Quantum Liouville theory and BTZ black hole entropy

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Abstract

In this paper I give an explicit conformal field theory description of (2+1)-
dimensional BTZ black hole entropy. In the boundary Liouville field theory I in-
vestigate the reducible Verma modules in the elliptic sector, which correspond to
certain irreducible representations of the quantum algebra $U_q(sl_2) \otimes U_q(sl_2)$. I show
that there are states that decouple from these reducible Verma modules in a similar
fashion to the decoupling of null states in minimal models. Because of the nonstan-
dard form of the Ward identity for the two-point correlation functions in quantum
Liouville field theory, these decoupling states have positive-definite norms. The ex-
licit counting from these states gives the desired Bekenstein-Hawking entropy in
the semi-classical limit when $q$ is a root of unity of odd order.

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1 Introduction

The classical laws of black hole mechanics together with the temperature of Hawking radiation suggest the identification of $A/4$ with the physical entropy of a black hole, where $A$ is the area of the horizon. A major goal of research in quantum gravity is to understand the statistical origin of this formula. Besides serving as a useful model for realistic black hole physics, the $(2+1)$-dimensional BTZ black hole [1] has also been found to be related to the near-horizon geometries of many high dimensional black hole solutions (see for example, [2–10]). Thus it is of strong importance to understand BTZ black hole entropy from first principles.

The asymptotic symmetry group [11] of $(2+1)$-dimensional gravity with negative cosmological constant $\Lambda = -1/l^2$ is generated by two copies of the Virasoro algebra, with classical central charge

$$c_L = c_R = \frac{3l}{2G},$$

where $G$ is the gravitational constant in $2+1$ dimension. Based on this fact, a simple derivation of the BTZ black hole entropy was given in [2,3] using Cardy’s formula [12,13,15], which states that the asymptotic density of states for a conformal field theory is given by

$$\rho(\Delta, \bar{\Delta}) \sim \exp \left\{ \frac{c_R - 24\Delta_0}{6} \Delta \right\} \exp \left\{ \frac{c_L - 24\bar{\Delta}_0}{6} \bar{\Delta} \right\},$$

where $\Delta$, $\bar{\Delta}$ are the eigenvalues of Virasoro generators $L_0$ and $\bar{L}_0$, and $\Delta_0$, $\bar{\Delta}_0$ the lowest eigenvalues. For the BTZ black hole [16,2],

$$M = (\Delta + \bar{\Delta})/l, \quad J = \Delta - \bar{\Delta},$$

where $M$ and $J$ are the mass and angular momentum of the black hole. Substituting into (1.2), assuming that

$$\Delta_0 = \bar{\Delta}_0 = 0,$$

we obtain the Bekenstein-Hawking entropy for BTZ black hole.

Unfortunately, such a derivation does not tell us what microscopic degrees of freedom contribute to the black hole entropy. The presence of the asymptotic conformal algebra strongly suggests that the asymptotic dynamics is described by a two-dimensional conformal field theory. We thus would like to have a more concrete understanding of the black hole entropy by explicitly counting the states in this boundary conformal field theory.

A good candidate for such a boundary conformal field theory is Liouville field theory. $(2+1)$-dimensional Einstein gravity with $\Lambda < 0$ can be reformulated as a Chern-Simons gauge theory with gauge group $SL(2, R) \times SL(2, R)$ [17,18], with gauge potentials

$$A^{(\pm)a} = \omega^a \pm e^a/l,$$
where \( e^a = e^a_{\mu} dx^\mu \) is the triad and \( \omega^a = \frac{1}{2} \epsilon^{abc} \omega_{\mu bc} dx^\mu \) is the spin connection. The Einstein-Hilbert action becomes

\[
I = I_{\text{CS}}[A^{(-)}] - I_{\text{CS}}[A^{(+)}],
\]

where

\[
I_{\text{CS}}[A] = \frac{l}{16\pi G} \int_M \text{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\}
\]

is the Chern-Simons action. The asymptotically AdS boundary condition \([11,19]\) reduces the asymptotic dynamics to a boundary Liouville theory \([19,20]\).

There have been questions whether Liouville field theory provide enough microstates for BTZ black hole entropy counting \([21]\). This issue is related to the spectrum of Liouville theory, which we will discuss in Section 3.6. After canonical quantization, the spectrum of Liouville theory consists of two different classes. In Seiberg’s notation, they are called the normalizable macroscopic states and the nonnormalizable Hartle-Hawking states respectively. The state-counting of the normalizable states is well understood. Define the effective central charge \([14,15]\) as

\[
c_{\text{eff}} = c - 24\Delta_0.
\]

The lowest Virasoro eigenvalue for normalizable states is \( \Delta_0 = (c - 1)/24 \) rather than 0, so \( c_{\text{eff}} = 1 \). Thus the density of states behaves like that of an ordinary scalar field, which does not provide enough states for entropy counting. However, the condition \((1.4)\) may be satisfied by the nonnormalizable Hartle-Hawking states, which correspond to local operator insertions. This suggests that in order to understand the state counting, we ought to investigate these Hartle-Hawking states instead.

The understanding of state-counting in Liouville field theory has further motivations, since Liouville theory can also be obtained near the horizon of an arbitrary black hole by dimensionally reducing to \( r - t \) plane \([22,23]\). There, similarly, the classical central charge of Liouville theory gives the Bekenstein-Hawking entropy of the black hole by applying Cardy’s formula. Thus the understanding of quantum Liouville theory and its explicit state counting could offer an explanation for the “universality” of the Bekenstein-Hawking entropy in different approaches to quantum gravity.

In this paper we first review in Section 2 classical Liouville field theory, which is closely related to the description of two-dimensional surfaces. We follow in Section 3 and 4 with a summary of the canonical quantization procedure proposed by Gervais and his collaborators \([24, 25, 27–34]\), which shows manifestly the underlying quantum algebra structure of the theory. In Section 5, we construct the Hartle-Hawking states corresponding to certain irreducible representations of the quantum algebra. The conformal weights of these states are of Kac form, and the Verma modules built on them are reducible. In order to define the norm of the states decoupling from these reducible Verma modules, Section 6 gives a discussion of the Ward identity of the two-point functions in Liouville field theory, whose difference with the standard form has a geometric origin. In Section 7 and 8 we show that these decoupling states can have positive-definite norms and that the corresponding Verma
modules are unitary irreducible representations of Virasoro algebra. When \( q \) is a root of unity of odd order, in the semi-classical limit these states give the Bekenstein-Hawking entropy.

2 Classical Liouville field theory

Here we give a quick review of classical Liouville field theory. Consider the Liouville action with Euclidean signature \([35,36]\)

\[
I_L = \frac{1}{4\pi} \int d^2x \sqrt{\hat{g}} \left[ \frac{1}{2} \hat{g}^{ab} \partial_a \phi \partial_b \phi + \frac{\mu}{2\gamma^2} e^{\gamma \phi} + \frac{1}{\gamma} R(\hat{g}) \phi \right],
\]

(2.1)

where \( \hat{g}_{ab} \) is the fixed background metric and \( \phi \) is the Liouville field. The coupling constant \( \gamma \) is related to the cosmological constant by \( \gamma^2 = 8G/l^2 \), and \( R \) is the scalar curvature of the background metric.

Classically the action (2.1) defines a conformal field theory invariant under the Weyl transformation

\[
\hat{g}_{ab} \rightarrow e^{2\rho} \hat{g}_{ab}, \quad \gamma \phi \rightarrow \gamma \phi - 2\rho.
\]

(2.2)

We also define a new field \( \varphi = \gamma \phi \), which will be convenient when we discuss the connection between Liouville theory and quantum geometry. In terms of \( \varphi \) the action becomes

\[
I_L = \frac{1}{8\pi \gamma^2} \int d^2x \sqrt{\hat{g}} \left[ \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + 4e^\varphi + R(\hat{g}) \varphi \right].
\]

(2.3)

Here \( \mu \) is set to be 4 by a shift in the value of \( \varphi \), so that each solution of the equation of motion corresponds to a two-dimensional surface with constant Gaussian curvature \( K = -1 \).

In terms of the complex coordinates,

\[
z = e^{\tau + i\sigma},
\]

(2.4)

the improved energy-momentum tensor \( T \) is given by

\[
T_{zz} = \frac{1}{\gamma^2} (\varphi_{zz} - \frac{1}{2} \varphi_z^2)
\]

(2.5)

with Fourier modes that satisfy the Virasoro algebra

\[
i\{L_m, L_n\}_{\text{P.B.}} = (m-n)L_{m+n} + \frac{c_{\text{cl}}}{12} (m^3 - m) \delta_{m,-n}
\]

(2.6)

with a classical central charge \( c_{\text{cl}} = 12/\gamma^2 \) (see, for example [35,36]).
elliptic solution  hyperbolic solution  parabolic solution

Figure 1: $\sigma$-independent Euclidean solutions

### 2.1 Classical solutions in Euclidean space

One can choose a local coordinate system such that $\hat{g}_{ab} = \delta_{ab}$. The equation of motion for $\varphi$, called the Liouville equation [37], is then

$$\partial_z \partial_{\bar{z}} \varphi = \frac{1}{2} e^\varphi. \quad (2.7)$$

By the uniformization theorem, each solution of (2.7)

$$e^{\varphi_{cl}(z)} dz d\bar{z} = 4 \frac{\partial A(z) \partial B(\bar{z})}{[A(z) + B(\bar{z})]^2} d\bar{z} d\bar{z}, \quad (2.8)$$

describes a two-dimensional surface with constant negative Gaussian curvature $K = -1$ conformally equivalent to a quotient of the Poincaré upper half plane $H$ by a discrete subgroup $\Gamma \in \text{PSL}(2, \mathbb{R})$, for some locally defined (anti-)holomorphic functions $A(z)(B(\bar{z}))$.

