trace requirement is responsible — and hence the correlated inconsistency of choosing $M_{ABCD}(X) = \eta_{ABCD}$ — let us look at the corresponding calculation for $L_2$. The steps are analogous and the $L_2$ result comes down to removing the $\delta_{12}\dot{Z}^1 \dot{Z}^2$ part in the $L_4$ result and adjusting some factors:

$$\Delta L_2 \propto e^6 R_{3456}\text{Str}(\dot{Z}^3 \text{Sym}(\dot{Z}^4 Z^5)). \quad (5.15)$$

Here we can make use of the identity:

$$\text{Str}(ABCD\ldots) = \text{Tr}(A\text{Sym}(BCD\ldots)). \quad (5.16)$$

This allows us to write $\Delta L_2$ as:

$$\Delta L_2 = e^6 R_{3456}\text{Str}(\dot{Z}^3 \dot{Z}^4 Z^5), \quad (5.17)$$

which is obviously zero. In the case of $\Delta L_4$ the identity in equation 5.16 is of no use to us, because of the extra $\delta_{12}\dot{Z}^1 \dot{Z}^2$ factor. Had we not insisted on a single trace result and used $M_{ABCD} = \eta_{AB}\eta_{CD}$, then the variation of $\Delta L_4$ would have had the structure:

$$\Delta L_4 = \Delta L_2 \delta_{12}\text{Str}(\dot{Z}^1 \dot{Z}^2) = 0. \quad (5.18)$$

So, as claimed, the single trace property spoils base-point independence. As a result of this we confirm the importance of the identity (5.10). $\nabla_E \nabla_F M_{ABCD}|_{\bar{X}}$ should not be zero if we insist on base-point independence.

Fortunately is it not terribly difficult to find an correction term to $L_4$ that renders the action base-point independent while keeping the correct $U(1)$ limit. One possible answer is:

$$L^C_4 = \alpha R_{1356}\delta_{24}\text{Str}(\dot{X}^1 \dot{X}^2 \text{Sym}(\dot{X}^4 X^5)\text{Sym}(\dot{X}^3 X^6)). \quad (5.19)$$

Since $R_{1356}$ is antisymmetric in 1 and 3, this result vanishes in the $U(1)$ limit. The variation of $L^C_4$ equals:

$$\Delta L^C_4 = \alpha R_{1356}\delta_{24}\text{Str}(\dot{Z}^1 \dot{Z}^2 \dot{Z}^3 \text{Sym}(\dot{Z}^4 Z^5))$$

$$+ \alpha R_{1356}\delta_{24}\text{Str}(\dot{Z}^1 \dot{Z}^2 \text{Sym}(\dot{Z}^4 Z^5)\dot{Z}^3 e^6)$$

$$= \alpha e^6 R_{1465}\delta_{24}\text{Str}(\dot{Z}^1 \dot{Z}^2 \dot{Z}^3 \text{Sym}(\dot{Z}^4 Z^5))$$

$$= 4\alpha \Delta L_4. \quad (5.20)$$

Requiring base-point independence determines the constant $\alpha$ to be $-\frac{1}{4}$.

The obvious next question is what is the value for $\nabla \ldots \nabla M_{ABCD}|_{\bar{X}}$ that corresponds to $L^C_4$. Since there is no $O(X)$ term in the $U(1)$ limit, we will try to maintain this property for
the non-abelian case: we require that there is no $\nabla_F M_{ABCD}|_X X^F$ term in the action. For the $\nabla_E\nabla_F M_{ABCD}|_X X^E X^F$ part we can only determine the part symmetric in $EF$ and $ABCD$ from $L_4^C$. It is determined by:

$$\nabla_E\nabla_F M_{ABCD}|_X X^E X^F \hat{X}^A \hat{X}^B \hat{X}^C \hat{X}^D = 4R_{1456}\delta_{23}\text{Str}(\hat{X}^1\hat{X}^2\text{Sym}(\hat{X}^4\hat{X}^5)\text{Sym}(\hat{X}^3\hat{X}^6)) \quad (5.21)$$

We write this schematically as (see appendix A for the notation):

$$\nabla_E\nabla_F M_{ABCD}|_X X^E X^F \hat{X}^A \hat{X}^B \hat{X}^C \hat{X}^D = \frac{1}{48}s_{(ABCD)(EF)}X^E X^F \hat{X}^A \hat{X}^B \hat{X}^C \hat{X}^D \quad (5.22)$$

with

$$s_{ABCDEF} = s_{123456} = 4R_{1456}\delta_{23}\Sigma_{1278}S^7_{45}S_{36}^8, \quad (5.23)$$

We already showed explicitly that the action thus obtained is base-point independent. However, consistency of the more general approach demands that base-point independence can also be shown by proving the following:

$$\epsilon^i\delta^{ab}(\nabla_{iab}\nabla_F M_{ABCD}|_X) = \epsilon^i(\nabla_i\nabla_F M_{ABCD}|_X). \quad (5.24)$$

