In this Note a social network model for opinion formation is proposed in which a person connected to \( q \) partners pays an attention \( 1/q \) to each partner. The mutual attention between two connected persons \( i \) and \( j \) is taken equal to the geometric mean \( 1/\sqrt{q_iq_j} \). Opinion is represented as usual by an Ising spin \( s = \pm 1 \), and mutual attention is given through a two-spin coupling \( J_{ij} = JQ/\sqrt{q_iq_j} \), \( Q \) being the average connectivity in the network. Connectivity diminishes attention and only persons with low connectivity can pay special attention to each other leading to a durable common (or opposing) opinion. The model is solved in “mean-field” approximation and a critical “temperature” \( T_c \) proportional to \( JQ \) is found, which is independent of the number of persons \( N \), for large \( N \).

**Key words:** sociophysics, random networks, opinion formation, Ising model

Recently Aleksiejuk et al.\[1\] proposed a social network model for opinion formation by introducing ferromagnetically coupled Ising spins on a Barabási-Albert network \[2\]. The strength of the couplings between linked nodes, \( J \), was taken to be uniform, independent of the number of connections to a node. Simulations of this model indicated the existence of a critical “temperature” below which opinion formation is possible (two-phase coexistence) and above which common opinion is unstable (disordered phase). A peculiar feature of this model is that simulations indicate that the critical temperature depends on the number of nodes, or system size, \( N \) in the manner \( T_c \propto \log(N) \), so that in the thermodynamic limit an initially imposed common opinion alwayspersists (in the absence of an external opposing “field”), no matter how weak the uniform coupling \( J \) between each pair of connected persons, or “partners” \[1\]. Incidentally, this peculiar divergence of \( T_c \) with \( N \) is missed in the simplest mean-field approximation to the model \[1\], but is captured correctly in an improved mean-field approach \[3–5\], and agrees with exact results for uncorrelated networks \[6\].
In the model of Aleksiejuk et al., the assumption of uniform coupling lends a lot of influence to nodes with many connections. Indeed, overturning the spin of a small set of most heavily connected nodes suffices for reversing the opinion of the entire network. It is, however, questionable whether a node, e.g., a person, with many partners (i.e., nodes to which he/she is linked) is able to influence all these partners as strongly as a person with only a few partners would be able to do. An intensive person-to-person discussion presumably creates a stronger tendency to form a durable common (or, for antiferromagnetic couplings, opposing) opinion than a one-to-many communication. Therefore we propose to attribute an “attention” to each person, inversely proportional to the number of partners. In network terms, a node with connectivity \( q_i \) is capable of maintaining an attention

\[
\alpha_i = \frac{1}{q_i}
\]  

(1)

towards each of its partners, the total attention per node being normalized to 1. Thus we assume that each person pays the same total attention, \( q_i \alpha_i = 1 \), to the exterior. If the average connectivity is denoted by \( Q \), \( q_i < Q \) signifies “special attention” and \( q_i > Q \) implies “little attention”. The model further assumes that in order to establish a strong mutual influence of opinion, the average of the attentions of two connected persons must be sufficiently big. Thus the average of their connectivities, \( q_i \) and \( q_j \), must be small enough. In particular, if both partners pay special attention to each other they are more likely to maintain durable agreement (or disagreement). In contrast, if they pay little attention to each other, opinion formation is difficult. An interesting mixed case is that of a person \( i \) paying special attention (\( q_i < Q \)) to TV or other mass media \( j \) (\( q_j >> Q \)). Although the person may quickly form an opinion dictated by the mass medium, this opinion is not strengthened by loyalty or peer-pressure considerations and may quickly “flip”. For example, the person soon realizes that while he/she may be especially devoted to TV (high \( \alpha_i \)), TV cares little or nothing about him/her individually (low \( \alpha_j \)), which weakens considerably the persuasive power of the medium.