Along a curve $z \to e^{2\pi i z}$, $A$ and $B$ transform by an $SL(2, \mathbb{R})$ transformation. Depending on the conjugacy class of the monodromy of $A(z)$ and $B(\bar{z})$, there are three classes of local solutions: elliptic, parabolic and hyperbolic.

1. **elliptic:** $A(z) \to (TR_\theta T^{-1})A(z)$, the curve surrounds a conical singularity on the surface,

2. **parabolic:** $A(z) \to (TP_\lambda T^{-1})A(z)$, the curve surrounds a puncture on the surface,

3. **hyperbolic:** $A(z) \to (TB_\epsilon T^{-1})A(z)$, the curve surrounds a handle of the surface,

where

$$R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad P_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad B_\epsilon = \begin{pmatrix} e^{\pi \epsilon} & 0 \\ 0 & e^{-\pi \epsilon} \end{pmatrix},$$

and $T$ is a $\text{PSL}(2, \mathbb{C})$ (Möbius) transformation of $A(z)$. $B(\bar{z})$ transforms in a similar fashion.

Here are some $\sigma$-independent examples of the classical solutions, shown in Fig. 1, which are:
1. elliptic solution:
\[ e^\varphi = \frac{4a^2}{(z\bar{z})^{1-a}[1 - (z\bar{z})^a]^2}, \quad T(z) = \frac{1}{z^2} \left( -\frac{a^2}{2\gamma^2} + \frac{1}{2\gamma^2} \right), \] (2.9)

2. parabolic solution:
\[ e^\varphi = \frac{4}{z\bar{z}[\log z\bar{z}]}^2, \quad T(z) = \frac{1}{z^2} \frac{1}{2\gamma^2}, \] (2.10)

3. hyperbolic solution:
\[ e^\varphi = \frac{4m^2}{z\bar{z} \left[ \sin \left( \frac{m\gamma}{2} \log z\bar{z} \right) \right]^2}, \quad T(z) = \frac{1}{z^2} \left( \frac{m^2}{2\gamma^2} + \frac{1}{2\gamma^2} \right). \] (2.11)

2.2 The classical \( SL(2, \mathbb{C}) \) symmetry

Consider the system defined on a flat cylinder, where \( \sigma \in [0, 2\pi) \) parametrizes space and \( \tau \) parametrizes imaginary time. Define light-cone coordinates \( x^\pm = \sigma \mp i\tau \).

A Liouville solution with periodic boundary condition can be rewritten in terms of the chiral fields \( \chi_i(x^+), \xi_i(x^-), i = 1, 2 \), with conformal weights \((-1/2, 0)\) and \((0, -1/2)\):
\[ e^{-\varphi} = \frac{1}{4}(\chi_2 \xi_1 + \chi_1 \xi_2)^2, \] (2.12)

where \( \chi_i(x^+), \xi_i(x^-) \) are two pairs of real solutions of the Schrödinger equations
\[ (\partial_+^2 - \frac{\gamma^2}{2} T_{++}) \chi_i = 0 \quad i = 1, 2 \]
\[ (\partial_-^2 - \frac{\gamma^2}{2} T_{--}) \xi_i = 0, \] (2.13)

with unit Wronskians \( \chi_1 \chi_2' - \chi_1' \chi_2 = 1, \xi_1 \xi_2' - \xi_1' \xi_2 = 1 \). The real periodic potentials \( T_{\pm\pm} = \frac{1}{\gamma}(\varphi_{\pm\pm} + \frac{1}{2}\varphi_{\pm\pm}^2) \) satisfy \( T_{\pm\pm}(\sigma + 2\pi) = T_{\pm\pm}(\sigma) \).

Note that a constant Möbius transformation
\[ \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \begin{pmatrix} \xi_1 \\ -\xi_2 \end{pmatrix} \rightarrow -\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1 \\ -\xi_2 \end{pmatrix} \] (2.14)

preserves both \( e^{-\varphi} = \frac{1}{4}(\chi_2 \xi_1 + \chi_1 \xi_2)^2 \) and the unit Wronskians. For \( 2j \) a positive integer, we have primary fields \( e^{-j\varphi} \) with weights \((-j, -j)\) expressed as
\[ e^{-j\varphi} = \left( \frac{1}{4} \right)^2 (\chi_2 \xi_1 + \chi_1 \xi_2)^{2j} \]
\[ = \left( \frac{1}{4} \right)^2 \sum_{m=-j}^{j} (-1)^{j-m} \psi_m^j(x^+) \psi_{-m}^j(x^-), \] (2.15)
where the fields
\[
\psi^j_m(x^+) = \sqrt{\left(\frac{2j}{j-m}\right)} \chi^j_{1m} x^m, \quad \psi^j_{-m}(x^-) = \sqrt{\left(\frac{2j}{j-m}\right)} \xi^j_{1m} (-\xi^j_2)^{j-m}
\] (2.16)
transform under \(SL(2,C)\) transformations like the spin-\(j\) representation with finite dimension \(2j+1\). For \(2j\) a negative integer, on the other hand, the decomposition into chiral fields is infinite. These fields also form \(SL(2,C)\) representation.

3 Canonical quantization

3.1 Bäcklund transformation

For canonical quantization, we may work at imaginary time \(\tau = 0\) without loss of generality. Because the potential \(T\) of the Schrödinger equation (2.13) is periodic, \(T(\sigma + 2\pi) = T(\sigma)\), the solution has periodicity
\[
\chi_i(\sigma + 2\pi) = M^j_i \chi_j(\sigma),
\] (3.1)
where \(M^j_i\) is the monodromy matrix of the Schrödinger equation, which is related to the monodromy property of the classical solution. For the hyperbolic and elliptic solutions, we can use an \(SL(2,C)\) transformation to diagonalize \(M^j_i\) so that \(\chi_i\) (and \(\xi_i\)) becomes periodic up to a multiplicative constant
\[
\chi_1(\sigma + 2\pi) = e^{\gamma\pi p^{(i)}_0} \chi_1(\sigma), \quad \chi_2(\sigma + 2\pi) = e^{\gamma\pi p^{(2)}_0} \chi_2(\sigma),
\] (3.2)
where \(p^{(i)}_0\) are real for hyperbolic solution and imaginary for elliptic solution. We will not consider the parabolic solution in this paper. A priori \(\chi_i, \xi_i\) do not need to be real, as long as the resulting \(\varphi\) field is real.

The chiral fields \(\chi_i\) can be written in terms of bosonic fields \(q^{(i)}\) as
\[
\chi_i = \exp\left(\frac{q^{(i)}}{2}\right).
\] (3.3)

With the chosen periodicity (3.2) the fields \(q^{(i)}\) and \(\bar{q}^{(i)}\) can be expanded in Fourier series:
\[
q^{(i)}(\sigma) = q^{(i)}_0 + p^{(i)}_0 \sigma + i \sum_{m \neq 0} a^{(i)}_m e^{-im\sigma}/m, \quad i = 1, 2,
\] (3.4)
and it can be shown that the Fourier modes \([39,24–27,32]\) satisfy the Poisson brackets
\[
\{a^{(i)}_n, a^{(j)}_m\}_{\text{P.B.}} = -in \delta_{n,-m}.
\] (3.5)
Assuming in addition that \(\{q^{(i)}_0, p^{(i)}_0\}_{\text{P.B.}} = 1\), the Liouville field \(\varphi(\sigma)\) is thus related to the free fields \(q^{(i)}(\sigma)\) and the anti-chiral counterparts \(\bar{q}^{(i)}(\sigma)\) through a classical canonical transformation (a Bäcklund transformation).
The stress energy tensor $T_{++}$ can be expressed in terms of either set of the free fields [26,27] as

$$T_{++} = \frac{1}{\gamma} [q^{(1)\prime\prime} + \frac{\gamma}{2} (q^{(1)\prime})^2] = \frac{1}{\gamma} [q^{(2)\prime\prime} + \frac{\gamma}{2} (q^{(2)\prime})^2].$$  (3.6)

The conformal generators can be written in either set of the Fourier modes, and give a Poisson-bracket realization of the Virasoro algebra (2.6).

