A unified model for vortex-string network evolution

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We describe and numerically test the velocity-dependent one-scale (VOS) string evolution model, a simple analytic approach describing a string network with the averaged correlation length and velocity. We show that it accurately reproduces the large-scale behaviour (in particular the scaling laws) of numerical simulations of both Goto-Nambu and field theory string networks. We explicitly demonstrate the relation between the high-energy physics approach and the damped and non-relativistic limits which are relevant for condensed matter physics. We also reproduce experimental results in this context and show that the vortex-string density is significantly reduced by loop production, an effect not included in the usual ‘coarse-grained’ approach.

INTRODUCTION

Vortex-lines or topological strings can appear in a wide range of physical contexts, ranging from cosmic strings in the early universe to vortex-lines in superfluid helium (for reviews see ref.4–6). Gaining a quantitative understanding of their important effects represents a significant challenge because of their nonlinear nature and interactions and because of the complexity of evolving string networks. Considerable reliance, therefore, has been placed on numerical simulations but unfortunately these turn out to be technically difficult and very computationally costly. This provides strong motivation for alternative analytic approaches, essentially abandoning the detailed ‘statistical physics’ of the string network to concentrate on its ‘thermodynamics’. Here we present one such model for string network evolution, the velocity-dependent one-scale (VOS) model4–6, and demonstrate its quantitative success by direct comparison with numerical simulations. We are able to describe the scaling laws and large-scale properties of string networks in both cosmological and condensed matter settings.

The first assumption in this analysis is to ‘localise’ the string so that we can treat it as a one-dimensional line-like object. This is clearly a good assumption for gauged strings, such as magnetic flux lines, but may seem more questionable for strings possessing long-range interactions, such as global strings or superfluid vortex lines. However, as we shall see, we will be able to establish good agreement between the VOS model and simulations in both ‘local’ and ‘global’ cases. The second step is to average the microscopic string equations of motion to derive the key evolution equations for the average string velocity and correlation length L. This is a generalization of Kibble’s original ‘one-scale’ model4, and has been described elsewhere6.

We make a detailed comparison between the VOS model and numerical simulations – currently the world’s largest and highest resolution – using both direct field theory simulations of magnetic flux-lines, as well as simulations of Nambu strings, treating them as localised line-like objects. We are able to demonstrate good agreement between all three approaches, thus underpinning the important assumptions required for the VOS model. Fixing a single parameter, we are able to provide a good description of cosmic strings throughout the history of the universe, from the friction-dominated regime after their formation, through the radiation-matter transition and into the accelerating epoch today. Significantly, we believe the model also describes the same average features of an evolving vortex-line tangle in a condensed matter context, reproducing the expected and experimentally observed scaling law. The present VOS model in this context averages over both the background fluid friction and the Magnus force, but we believe it can be adapted further to incorporate other physical effects.

THE VOS STRING NETWORK MODEL

It is interesting to contrast the condensed matter and high-energy physics approaches to vortex string dynamics. In high-energy physics one uses, whenever possible, a one-dimensional view (obtained by integrating over radial modes of the vortex solution), described by the Nambu action. In this approximation the string equations of motion are4–6

\[
\ddot{x} + (1 - \dot{x}^2) \frac{\dot{x}}{\ell_d} = \frac{1}{\epsilon} \left( \frac{x'}{\epsilon} \right)' + \zeta x \wedge \frac{x'}{\epsilon} \tag{1}
\]

\[
\dot{\epsilon} + \frac{\dot{x}^2}{\ell_d^2} \epsilon = 0 \tag{2}
\]

where x is the string position, \(\epsilon\) is a measure of the energy along the string, and \(\ell_d = 2\mathcal{H} + \mu/\beta T^3\) is a damping length scale; time derivatives are with respect to conformal time (for condensed matter purposes this is identical to physical time). All terms can be rigorously derived...
from the Nambu action, except the last one in [1] which comes from the Kalb-Ramond action and describes the Magnus force arising when a global string moves through a relativistic superfluid background [8] whose density is parametrized by $\zeta$ [8, 11].