Along this line of reasoning we advocate that reciprocity of attention is important for durable opinion formation and an interaction is proposed which is proportional to the mean - for calculational simplicity the geometric mean - of the attentions of the connected persons,

\[
J_{ij} = J Q \sqrt{\alpha_i \alpha_j}
\]  

(2)

Note that the average coupling equals \( J \), within a mean-connectivity approximation (\( q_i = Q \)), which in the present social context is called a mean-attention approximation. Taking the geometric mean leads to separable couplings, which usually facilitates calculations drastically (cf. separable spin-glass models [7]).
Also note that, in this symmetric model, there is no directionality in the couplings, $J_{ij} = J_{ji}$. At this stage no distinction is made between attention as “speaker” or as “listener”.

For a start, it is straightforward to apply a double mean-field approximation as follows. For a given network realization (quenched randomness) the exact self-consistent “equation of state” reads

$$< s_i > = < \tanh\left( \sum_{j=1}^{q_i} \frac{J_{ij}}{k_B T} s_j \right) >,$$  

(3)

where the bracket denotes thermal average and $k_B$ is the Boltzmann constant. Now we perform the quenched random average over all networks simply in the mean-attention approximation $\alpha_i \approx 1/Q$, and we also invoke the mean-opinion approximation $s_i \rightarrow < s_i > \approx S$, with $-1 < S < 1$, where $S$ is the average opinion. This gives

$$S = \tanh(SJQ/k_B T),$$  

(4)

which leads to the familiar critical “temperature” $T_c = JQ/k_B$ proportional to the average connectivity $Q$.

In a more refined step we apply the improved mean-field approach of Bianconi [3] and Leone et al.[4]. In this calculation an equation of state for the mean local opinion $< s_i >$ is found, of the form

$$< s_i > = \tanh\left( \sum_{j=1}^{N} \frac{|J_{ij}|}{kT} < s_j > \right),$$  

(5)

where the square brackets denote the quenched random average over network realizations. Note that now the sum runs over all nodes. Specifically, for the present model, and assuming a Barabási-Albert network,

$$[J_{ij}] = J_{ij} p_{ij} = \frac{J}{N} \sqrt{q_i q_j},$$  

(6)

where $p_{ij}$ is the probability that nodes $i$ and $j$ are linked.

It is then natural to define the order parameter

$$\hat{S} = \frac{1}{\sqrt{QN}} \sum_{j=1}^{N} \sqrt{q_j} < s_j >$$  

(7)
Following Bianconi’s continuum approximation [3] the critical temperature is then, for large $N$, obtained from the linearized equation

$$\hat{S} = \frac{1}{N} \int d n' \frac{J}{k_B T} \hat{S} q(n'),$$

(8)

with, for the Barabási-Albert network in the large-$N$ limit [2],

$$q(n) \sim \frac{Q}{2} \sqrt{\frac{N}{n}},$$

(9)

where $q(n)$ stands for the connectivity of node $n$. This also leads to the same result $T_c = JQ/k_B$.

In view of the fact that none of the spins on this network exerts an anomalously strong influence on other parts of the network, one may expect that this model behaves normally, so that, when fluctuations are taken into account, a finite critical temperature results which is independent of $N$, in the large network limit. Indeed, in the model of Aleksiejuk et al. a highly connected spin $s_i$ exerts a local field of strength $J$ on a large set of $q_i$ other spins. Consequently, for large $N$ such centers of massive influence may well induce significant deviations from a “well-behaved” thermodynamic limit, as is corroborated by the simulations [1].

In sum, an opinion formation model has been proposed in which each person can devote a fixed total amount of attention to others, distributing this attention equally over all partners. The Special Attention Network presented here attenuates the strong influence exerted by highly connected nodes in networks with uniform couplings $J$, by introducing a detailed local compensation of high connectivity by weak interaction. Therefore, we conjecture that the essential feature $T_c \propto JQ$ captured here in the mean-field approximation(s) holds true also for the fluctuating spin model on any quenched random Special Attention Network, scale-free or not, with finite mean connectivity $Q < \infty$. Careful Monte Carlo simulation and/or more sophisticated analytical calculation will be needed to verify this.

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