### 3.2 Canonical quantization

After the Bäcklund transformation, since $q^{(i)}(\sigma)$ and $\bar{q}^{(i)}(\sigma)$ are free fields, the quantization is straightforward. All the complications of the interacting theory are in the Bäcklund transformation.

Classically the two free fields $q^{(1)}(\sigma)$ (or $\bar{q}^{(1)}(\sigma)$) and $q^{(2)}(\sigma)$ (or $\bar{q}^{(2)}(\sigma)$) are not independent. We could express one of the free field in terms of the other and proceed with the canonical quantization in terms of one free field as in [38,39]. Instead, we will follow the canonical quantization procedure of Gervais and Neveu [24–34], where the two free fields remain symmetric in the quantization, so that the quantum algebra structure becomes clear.

The basic idea of [24–34] is quite natural. Classically we have the relation (3.6), where the description in terms of two sets of fields are equivalent. Now impose canonical quantization conditions such that the two sets of the free fields remain symmetric:

$$[q_0^{(i)}, p_0^{(i)}] = i, \quad [a_n^{(i)}, a_m^{(i)}] = n \delta_{n,-m},$$  (3.7)

$$N^{(1)} [q^{(1)\prime\prime} + \frac{\gamma}{2} (q^{(1)\prime})^2] = N^{(2)} [q^{(2)\prime\prime} + \frac{\gamma}{2} (q^{(2)\prime})^2],$$

$$p_0 \equiv p_0^{(1)} = -p_0^{(2)}.\quad (3.9)$$

In (3.8) $N^{(i)}$ define normal orderings for the two sets of fields. Condition (3.9) ensures that the Wronskians will not change when $\sigma \to \sigma + 2\pi$.

The Virasoro generators can then be written in terms of either set of the normal-ordered creation-annihilation operators [26,27]

$$T_0 = N^{(i)} \left( \frac{1}{2} p_0^{(i)} a_0^{(i)} + \sum_{n=1}^{\infty} a_{-n}^{(i)} a_n^{(i)} \right),$$

$$T_m = N^{(i)} \left( p_0^{(i)} a_m^{(i)} - \frac{i}{\gamma} m a_m^{(i)} + \frac{1}{2} \sum_{n \neq 0,m} a_{-m-n}^{(i)} a_n^{(i)} \right), \quad m \neq 0,$$  (3.10)

and satisfy

$$[T_m, T_n] = (m-n)(T_{m+n} - \frac{1}{24} \delta_{m,-n}) + \frac{1}{12} \left( 1 + \frac{12}{\gamma^2} \right) m^3 \delta_{m,-n}$$

(3.11)

with quantum central charge

$$c = 1 + \frac{12}{\gamma^2}.\quad (3.12)$$
Mapped onto the complex plane, the operators
\[ L_0 = T_0 + \frac{1}{2\gamma^2}, \quad L_m = T_m, \] (3.13)
satisfy the standard Virasoro algebra.

### 3.3 Quantum coupling constant

After canonical quantization, the conformal weight of the vertex operator \( e^{\alpha \phi} \) is shifted from the classical value to
\[ \Delta = \bar{\Delta} = -\frac{\gamma^2}{2} \left( \alpha - \frac{1}{\gamma^2} \right)^2 + \frac{1}{2\gamma^2}, \] (3.14)

It is convenient to define a quantum coupling constant \( \tilde{\gamma} \) [28,29]. The quantum operator corresponding to the metric is no longer \( e^\phi \) but is renormalized to \( e^{\nu \phi} \), with conformal weight \( (1,1) \), such that \( e^{\nu \phi} dz d\bar{z} \) remains conformally invariant. The quantum coupling constant is defined by \( \tilde{\gamma} = \nu \gamma \), and satisfies the equation
\[ \frac{\tilde{\gamma}}{\gamma} - \frac{1}{2} \tilde{\gamma}^2 = 1, \] (3.15)
which has two solutions:
\[ \tilde{\gamma}_\pm = \frac{1 \pm \sqrt{1 - 2\gamma^2}}{\gamma}. \] (3.16)

It is this quantum coupling constant \( \tilde{\gamma} \) which will appear in the deformation parameters in the quantum algebra.

In terms of the quantum coupling constant \( \tilde{\gamma} \), the central charge (3.12) now becomes
\[ c = 1 + 3 \left( \frac{2}{\tilde{\gamma}} + \tilde{\gamma} \right)^2, \] (3.17)
or equivalently
\[ \tilde{\gamma}^2 = \frac{c - 13 \pm \sqrt{(c - 1)(c - 25)}}{6}. \] (3.18)

The renormalization of the coupling constant is the same as in [40].

In this paper we are interested in the region \( c > 25 \), the so-called “weak-coupling region”, which corresponds to the semi-classical region \( l \gg G \). In this region, both \( \gamma \) and \( \tilde{\gamma} \) are real. In the following we will write \( \tilde{\gamma}_- \) as \( \tilde{\gamma} \) and \( \tilde{\gamma}_+ \) as \( 2/\tilde{\gamma} \).
3.4 The nonexistence of $SL(2, C)$ invariant vacuum

Define the operator

$$\varpi = i \frac{2}{\gamma} p_0,$$

where the zero-mode $p_0$ is defined in (3.9). For $c > 25$ and $\gamma$ real, since the spectrum $p_0$ of $p_0$ is purely imaginary in the elliptic sector, the spectrum $\varpi$ of $\varpi$ is real. In the hyperbolic sector, on the other hand, the spectrum of $\varpi$ is purely imaginary.

Introduce a basis of zero mode eigenstates $|\varpi, 0\rangle$ such that

$$\varpi |\varpi, 0\rangle = a^{(i)}_n |\varpi, 0\rangle = 0, \quad n > 0, \quad i = 1, 2.$$

These are highest-weight states, satisfying

$$L_0 |\varpi, 0\rangle = \left[ \frac{1}{2\gamma^2} + \frac{1}{2} p_0^{(1)} \right] |\varpi, 0\rangle = \left[ \frac{1}{2\gamma^2} + \frac{1}{2} p_0^{(2)} \right] |\varpi, 0\rangle,$$

$$L_n^{(1)} |\varpi, 0\rangle = L_n^{(2)} |\varpi, 0\rangle = 0, \quad n > 0.$$

The Hilbert space is a direct sum of Verma modules (not necessarily irreducible) which are obtained by applying either $L_{-n}^{(1)}$ or $L_{-n}^{(2)}$ for $n > 0$ on the highest-weight states labeled by $\varpi$. The two chiral Verma modules generated by $p_0^{(1)}$ and $p_0^{(2)}$ coincide, since the highest weights only depends upon $p_0^{(1)} = p_0^{(2)}$.

The ground state structure of quantum Liouville theory is rather unconventional. The canonical quantization condition (3.8) can be solved perturbatively [27], yet the (quantum) series expansion for one set of oscillators in terms of the other has manifest poles and breaks down at certain values of zero modes $p_0 = im/\gamma$, $m \in \mathbb{Z}\{0\}$.

Classically for these values of $p_0$, the Liouville field $\varphi$ cannot be real. The value $p_0 = i/\gamma$ corresponding to $\Delta = 0$ is included in this set of zero modes, which means that the $SL(2, C)$ invariant vacuum is not included in the spectrum. We can see this explicitly [41] from the fact that the $L_{-1}$ (translation) operator acting on a ground state

$$L_{-1}^{(i)} |\varpi\rangle = \left( p_0^{(i)} a_{-1}^{(i)} + \frac{i}{\gamma} a_{-1}^{(i)} + \frac{1}{2} \sum_{n \neq 0, -1} a_{-1-n}^{(i)} a_n^{(i)} \right) |\varpi\rangle = 0$$

implies that $p_0^{(1)} = p_0^{(2)} = -i/\gamma$, which contradicts with the condition that $p_0^{(1)} = -p_0^{(2)}$.

D’Hoker and Jackiw [42] argue from the quantum equation of motion

$$\nabla \varphi = 2 e^{i\varphi}$$

that no translationally invariant normalizable vacuum $|0\rangle$ exists in Liouville theory, since that would imply that

$$\langle 0 | 2 e^{i\varphi} | 0 \rangle = \langle 0 | \nabla \varphi | 0 \rangle = 0,$$

violating the formal positivity of the exponential. We will see in Section 6 that the nonexistence of $SL(2, C)$-invariant vacuum in the spectrum has important consequences in the whole Hilbert space structure of the quantum theory.
3.5 Construction of Hartle-Hawking states

In the framework of standard conformal field theory, a basic assumption is that all highest-weight states are generated by applying a primary field to the $SL(2,C)$-invariant vacuum $|0\rangle$. As we already discussed above, the situation is not standard in Liouville theory because of the absence of the $SL(2,C)$-invariant vacuum. The nonexistence of such a vacuum in the spectrum means that the standard map from an operator $\mathcal{O}$ to the state $\mathcal{O}(z=0)|0\rangle$ cannot be used here. Instead, we can create a Hartle-Hawking state by performing a path integral on a disk $D$. To evaluate such a path integral, the boundary conditions for the field $\phi$ must be specified on the boundary of the disk. Insertion of the operator $\mathcal{O}$ on the disk $D$ gives the wavefunction $\Psi_{\mathcal{O}}$ for the operator $\mathcal{O}$

$$\Psi_{\mathcal{O}}[\phi_b] = \int_{\phi|_{D=\phi_b}} [d\phi] e^{-I[\phi]} \mathcal{O}.$$ (3.26)

The insertion of the vertex operator $\mathcal{O} = e^{\alpha \phi}$ creates a state with purely imaginary zero mode $p_0 = i(\gamma_c - 1/\gamma)$, corresponding to a highest-weight state in the elliptic sector.