In order to check this we have to know what the right-hand part of the previous equation is. It corresponds to the variation of the $\nabla_F M_{ABCD}|_X X^F$ term. Since this term vanishes, we conclude that the right-hand side of equation 5.24 should be zero in the action. Recall that $\nabla_E\nabla_F M_{ABCD}|_X$ can receive several contributions, of which only the $S$ part contributes to the action. Of course, if it vanishes as a single formal tensor, that would be the more satisfactory. Using the explicit representation for $s_{ABCDEF}$ (equation 5.23), we find for the left-hand side of eq. (5.24):

$$\epsilon^i\delta^{ab}S_{1234 iab 6} = \epsilon^i\delta^{ab}(R_{14i6}\delta_{23}\Sigma_{1278}S^7_{4ab}S_{36}^8 + \ldots)
= \epsilon^i(R_{14i6}\delta_{23}\Sigma_{1248}S_{36}^8 + \ldots)
\propto \epsilon^i(R_{14i6}\delta_{23}\Sigma_{1248}S_{36}^8 + \epsilon^iR_{41i6}\delta_{23}\Sigma_{1248}S_{36}^8 + \ldots) = 0. \quad (5.25)$$

This confirms the base-point independence. The other contributions to $\nabla\nabla M$, such as the parts anti-symmetric in $E, F$, are of no concern since these vanish in the action. It is, however, possible to add other corrections terms to $M$ such that equation 5.24 is zero, even as a formal tensor.

Presumably these results can be generalized to arbitrary order in the velocity $\dot{X}$. However, the extension to higher order in $X$ is far from trivial due to the complications from the inability to put $\nabla \ldots \nabla M$ to zero in matrix geometry. For every order in $X$ we have to find the appropriate tensor $\nabla\nabla \ldots \nabla M|_X$, which is beyond our present capability. The fact that we are able to do so for fourth order in derivatives does give confidence that this is possible. Our computation does explicitly show that at the first corrective order in derivatives the symmetrized trace approximation is no longer consistent with the single trace requirement in the presence of gravity.
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A Notation

- Riemann Normal Coordinates
  We write fluctuations in a metric $g_{ij}$ as $h_{ij}$: $g_{ij} = \eta_{ij} + h_{ij}$. The moments of $g$ evaluated at the base-point $p$:
  \[
  \partial_{k_1} \ldots \partial_{k_n} h_{ij} = \frac{(n-1)}{(n+1)!} \nabla_{(k_1} \ldots \nabla_{k_{n-1}} \nabla \eta_{i|j|k_n} + i \leftrightarrow j = \frac{(n-1)}{(n+1)} R_{k_1 ij k_2 \ldots k_n} + i \leftrightarrow j.
  \]
  Note that parenthetical symmetrization of $n$ objects has weight $n!$ instead of the usual weight 1, i.e.:
  \[
  \nabla_{(k_1} \nabla_{k_2)} = \nabla_{k_1} \nabla_{k_2} + \nabla_{k_2} \nabla_{k_1}.
  \]

- Matrix Geometry
  Capital letters refer to a multi-index notation in which a matrix $X^i$ is represented as $X^I = X^{i\alpha\beta}$. Sometimes it is easier to work in a local-Lorentz frame, in this case the matrix is written as: $X^A = X^{iab}$. The vielbein relating the $X^I$ and the $X^A$,
  \[
  X^A = E^A_i X^I \Rightarrow X^{iab} = \sum_{i\alpha\beta} E^{iab}_{i\alpha\beta} X^{i\alpha\beta},
  \]
  has a convenient flat space representation:
  \[
  E^A_i|_{\text{flat}} = \delta_I^A = \delta_i^\alpha \delta_a^\beta.
  \]
  The flat metric is defined in such a way that:
  \[
  \text{tr}(X^i X^j) \delta_{ij} = \eta_{IJ} X^I X^J \Rightarrow \eta_{IJ} = \eta_{i\alpha\beta, j\gamma\delta} = \delta_{ij} \delta_{\alpha\delta} \delta_{\beta\gamma}.
  \]
  The metric for curved space is the given by:
  \[
  G_{IJ} = E^A_i E^B_j \eta_{AB}
  \]
  The Riemann tensor in matrix geometry evaluated at the base-point $X = p$ is:
  \[
  R_{ijkl}(p) X^A X^B Y^K Y^L = R_{ijkl}(p) \text{Str}(X^i X^j Y^k Y^l).
  \]
  Here Str is the symmetrized trace (symmetrization has weight one).
The object $\Sigma$ is defined such that:

$$\Sigma_{a_1b_1,\ldots,a_nb_n}O^{i_1a_1b_1}\cdots O^{i_na_nb_n} = \text{Str}(O^{i_1}\cdots O^{i_n}).$$

Another object $S$ takes a set of matrices and combines them (symmetrically) into one matrix:

$$S_{a_1b_1}^{a_2b_2\ldots a_nb_n} O^{i_2a_2b_2}\cdots O^{i_na_nb_n} = \text{Sym}(O^{i_2}\cdots O^{i_n})^{a_1b_1}$$  \hspace{1cm} (A.1)

References