Now let us consider two limits of the equations of motion that are relevant in condensed matter systems. Firstly, if the damping term dominates, we find after some algebra (note that the Magnus force is not included, since it’s not dissipative)

$$\frac{\dot{x}}{\ell_d} = -\frac{1}{x'^2} \left[ x' \wedge (x' \wedge x'') \right], \quad (3)$$

where the right-hand side is the friction force term, e.g., on a superfluid vortex [2, 8, 14, 15]. Secondly, let us consider the non-relativistic limit but without damping. In this case we find

$$\dot{x} = \frac{1}{\zeta x'^2} x' \wedge x'', \quad (4)$$

which is the equation describing the frictionless motion of a vortex filament in an unbounded fluid [2, 11].

By combining these two terms we can therefore reproduce the equation of motion obtained by Schwarz [2, 10], which used an effective and more phenomenological 1D approach (based on a coarse-grained order parameter) to obtain the terms one by one. Note however that the Schwarz equation has further (subdominant) terms, coming from non-local and boundary contributions (which we have neglected altogether).

Now, it is well known experimentally [2, 8] that vortices in such systems evolve following a scaling law $L \propto t^{1/2}$, and the same is true for example for defects in liquid crystals [11]. In fact this is expected to hold for any phase ordering with a non-conserved order parameter (and in this context it is sometimes referred to as the Lifshitz-Allen-Cahn equation). We will see below that this scaling law can be easily derived from an averaged version of the evolution equations [12]—a process that might be compared to coarse-graining—in addition to also being relevant in high-energy contexts.

The VOS model has been described in detail elsewhere [4, 8], so here we limit ourselves to a brief summary. Any string network divides fairly neatly into two distinct populations, long (or ‘infinite’) strings and small closed loops. A phenomenological term must then be included to account for the loss of energy from long strings by the production of loops, which are much smaller than $L$—this is the ‘loop chopping efficiency’ parameter $\tilde{c}$. A further phenomenological term (characterized by a strength $\Sigma$ and a characteristic length scale $L_d$) is also included to account for radiation back-reaction effects. Note that all these parameters are constants—refer to [2, 8, 12] for details. This term is negligible for GUT-scale Goto-Nambu string networks, but can be relevant in other cases.

By suitably averaging Eqs. [12] one can obtain the following evolution equations

$$2 \frac{dL}{dt} = 2HL + \tilde{c}v + \frac{L}{d} v^2 + 8\Sigma v^6 \exp \left( -\frac{L}{dL} \right), \quad (5)$$

$$\frac{dv}{dt} = (1 - v^2) \left( \frac{k(v)}{L} - \frac{v}{\ell_d} \right); \quad (6)$$

here $H$ is the Hubble parameter (relevant for cosmology) and $k$ is a velocity-dependent phenomenological (but otherwise universal) parameter, called the ‘momentum parameter’, given by [8]

$$k(v) = \frac{2\sqrt{2}}{\pi} (1 - v^2)(1 + 2\sqrt{2}v^3) \frac{1 - 8v^6}{1 + 8v^6}; \quad (7)$$

its non-relativistic limit is $k_{nr}(v) \sim 2\sqrt{2}\pi$. We can now take the condensed matter (non-relativistic) limit of the VOS model. All we need to do is set $H = 0$ and the damping length to a constant. One finds a stable attractor solution

$$L = \sqrt{1 + \tilde{c}(\ell_d)^{1/2}} \quad v = \frac{\ell_d}{L}. \quad (8)$$

We have thus shown that our Goto-Nambu based microscopic equations of motion and our averaged version of them successfully reproduce known condensed matter results, respectively the Schwarz equation and the $L \propto t^{1/2}$ scaling law. Note that our solution demonstrates the importance of loop production—although this is not usually included in theoretical or numerical analyses in the condensed matter context, it has been observed in experiments [11]. We will now further test our averaged model in the context of field theory simulations and cosmolgy.