3.6 Spectrum of the Hilbert space

The spectrum of the theory thus includes different sectors depending on the values of the zero modes $p_0$:

- hyperbolic sector: $p_0$ real, $\Delta = 1/(2\gamma^2) + p_0^2/2 > 1/(2\gamma^2)$,
- elliptic sector: $p_0$ imaginary, $\Delta = 1/(2\gamma^2) + p_0^2/2 < 1/(2\gamma^2)$. (3.27)

The distinction between these two sectors can be made clear [35] in the mini-superspace approximation of the theory, where Liouville theory is described by a quantum mechanics problem of $\sigma$-independent $\phi_0$. The states in the hyperbolic sector are labeled by the real continuous parameter $p_0$, and the wave function is normalizable in the limit $\phi_0 \to -\infty$, when interaction term vanishes. The states that correspond to local vertex operator insertions are in the elliptic sector of the theory and lead to eigenfunctions of the Hamiltonian which diverge as $\phi_0 \to -\infty$ because of the imaginary value of $p_0$. Thus these Hartle-Hawking states are called nonnormalizable states.

The general solutions of (2+1)-dimensional gravity with $\Lambda < 0$ can be classified in terms of the spectrum of Liouville theory. The Fefferman-Graham expansion [43] of the metric which solves the Euclidean Einstein’s equation is completely defined by the geometry on the boundary [44–46]. Choosing the asymptotic geometry to be an infinite cylinder, the complete expression of a locally AdS metric is

$$ds^2 = 4G_l(\Lambda d\omega^2 + \bar{\Lambda} d\bar{\omega}^2) + (e^{2\rho} + 16G^2_l \Lambda \bar{L} e^{-2\rho})d\omega d\bar{\omega} + l^2 d\rho^2,$$ (3.28)
where \( \{\omega,\bar{\omega},\rho\} \) are coordinates such that the boundary is located at \( e^\rho \to \infty \), and \( \omega, \bar{\omega} \) are complex coordinates on the boundary.

When \( L \) and \( \bar{L} \) have constant values \( L_c \) and \( \bar{L}_c \), we can parametrize them as

\[
L_c = \frac{1}{2}(Ml + J), \quad \bar{L}_c = \frac{1}{2}(Ml - J).
\]  

(3.29)

For \( Ml > |J| \), the metric (3.28) is globally isometric to the Euclidean \((2+1)\)-dimensional black hole of mass \( M \) and angular momentum \( J \). For \( J = 0 \) and \( M = -1/8G \) the metric reduces to Euclidean anti-de Sitter space. When \( Ml < |J| \), the metric can be described as a conical singularity, which has an ADM mass lying between the anti-de Sitter value of \(-1/8G\) and the extremal BTZ black hole value of zero.

The functions \( L \) and \( \bar{L} \) in the exact solution are given by the energy-momentum tensor in the boundary Liouville field theory \([45, 46]\). Comparing the spectrum (3.27) with the values of \( L_c \) and \( \bar{L}_c \), we see that the hyperbolic sector of Liouville field theory correspond to black hole solutions, while the elliptic sector give solutions which behave like conical singularities.

4 The underlying quantum algebra structure for Liouville theory

We have seen from (2.15) that under \( SL(2,C) \) transformations the chiral fields \( \psi^j_m(\sigma) \) transforms classically in the spin \( j \) representation with finite dimension \( 2j + 1 \), and that \( e^{-j\varphi} \) is a group invariant. Gervais and Neveu \([28–34]\) have shown that this group structure is replaced by a quantum algebra after canonical quantization.

4.1 The exchange algebra and fusion of chiral vertex operators

Define normal-ordered chiral vertex operators

\[
\psi_i = d_i(\omega)N^{(i)}(\exp(\frac{\tilde{\gamma}}{2}q^{(i)}))
\]  

(4.1)

as normalized solutions of the quantum version of eqn. (2.13), where \( d_i(\omega) \) is normalization factor depending only on zero-modes. The \( \psi_i \) fields satisfy the exchange algebra

\[
\psi_i(\sigma)\psi_j(\sigma') = \sum_{k=1,2,l=1,2} S^{kl}_{ij}(\omega,\sigma - \sigma')\psi_k(\sigma')\psi_l(\sigma).
\]  

(4.2)

It is shown in \([28]\) that the above \( S^{kl}_{ij} \) satisfies the Yang-Baxter equations \([47–49]\)

\[
\sum_{\rho,\lambda,\mu} S^{\lambda\mu}_{jk}(\omega + \varepsilon_l,\sigma_2 - \sigma_3) S^{\rho\mu}_{ij}(\omega,\sigma_1 - \sigma_3) S^{mn}_{\rho\lambda}(\omega + \varepsilon_1,\sigma_1 - \sigma_2)
\]

\[
= \sum_{\rho,\lambda,\mu} S^{\rho\lambda}_{ij}(\omega,\sigma_1 - \sigma_2) S^{\mu\rho}_{kl}(\omega + \varepsilon_\mu,\sigma_1 - \sigma_3) S^{lm}_{\mu\lambda}(\omega,\sigma_2 - \sigma_3),
\]

(4.3)

\( \varepsilon_2 = -\varepsilon_1 = 1, \)
due to the associativity of the products of three $\psi$ fields. However they depend upon the zero modes $\varpi$, which can be shifted by the $\psi$ fields since

$$\varpi e^{\frac{\gamma}{2}q(i)} = e^{\frac{\gamma}{2}q(i)} (\varpi \pm 1).$$  \hfill (4.4)

Gervais [33] showed that by taking operator product of $\psi_i, i = 1, 2$, one can generate chiral fields

$$\psi^{\mu,\nu} \sim N(1)(e^{\mu \frac{\gamma}{2}q(1)})N(2)(e^{\nu \frac{\gamma}{2}q(2)}),$$  \hfill (5.5)

with integer $\mu, \nu$. It is convenient to adopt the notation:

$$\psi^{(j)}_m \equiv \psi^{j-m,j+m}; \quad \mu + \nu = 2j, \quad \nu - \mu = 2m,$$  \hfill (5.6)

with $\psi^{(1/2)}_{-1/2} = \psi_1$ and $\psi^{(1/2)}_{1/2} = \psi_2$. The positive half-integer $j$ determines the conformal weight of $\psi^{(j)}_m$:

$$\Delta_j = -j - \frac{\gamma^2}{2} j(j + 1).$$  \hfill (5.7)

The operators $\psi^{(j)}_m$ are closed under OPE and braiding, but the fusion coefficients and $R$-matrix elements depend on the zero modes $\varpi$ and thus do not commute with the $\psi^{(j)}_m$. The quantum-group structure can be exhibited more explicitly, once one changes to another basis of Hermitian chiral fields [33]

$$\xi^{(j)}_M(\sigma) = \sum_{-j \leq m \leq j} a^{m}_M(j, \varpi) \psi^{(j)}_m(\sigma), \quad -j \leq M \leq j.$$  \hfill (5.8)

It was shown in [33] that the exchange algebra of these operators is

$$\xi^{(j)}_M(\sigma) \xi^{(j')}_M(\sigma') = \sum_{-j \leq N \leq j, -j' \leq N' \leq j'} (j,j')^{N,N'}_{M,M'} \xi^{(j')}_{N'}(\sigma') \xi^{(j)}_N(\sigma),$$  \hfill (5.9)

where $(j, j')^{N,N'}_{M,M'}$ is coefficient of the universal $R$-matrix of quantum algebra $U_h(sl_2)$

$$R = e^{hH \otimes H/2} \sum_{n=0}^{\infty} \frac{1}{[n]!} q^n (1-q^{-2})^n J^n_+ \otimes J^n_-,$$  \hfill (5.10)

with

$$h = i\pi \frac{\gamma^2}{2},$$  \hfill (5.11)

where $q = e^{-h}$ for $0 < \sigma < \sigma'$ and $q = e^h$ for $0 < \sigma' < \sigma < \pi$, and

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$  \hfill (5.12)
$H$, $J_+$ and $J_-$ are the generators of the the formal deformation algebra $U_q(sl_2)$ of the universal enveloping algebra $U(sl_2)$. The algebra $U_q(sl_2)$ (see [50–52] and the references therein) is an $h$-adic algebra defined over the ring $\mathbb{C}[[h]]$ of formal power series with the defining relations$^1$

$$[H, J_{\pm}] = \pm 2J_{\pm}, \quad [J_+, J_-] = [H] = \pm \frac{e^{H} - e^{-H}}{e^{h} - e^{-h}}. \quad (4.13)$$

It admits a unique $h$-adic Hopf algebra structure and is quasitriangular (see Appendix A for some basic definitions regarding quantum algebras).

The short-distance operator-product expansion of the $\xi$ fields is of the form [33]

$$\xi_{M_1}^{(j_1)}(\sigma) \xi_{M_2}^{(j_2)}(\sigma') = \sum_{j = |j_1 - j_2|}^{j_1 + j_2} \left\{ (1 - e^{-i(\sigma - \sigma')})^2 \Delta(j) - \Delta(j_1) + \Delta(j_2) \times \right.$$  

$$\left. \times (j_1, M_1; j_2, M_2; j_1; j; M_1 + M_2) \left( \xi_{M_1 + M_2}^{(j)}(\sigma) + \text{descendants} \right) \right\}, \quad (4.14)$$

where $(j_1, M_1; j_2, M_2; j_1; j; M_1 + M_2)$ are the Clebsh-Gordan coefficients of $U_q(sl_2)$, and $\Delta(j) = -j - \frac{2}{\pi} j(j + 1)$ is the conformal weight of $\xi_{M}^{(j)}$, assuming that there is no maximum value of $j$.

### 4.2 Quantum algebras $U_q(sl_2)$ and representations with $q$ as roots of unity

Let $q$ be a fixed complex number such that $q \neq 0$ and $q^2 \neq 1$. Denote by $U_q(sl_2)$ the quantum algebra [50–52] over $\mathbb{C}$ with four generators $J_+, J_-, K, K^{-1}$ satisfying the defining relations

$$KK^{-1} = K^{-1}K = 1, \quad [J_+, J_-] = \frac{K - K^{-1}}{q - q^{-1}}, \quad KJ_\pm K^{-1} = q^\pm 2J_\pm, \quad (4.15)$$

on which exists a unique Hopf algebra structure with comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$ such that

$$\Delta(J_+) = J_+ \otimes K + 1 \otimes J_+, \quad \Delta(J_-) = J_- \otimes 1 + K^{-1} \otimes J_-, \quad \Delta(K) = K \otimes K,$$

$$\varepsilon(K) = 1, \quad \varepsilon(J_\pm) = 0,$$

$$S(K) = K^{-1}, \quad S(J_+) = -J_+K^{-1}, \quad S(J_-) = -KJ_-.$$

(4.16)

Note that the Hopf algebras $U_q(sl_2)$ and $U_{q^{-1}}(sl_2)$ are isomorphic.

The quantum algebra $U_q(sl_2)$ does not admit a universal $R$-matrix since it is not quasitriangular, but it is a braided Hopf algebra [53] such that the condition of quasitriangularity is generalized to the existence of an automorphism of $U_q(sl_2) \otimes U_q(sl_2)$. Because of an injective homomorphism $i : U_q(sl_2) \rightarrow U_h(sl_2)$ such that

$$i(K) = e^{H}, \quad i(K^{-1}) = e^{-H}, \quad i(q) = e^{h}, \quad (4.17)$$

$^1$Note that in Gervais and Neveu’s work the definition of $U_h(sl_2)$ is slightly different, such that their generator $J_3$ is twice the generator $H$ defined here.
the universal $R$-matrix (4.10) of the quasitriangular algebra $U_h(sl_2)$ gives the exchange algebra of two representations of $U_q(sl_2)$ of the type that we shall discuss below. Equations (4.10) and (4.14) show the quantum algebra structure of the chiral fields $\xi_{M}^{(j)}(\sigma)$, with the deformation parameter

$$ q = e^{i\pi \gamma^2/2}. \quad (4.18) $$

For real positive $q$ the representation theory of the quantum algebra is essentially the same as that of the corresponding Lie algebra [50–52]. However, the special case when $q$ is a root of unity, $q = \exp(i\pi p)$, $p \in \mathbb{Z}$, is different. Define

$$
\begin{cases}
  p' = p & \text{for odd } p \\
  p' = p/2 & \text{for even } p.
\end{cases} 
\quad (4.19)
$$

Then all of the irreducible representations have finite dimension of at most $p'$.

At roots of unity, the quantum Casimir invariant

$$ C_q = J_+ J_- + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2} = J_- J_+ + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2} \quad (4.20) $$

is no longer the only invariants of $U_q(sl_2)$. The center of the algebra is generated by

$$ C_q, \: (J_+)^{p'}, \: (J_-)^{p'}, \: (K)^{p'}, \: (K)^{-p'}. \quad (4.21) $$

These invariants are dependent, and there exists a polynomial relation among them. Hence irreducible representations of $U_q(sl_2)$ have three independent labels, which we may take to be the eigenvalues of $(J_+)^{p'}, \: (J_-)^{p'}, \: (K)^{p'}$ with the Casimir invariant $C_q$ determined by them.

Let $T$ denote an irreducible representation of $U_q(sl_2)$ on a vector space $V$. Concerning the values of the scalars $T(J_+)^{p'}, T(J_-)^{p'}$, when $q$ is a root of unity, the irreducible representations of $U_q(sl_2)$ fall into three classes [50–52]: nilpotent, cyclic and semicyclic representations. In this paper we will only discuss the discrete nilpotent representations $T_{\omega j}$, defined as follows [50]:

Let $j$ be an nonnegative integer or half-integer and let $\omega \in \{+1, -1\}$. Let $V_j$ be a $(2j + 1)$-dimensional complex vector space with basis $e_M$, $M = -j, -j + 1, ..., j$. For notational convenience, set $e_{j+1} = e_{-j-1} = 0$. Define operators $T_{\omega j}(J_+), T_{\omega j}(J_-), T_{\omega j}(K)$ acting on $V_j$ by

$$
T_{\omega j}(J_+) e_M = \sqrt{|j - M||j + M + 1|} e_{M+1}, \\
T_{\omega j}(J_-) e_M = \omega \sqrt{|j + M||j - M + 1|} e_{M-1}, \\
T_{\omega j}(K) e_M = \omega q^{2M} e_M, \quad (4.22)
$$

with $[n]$ defined as in eqn. (4.12). It can be shown that these operators satisfy the defining relations of the algebra, and hence define a representation $T_{\omega j}$ of the algebra $U_q(sl_2)$ on $V_j$.

For $q$ a root of unity, the representation $T_{\omega j}$ is irreducible if and only if $2j < p'$. The representation $T_{\omega j}$, $\omega \in \{+1, -1\}$, $j = 0, \frac{1}{2}, 1, ..., \frac{p'-1}{2}$, are pairwise inequivalent and satisfy the condition $T(J_+^{p'}) = T(J_-^{p'}) = 0$. 

14
5 Hartle-Hawking states

5.1 Vertex operator as quantum group invariant

In Section 4 we discussed the quantum algebra structure of the chiral fields $\xi_{M}^{(j)}(\sigma)$. The discussion equally applies to the chiral fields defined with the other solution of eqn. (3.15), with $\tilde{\gamma}$ replaced by $2/\tilde{\gamma}$. The chiral fields $\tilde{\xi}_{M}^{(j)}(\sigma)$ can be constructed, and exhibit similar quantum algebra structure with the deformation parameter

$$\tilde{q} = \exp(\imath 2\pi/\tilde{\gamma}^2).$$

(5.1)

Consider the fields $\xi_{M}^{(j)}$ and $\tilde{\xi}_{M}^{(j)}$ as representations of $U_{q}(sl_{2})$ and $U_{q}(sl_{2})$ of type $T_{\omega j}$, on linear spaces over $\mathbb{C}$ (will here write $T_{\omega j}(J_{\pm})$ simply as $J_{\pm}$, etc.):

$$J_{+} \xi_{M}^{(j)} = \sqrt{[j - M][j + M + 1]} \xi_{M+1}^{(j)}, \quad \tilde{J}_{+} \tilde{\xi}_{M}^{(j)} = \sqrt{[j - \tilde{M}][j + \tilde{M} + 1]} \tilde{\xi}_{M+1}^{(j)},$$

$$J_{-} \xi_{M}^{(j)} = \omega \sqrt{[j + M][j - M + 1]} \xi_{M-1}^{(j)}, \quad \tilde{J}_{-} \tilde{\xi}_{M}^{(j)} = \tilde{\omega} \sqrt{[j + \tilde{M}][j - \tilde{M} + 1]} \tilde{\xi}_{M-1}^{(j)},$$

$$K \xi_{M}^{(j)} = \omega q^{2M} \xi_{M}^{(j)}, \quad \tilde{K} \tilde{\xi}_{M}^{(j)} = \tilde{\omega} q^{2\tilde{M}} \tilde{\xi}_{M}^{(j)}.$$  

(5.2)

Following the discussion by Gervais in [54, 55], for half-integer $j$ and $\tilde{j}$, the chiral field $\xi_{M}^{(j, \tilde{j})}$ of type $T_{\omega j}$ is constructed by fusion of hatted and unhatted chiral fields

$$\xi_{M}^{(j)} \tilde{\xi}_{M}^{(\tilde{j})} \sim e^{\imath \pi(Mj - \tilde{M}\tilde{j})} \xi_{M}^{(j, \tilde{j})}.$$  

(5.3)

The corresponding quantum algebra structure of the general chiral fields was observed in [54] to be $U_{q}(gl_{2}) \circ U_{q}(sl_{2})$, where $\circ$ denotes some kind of graded tensor product, since the hatted and unhatted fields commute up to a simple phase when $j$ and $\tilde{j}$ are half-integers. For continuous spins, however, the commutation becomes nontrivial, and $j$, $\tilde{j}$ lose their individual meanings since $j$ and $\tilde{j}$ can no longer be separated [56, 57]. Instead, one must introduce the effective spins [57]

$$j^{e} = j + \frac{2}{\tilde{\gamma}^2} \tilde{j}, \quad M^{e} = M + \frac{2}{\tilde{\gamma}^2} \tilde{M},$$  

(5.4)

which are appropriate quantum numbers in this case. The fusion and braiding may be written in terms of these effective spins, which were shown to consistently include representations that are semi-infinite [56, 57]. For half-integer $j$ and $\tilde{j}$, it is equivalent to specify either $(j, \tilde{j})$ or $j^{e}$.