**FIELD THEORY SIMULATIONS**

In a previous paper [12] we have described the evolution of string networks in full 3D field theory simulations of the Abelian-Higgs model. This is a relativistic analogue of the Ginzburg-Landau theory of superconductivity. Before reviewing the comparisons between these simulations and the VOS model, it is worthwhile noting a byproduct of this work which relates to vortex tangles in condensed matter. In order to create quiescent initial conditions for string evolution in these simulations, instead of the relativistic equations, we began by solving the corresponding diffusive equations (refer to [12] for details). This was essentially equivalent to the non-relativistic evolution of magnetic flux-lines in a friction-dominated regime. Measurements of the string correlation length for the evolving network revealed a clear $L \propto t^{1/2}$ scaling, as illustrated in Fig. [10]. As well as confirming the VOS model prediction in this case [5], this has already been observed experimentally [11].
The relativistic evolution of the string networks also clearly revealed the predicted scaling laws and, remarkably, the correlation length and velocities for all the simulations had a good asymptotic fit from the VOS model using the single parameter $\tilde{c} \approx 0.57$. This fit was universal regardless of whether the simulations were in flat space or in an expanding universe, or whether matter or radiation eras, as illustrated in fig. 2. There is no degeneracy here between the chopping efficiency $\tilde{c}$ which determines the asymptotics and the massive radiation parameters $\Sigma = 0.5$ and $L_d = 4\pi$ which only affect the initial conditions. However, for global strings with massless radiation $L_d \to \infty$ there is such a degeneracy between $\tilde{c}$ and $\Sigma$ because they act in the same manner asymptotically. However, assuming the same loop chopping efficiency $\tilde{c} = 0.57$ for local and global cases, requires a much higher damping coefficient $\Sigma = 1.1$ for the latter as expected for massless radiation [12]. (These results and fits are also in reasonable agreement with other recent simulations of global strings in ref. [13].) This excellent correspondence for both local and global strings appears to establish the validity of the two key (‘localization’ and ‘thermodynamic’) assumptions underlying the VOS model.

**HIGH-RESOLUTION GAUGE STRING SIMULATIONS**

We now compare our model with ultra-high resolution numerical simulations, the details of which will be reported elsewhere. Specifically, we discuss both the approach to the linear scaling regime in the radiation and matter epochs, and the transition between the two epochs. Note that in this case both friction and radiative back-reaction are negligible. We first compare the model with three different ultra-high resolution runs (75 points per correlation length, fixed physical resolution) in the radiation and matter epochs, see Fig. 3. For all 6 runs, the analytic model fit displayed corresponds to a loop chopping efficiency parameter $\tilde{c} = 0.23$. We find that this value also approximately reproduces the earlier results of Bennett and Bouchet [14] and Allen and Shellard [15]. Residual differences can be attributed to the much higher resolution of our runs. Earlier work [14] reported different scaling values, but numerical codes were substantially improved subsequently. Thus fixing this parameter via the radiation era, our model correctly predicts the matter era scaling large-scale properties without any further tampering with parameters. We estimate the loop chopping efficiency to have the value $\tilde{c}_{ren} = 0.23 \pm 0.04$. We emphasize that we expect this to be a ‘universal’ parameter, independent of the cosmological scenario in which the string network is evolving. However, if one performs analogous simulations in flat (Minkowski) spacetime, one does find a different value, $\tilde{c}_{ren} = 0.57 \pm 0.04$, which coincides with the value we found above for field theory simulations. This is because the amount of small scale structure present in Goto-Nambu expanding runs is much larger than that in field theory and/or Minkowski space runs, and this has an influence on the large-scale features of the network described by the model. Hence the two values can be regarded as the ‘renormalized’ and ‘bare’ chopping efficiencies. This is discussed at greater length in [12].

Now, given that we can reproduce the radiation and matter era scaling values, can we also reproduce the transition between them? Note that this point is not straightforward: nothing guarantees us, a priori, that the model will produce the correct timescale for the transition even
initial conditions, for \( \tilde{c} = 0.23 \).

If both its asymptotic limits are correct. Here it is not possible to do a single numerical simulation spanning all the relevant time interval, so we have performed a sequence of twelve different high-resolution runs (16 points per correlation length, fixed physical resolution) which together span the required range, see Fig. 4. Again we can see that the analytic model, with the previously determined parameter \( \tilde{c} = 0.23 \) can actually do a good job in predicting the overall timescale of the transition between the regimes. We also emphasise that a GUT-scale network at the present time has not quite yet reached the asymptotic matter-era scaling regime. This slow relaxation process has been pointed out before \([4]\), and is clearly of vital importance for an adequate analysis of the main assumptions on which the VOS model is based, notably string ‘locality’ and ‘thermodynamic’ averaging. Our results confirm that the dominant mechanisms affecting string network evolution are loop production and damping, whether from friction or radiation depending on the context. In condensed matter systems we find that the two are of comparable magnitude, despite loop production being neglected in the usual treatments. For global strings, loop production is dominant but radiation damping can significantly affect the network density.

The outstanding issue raised by comparisons of gauged string networks in Nambu and field theory simulations remains the modelling of small-scale features. A more detailed analysis of the simulations provides some valuable insights into this problem, but this will be the subject of a forthcoming publication.

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