For half-integer $j$ and $\tilde{j}$, the complete quantum algebra action on the $\xi$ family is given by

$$J_{+} \xi_{M}^{(j, \tilde{j})} = \sqrt{[j - M][j + M + 1]} \xi_{M+1}^{(j, \tilde{j})}, \quad \tilde{J}_{+} \tilde{\xi}_{M}^{(j, \tilde{j})} = \sqrt{[j - \tilde{M}][j + \tilde{M} + 1]} \tilde{\xi}_{M+1}^{(j, \tilde{j})},$$

$$J_{-} \xi_{M}^{(j, \tilde{j})} = \omega \sqrt{[j + M][j - M + 1]} \xi_{M-1}^{(j, \tilde{j})}, \quad \tilde{J}_{-} \tilde{\xi}_{M}^{(j, \tilde{j})} = \tilde{\omega} \sqrt{[j + \tilde{M}][j - \tilde{M} + 1]} \tilde{\xi}_{M-1}^{(j, \tilde{j})},$$

$$K \xi_{M}^{(j, \tilde{j})} = \omega q^{2M} \xi_{M}^{(j, \tilde{j})}, \quad \tilde{K} \tilde{\xi}_{M}^{(j, \tilde{j})} = \tilde{\omega} q^{2\tilde{M}} \tilde{\xi}_{M}^{(j, \tilde{j})}.$$  

(5.5)
with the restriction that \( \omega = \hat{\omega} \), in order for the two descriptions in terms of \((j, \bar{j})\) and \(j^e\) to be equivalent.

The reconstruction of the Liouville field as a quantum invariant was discussed in [55], as the quantum analog of the classical expression (2.15):

\[
e^{-\left(j + \frac{\bar{j}}{2}\right) \hat{\gamma}^\phi} = c_{jj} \sum_{MM} \frac{1}{[2j][2\bar{j}]!} (-1)^{j-M+\bar{M}} q^{(j-M)(j+M-1)} q^{(j-M\bar{M})(j+M-1)} \times
\]

\[
\times \xi^{(jj)}_{MM}(x_+)^i_{-M-M}(x_-),
\]

(5.6)

where \( c_{jj} \) is a constant depending on \( j, \bar{j} \) and \( \bar{\gamma} \). The quantum algebra structure of the field (5.6) is of the type \( U_q(sl_2) \otimes U_q(sl_2) \otimes U_q(sl_2) \otimes U_q(sl_2) \), and the field has negative conformal weight

\[
\Delta = \frac{c - 1}{24} - \frac{1}{24}(j + \bar{j} + 1)\sqrt{c - 1} - (j - \bar{j})\sqrt{c - 25}^2.
\]

(5.7)

The anti-chiral fields \( \xi^{(jj)}_{-M-M}(x^-) \) has a quantum group structure similar to eqn. (5.5). Define the quantum algebra generators \( J_\pm \) and \( K \) by the coproduct

\[
J_+ = J_+ \otimes \bar{K} + 1 \otimes J_+, \quad J_- = J_- \otimes 1 + K^{-1} \otimes J_-, \quad K = K \otimes \bar{K},
\]

\[
J_+ = \hat{J}_+ \otimes \tilde{K} + 1 \otimes \hat{J}_+, \quad \hat{J}_- = \hat{J}_- \otimes 1 + \tilde{K}^{-1} \otimes \hat{J}_-, \quad \tilde{K} = \tilde{K} \otimes \tilde{K},
\]

(5.8)

which naturally satisfies the commutation relations (4.15). For values \( \bar{\omega} = 1, \omega = \pm 1 \), we then have

\[
J_\pm e^{-\left(j + \frac{\bar{j}}{2}\right) \hat{\gamma}^\phi} = 0, \quad K e^{-\left(j + \frac{\bar{j}}{2}\right) \hat{\gamma}^\phi} = \omega e^{-\left(j + \frac{\bar{j}}{2}\right) \hat{\gamma}^\phi},
\]

\[
\hat{J}_\pm e^{-\left(j + \frac{\bar{j}}{2}\right) \hat{\gamma}^\phi} = 0, \quad \hat{K} e^{-\left(j + \frac{\bar{j}}{2}\right) \hat{\gamma}^\phi} = \omega e^{-\left(j + \frac{\bar{j}}{2}\right) \hat{\gamma}^\phi},
\]

(5.9)

implying that the quantized Liouville field is a quantum-group invariant.

### 5.2 Reducible Verma modules and singular states

We now construct the highest-weight representations of the Virasoro algebra corresponding to the local insertions of the vertex operators (5.6). Since the holomorphic and antiholomorphic components of the overall algebra decouple, representations are obtained by taking their tensor products. Denote by \( V(c, \Delta) \) and \( \bar{V}(c, \bar{\Delta}) \) the Verma modules generated by the sets \( \{L_n\} \) and \( \{\bar{L}_n\} \) with central charge \( c \) and highest weights \( \Delta \) and \( \bar{\Delta} \). The Hilbert space in general is a direct sum of the tensor products of all conformal dimensions of the theory:

\[
\sum_{\Delta, \bar{\Delta}} V(c, \Delta) \otimes \bar{V}(c, \bar{\Delta}).
\]

(5.10)

It may happen that the representations of the Virasoro algebra comprising the states

\[
|\Delta, \{\lambda\}\rangle \equiv L_{-k_1}^{\lambda_1} L_{-k_2}^{\lambda_2} ... L_{-k_n}^{\lambda_n} |\Delta\rangle, \quad \lambda_i > 0
\]

(5.11)
are reducible, that is, there is a submodule that is itself a representation of the Virasoro algebra. Such a submodule is generated from a highest-weight state \( |\delta\rangle \), such that
\[
L_n |\delta\rangle = 0 \quad n > 0,
\]
although this state is also of the form (5.11). Such a state generates its own Verma module and is called a singular state. In the case of minimal models \([58]\) it is also called a null state, since its norm defined with respect to the inner product is
\[
\langle \delta | \delta \rangle = 0.
\]

Consider two partitions of level \( l \):
\[
\sum \lambda_k k = \sum \lambda'_k k = l.
\]
The result is proportional to \( |\Delta\rangle \):
\[
L^\lambda_{k_1} \ldots L^\lambda'_{k_n} |\Delta, \{\lambda\}\rangle = M^{(l)}_{\{\lambda'\},\{\lambda\}} |\Delta\rangle,
\]
where \( M^{(l)}_{\{\lambda'\},\{\lambda\}} \) is a polynomial in \( \Delta \) and \( c \). If \( M^{(l)} \) has a zero eigenvalue, then there exists a singular state (5.12) at level \( l \), and the representation on the subspace generated by (5.11) is reducible.

There exists a general formula, due to Kac \([59,60]\), for the determinant of the \( M^{(l)} \), the Kac determinant:
\[
\det M^{(l)} = \alpha_l \prod_{r,s \geq 1, rs \leq l} [\Delta - \Delta_{r,s}(c)]^{p(l-rs)},
\]
where \( p(l-rs) \) is the number of partitions of the integer \( l-rs \) and \( \alpha_l \) is a positive constant independent of \( \Delta \) and \( c \).

The function \( \Delta_{r,s}(c) \) can be written as
\[
\Delta_{r,s} = \frac{c - 1}{24} + \frac{1}{4} \left[ r\alpha_+ + s\alpha_- \right]^2,
\]
with
\[
\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}.
\]
When \( \Delta = \Delta_{r,s} \), there exist singular vectors with dimensions
\[
\Delta_\delta = \Delta_{r,s} + rs.
\]

In Liouville field theory the Hartle-Hawking states corresponding to vertex operators (5.6) have negative conformal weights (5.7), which can be put in Kac form
\[
\Delta_{J,j} = \frac{c - 1}{24} + \frac{1}{4} \left[ 2\hat{J}\alpha_+ + 2J\alpha_- \right]^2,
\]
where in this case
\[ \alpha_+ = i\sqrt{2} \frac{\tilde{\gamma}}{\gamma}, \quad \alpha_- = i\sqrt{2} \frac{\tilde{\gamma}}{\gamma}. \] (5.20)

\( J \) and \( \hat{J} \) are related to the spins \( j, \hat{j} \) of the quantum algebra \( U_q(sl_2) \odot U_q(sl_2) \) by
\[ J = j + 1/2, \quad \hat{J} = \hat{j} + 1/2. \] (5.21)

We see that at level \( l = 4J\hat{J} \) there are singular states with \( \Delta_\delta = \Delta_{j,\hat{j}} + l \).

6 Conformal Ward identity

We will see in Section 7 that just as in minimal models, the singular states discussed in Section 5.2 decouple from the conformal families. In order to understand this and to define the norm of these decoupling states, we need to understand the Ward identity in quantum Liouville theory for two-point correlation functions. We will follow the perturbative treatment of the geometrical approach to two-dimensional quantum gravity by Takhtajan [61–65], which has a deep connection with the uniformation problem going back to Poincaré’s theorem [66].

The correlation functions of the vertex operators of Liouville field theory are represented by functional integrals with the Liouville action over all Riemannian metrics in a given conformal class with prescribed conical singularities at inserted points. It was first observed by Polyakov (see [65] for details) that at the semi-classical level the Ward identities of the quantum Liouville theory establish a non-trivial relation between the accessory parameters and the critical value of the Liouville action functional. In the series of work by Takhtajan [61–65], for the region \( c > 1 \), the validity of BPZ conformal Ward identity for \( n \geq 3 \) puncture operators was proven perturbatively, with the quantum central charge \( c = 1 + 12/\gamma^2 \), in agreement with the result of the algebraic approach (3.12).

The proof can be easily generalized to the case of \( n \geq 3 \) conical singularities [67]. However, we shall see that in the case of two conical singularities, the Ward identity takes a different form, which is related to the geometrical property of a sphere with two conical singularities.

6.1 Facts about the sphere with two conical singularities

We will need the following facts about the Riemann sphere with two conical singularities.

Let \( X \) be a Riemann sphere with two conical singularities at \( z = 0 \) and \( z = \infty \). Following Troyanov [68], a (conformal) metric \( ds^2 \) on a Riemann surface \( S \) has a conical singularity of order \( \beta \) (\( \beta \) a real number > -1) at a point \( p \in S \) if in some neighbourhood of \( p \)
\[ ds^2 = e^{\varphi}|dz|^2, \] (6.1)

where \( z \) is a coordinate of \( S \) defined in this neighbourhood and \( \varphi \) is a function such that
\[ \varphi(z) - 2\beta \log|z - z(p)| \] (6.2)
is continuous at $p$.

A projective connection $\eta(z)$ on a Riemann surface $S$ is defined as a rule which associates to each local uniformizer $z$ on $S$ a meromorphic quadratic differential

$$\eta(z) = T(z)dz^2$$

(6.3)

defined in the domain of $z$, in such a way that under a holomorphic change of coordinates

$$\eta(w) = \eta(z) + \{z, w\}dw^2,$$

$$T(w(z)) = T(z)\left(\frac{dz}{dw}\right)^2 + \{z, w\},$$

(6.4)

where $\{.,\}$ denotes the Schwarzian derivative:

$$\{f, w\} = f'''f - \frac{3}{2} f'' f'.$$

(6.5)

The projective connection $\eta$ has a regular singularity of weight $\Delta$ if

$$\eta = \left(\frac{\Delta}{z^2} + \frac{d}{z} + T_1(z)\right), \quad T_1 \text{ holomorphic},$$

(6.6)

where $z$ is a uniformizer at $p$ such that $z(p) = 0$. This definition of weight is independent of the choice of uniformizer.

If $ds^2 = e^\varphi|dz|^2$ is a (conformal) metric of constant curvature on $S$ with conical singularities of order $\beta_1, \beta_2, ..., \beta_n$ at $p_1, p_2, ..., p_n$, then

$$T(z) = \varphi_{zz} - \frac{1}{2} \varphi_z^2,$$

(6.7)

where $\varphi$ is a solution of Liouville equation with constant positive curvature.

Let $\eta$ be a projective connection on $S^2$ with regular singularities at $z = 0$ and $z = \infty$. Then we have (in the standard coordinate $z$):

$$\eta(z) = \frac{\Delta}{z^2}dz^2, \quad \Delta \in \mathbb{C}.$$  

(6.8)

In particular, both singularities have the same order $\beta$ and weight

$$\Delta = -\frac{\beta(\beta + 2)}{2}.$$  

(6.9)

### 6.2 Correlation functions defined on a sphere with two conical singularities

For the case of two vertex operators, the correlation function is given by functional integral that diverges, so we need to define the functional integral for fixed area $A = \int e^\varphi d^2 z$, following [69, 70, 35]:

$$\langle X \rangle = \int dA \langle X \rangle_A e^{-A/2\gamma^2},$$  

(6.10)
where the functional integral for fixed area $A$ is defined as

$$\langle X \rangle_A = \int_{\mathcal{C}(X)} \mathcal{D}\phi e^{-\frac{1}{2\pi\gamma^2}I_L^{(A)}} \delta\left( \int d^2z e^{\phi} - A \right).$$ \hspace{1cm} (6.11)

Here $\mathcal{C}(X)$ denotes the class of smooth conformal metrics on $X$ with one conical singularity at $z = 0$, another at $z = \infty$, both have asymptotics (6.2). These conditions imply that the Liouville action with fixed area diverges. A properly regularized Liouville action $I_L^{(A)}$ [61–65,71] with fixed area contains boundary terms around singularities,

$$I_L^{(A)} = \lim_{\epsilon \to 0} \left\{ \int_{X_\epsilon} |\partial \varphi|^2 d^2z + (2\pi - 2) \log \epsilon \right\},$$ \hspace{1cm} (6.12)

where $X_\epsilon = X \setminus \{ r_1 < \epsilon \} \cup \{ r_2 > 1/\epsilon \}$.

Similarly, correlation functions of the energy-momentum tensor in the presence of conical singularities are defined by

$$\langle \prod_{i=1}^k T(z_i) \prod_{j=1}^l \bar{T}(\bar{w}_j)X \rangle \equiv \int dA \left\{ \prod_{i=1}^k T(z_i) \prod_{j=1}^l \bar{T}(\bar{w}_j)X \right\} e^{-A/(2\pi\gamma^2)},$$ \hspace{1cm} (6.13)

where

$$\langle I \rangle_A \equiv \langle \prod_{i=1}^k T(z_i) \prod_{j=1}^l \bar{T}(\bar{w}_j)X \rangle_A$$

$$= \int_{\mathcal{C}(X)} \mathcal{D}\phi \left\{ \prod_{i=1}^k T(z_i) \prod_{j=1}^l \bar{T}(\bar{w}_j) \right\} e^{-1/(2\pi\gamma^2)} \int d^2z |\partial \varphi|^2 \delta\left( \int d^2z e^{\phi} - A \right).$$ \hspace{1cm} (6.14)

As in standard quantum field theory, define the normalized connected correlation function

$$\langle \langle I \rangle \rangle_A \equiv \langle \langle \prod_{i=1}^k T(z_i) \prod_{j=1}^l \bar{T}(\bar{w}_j)X \rangle \rangle_A$$ \hspace{1cm} (6.15)

by the following inductive formula:

$$\langle \langle I \rangle \rangle_A = \frac{\langle I \rangle_A}{\langle X \rangle_A} - \sum_{r=2}^{n} \sum_{I_1 \cup \cdots \cup I_r} \langle \langle I_1 \rangle \rangle_A \cdots \langle \langle I_r \rangle \rangle_A,$$ \hspace{1cm} (6.16)

where the summation goes over all representations of the set $I$ as a disjoint union of the subset $I_1 \cup \cdots \cup I_r$. For example,

$$\langle \langle T(z)X \rangle \rangle_A = \langle \langle T(z)X \rangle \rangle_A,$$ \hspace{1cm} (6.17)

$$\langle \langle T(z)T(w)X \rangle \rangle_A = \frac{\langle \langle T(z)T(w)X \rangle \rangle_A}{\langle X \rangle_A} + \langle \langle T(z)X \rangle \rangle_A \langle \langle T(w)X \rangle \rangle_A.$$ \hspace{1cm} (6.18)
The generating functional for correlation functions of the stress-energy tensor (6.15) is introduced by the following expression:

\[ Z(\mu, \bar{\mu}; X)_A = \frac{Z(\mu, \bar{\mu}; X)_A}{(X)_A}, \]  

(6.19)

where

\[ Z(\mu, \bar{\mu}; X)_A = \int_d \mathcal{D}\varphi \delta(\int d^2 z e^\varphi - A) \times \]

\[ \times \exp \left[ -\frac{1}{2\pi\gamma^2} I_L^{(A)} + \text{p.v.} \int_X (T(\varphi)\mu + \bar{T}(\varphi)\bar{\mu}) \right], \]  

(6.20)

where the external sources are represented by Beltrami differentials \( \mu \) on the Riemann surface \( X \). The generating functional for the normalized connected multipoint correlation function is defined in a standard fashion:

\[ \frac{1}{\gamma^2} W(\mu, \bar{\mu}; X)_A = \log Z(\mu, \bar{\mu}; X)_A, \]  

(6.21)

so that

\[ \gamma^2 \langle \prod_{i=1}^k T(z_i) \prod_{j=1}^l \bar{T}(\bar{w}_j)X \rangle_A = \left. \frac{\delta^{k+l} W(\mu, \bar{\mu}; X)_A}{\delta \mu(z_1) \cdots \delta \mu(z_k) \delta \bar{\mu}(\bar{w}_1) \delta \bar{\mu}(\bar{w}_l)} \right|_{\mu = \bar{\mu} = 0}. \]  

(6.22)

6.3 Semi-classical Ward Identity for two-point functions

In the semi-classical limit the correlation function is dominated by the contribution from the solution \( \varphi_{cl} \) to the classical Liouville equation. Before evaluating the functional integral, first we calculate the classical solution \( \varphi_{cl} \) around which we will do the expansion. Let us look for classical solution corresponding to a Riemann sphere \( S^2 \) with two conical singularities at \( z = 0, \infty \), both of order \( \beta \). The Gauss-Bonnet formula for such a surface with positive constant curvature gives

\[ \frac{K}{2\pi} A = \chi(S^2) + \sum_{i=1}^2 \beta_i, \]  

(6.23)

where \( K \) is the Gaussian curvature, and \( \chi = 2 \) is the Euler characteristic of \( S^2 \). \( \varphi_{cl} \) is then the solution of the Liouville equation with constant positive curvature

\[ \partial \bar{\partial} \varphi = -\frac{K}{2} e^\varphi = -\frac{\pi}{A} (2 + \sum_{i=1}^2 \beta_i) e^\varphi. \]  

(6.24)

We will choose \( K = 1 \) and fixed area \( A_0 = 2\pi(2 + \sum \beta_i) \), but any choice of \( A \) can be realized by a shift in \( \varphi \). With this choice the Liouville equation for the classical solution becomes

\[ \partial \bar{\partial} \varphi = -\frac{1}{2} e^\varphi. \]  

(6.25)
In order to expand the Liouville action around the classical solution, first let
\[ \varphi = \varphi_{cl} + \delta \varphi. \]  
(6.26)

The expansion of the Liouville action (6.12) is then
\[ I^{(A_0)}_L(\varphi_{cl} + \delta \varphi) = I^{(A_0)}_L(\varphi_{cl}) - \int_X (\delta \varphi)(L_0 + 1/2)(\delta \varphi) \, d\rho - \sum_{k=3}^{\infty} \frac{1}{k!} \int_X (\delta \varphi)^k \, d\rho, \]  
(6.27)

where \( d\rho = e^{\varphi_{cl}} d^2z \) is the volume form of the metric, and
\[ L_0 = e^{-\varphi_{cl}} \partial \bar{\partial} \]  
(6.28)
is the Laplacian operator on \( X \). The inverse of \((2L_0 + 1)\) is given by the Green’s function \( G(z, z') \). The functional integral (6.11) now becomes
\[ \langle X \rangle_{A_0} = \int C(X) \mathcal{D}[\delta \varphi] \exp \left\{ -\frac{1}{2\pi \gamma^2} \int_X (\gamma^2 \delta/\delta \xi \delta k) \, d\rho \right\} \times \]
\[ \times \left( \int C(X) \mathcal{D}[\delta \varphi] \exp \left\{ \int d^2z e^{\varphi_{cl}} (e^{\delta \varphi} - 1) \right\} \right) \times \]
\[ \times \exp \left\{ \frac{1}{2\pi \gamma^2} \int_X \left[ (\delta \varphi)'(L_0 + 1/2 - \pi e^{-\varphi_{cl}}(\partial \mu \bar{\partial} + \bar{\partial} \mu \partial))(\delta \varphi)' \right] \, d\rho \right\} \times \]
\[ \times \exp \left\{ -\frac{\pi}{\gamma^2} \int_X \xi \left[ 1 - 2\pi G e^{-\varphi_{cl}}(\partial \mu \bar{\partial} + \bar{\partial} \mu \partial) \right]^{-1} \xi \, d\rho \right\} \]
(6.30)

where
\[ \xi = e^{-\varphi_{cl}}(\omega + \bar{\omega}), \quad \omega = \mu_{zz} + (\varphi_{cl})_z \mu + (\varphi_{cl})_{zz} \mu, \]  
(6.31)
and
\[ \delta \varphi' = \delta \varphi + \frac{\pi \xi}{L_0 + 1/2 - \pi e^{-\varphi_{cl}}(\partial \mu \bar{\partial} + \bar{\partial} \mu \partial)}. \]  
(6.32)
Now look at the tree level value of \( \langle\langle T(z)X \rangle\rangle_{A_0} \) and \( \langle\langle T(z)T(w)X \rangle\rangle_{A_0} \), which are defined in (6.17) and (6.18). According to the definition (6.22),

\[
\langle\langle T(z)X \rangle\rangle_{A_0} = \left. \frac{1}{\gamma^2} \frac{\delta W(\mu, \bar{\mu}; X)_{A_0}}{\delta \mu(z)} \right|_{\mu=0},
\]

(6.33)

It is clear that

\[
\langle\langle T(z)X \rangle\rangle_{A_0 - \text{tree}} = T_{A_0 - \text{cl}}(z) = \left. \frac{1}{\gamma^2} \Delta \right|_{\mu=0},
\]

(6.34)

where \( \Delta \) is defined as in (6.9), due to the form of the projective connection (6.8) on \( X \).

This is of a different form from the case of \( n \geq 3 \) conical singularities [61–65]. For a Riemann sphere with \( n \geq 3 \) conical singularities of deficit angle \( \theta = 2\pi(1 - \alpha_i), i = 1, \cdots, n \), with

\[
\alpha_i < 1, \quad \sum_{i=1}^{n-1} \alpha_i > 2,
\]

(6.35)

there exists [72–74] a unique conformal metric \( ds^2 = e^{\varphi}|dz|^2 \) of constant curvature \( -1 \), where \( \varphi \) is a smooth function satisfying the Liouville equation (2.7) with certain asymptotics near the singular points. At the tree level [67]

\[
\langle\langle T(z)X \rangle\rangle_{\text{tree}} = T_{\text{cl}}(z) = \sum_{i=1}^{n-1} \left( \frac{h_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right),
\]

(6.36)

where \( h_i = \alpha_i(2 - \alpha_i) \). The complex number \( c_i \) are the famous accessory parameters, which are uniquely determined by the singular points \( z_1, \cdots, z_n \) and the set of orders \( \alpha_i \).

For the connected Ward identity

\[
\langle\langle T(z)T(w)X \rangle\rangle_{A_0} = \left. \frac{1}{\gamma^2} \frac{\delta^2 W(\mu, \bar{\mu}; X)_{A_0}}{\delta \mu(z) \delta \mu(w)} \right|_{\mu=0},
\]

(6.37)

at the tree level, the only terms of order \( 1/\gamma^2 \) come from

\[
\exp \left[ \frac{1}{2\pi\gamma^2} \sum_{k=3}^{\infty} \frac{1}{k!} \int_X \left( \gamma^2 \frac{\delta}{\delta \xi} \right)^k d\rho \right],
\]

(6.38)

acting on the first term in the expansion of the integral

\[
\exp \left\{ -\frac{\pi}{\gamma^2} \int_X \xi G[1 - 2\pi Ge^{-\varphi}(\partial \mu \partial + \bar{\partial} \bar{\mu} \bar{\partial})^{-1} \xi] d\rho \right\}.
\]

(6.39)

We obtain the following expression

\[
\langle\langle T(z)T(w)X \rangle\rangle_{A_0 - \text{tree}} = -\frac{2\pi}{\gamma^2} D_z D_w G(z, w).
\]

(6.40)

where \( D_z = \partial_z \partial_z - (\varphi_{cl})_z \partial_z \).
6.4 Ward identity on the sphere with two conical singularities

We will derive the Ward identity for the insertion of two vertex operators \( e^{-\varphi/\gamma} \) at \( z = 0, \infty \), which classically corresponds to \( \beta = 1 \). In this case two spheres are cut along a geodesic joining the north pole to south pole and then glued together [68], so that the tangent cones at the insertion point \( z = 0, \infty \) are of angles \( \theta = 4\pi \). There exists a map from \( z \)-plane \( X \) to \( \zeta \)-plane \( S^2 \),

\[
\zeta = z^2,
\]

which is a covering \( f : S^2 \to S^2 \).

The kernel of the integral operator \((2L_0 + 1)\) is given by the Green’s function \( G(z, z') \), which satisfies on \( X \) the PDE

\[
2G_{z\bar{z}}(z, z') + \exp[\varphi_{cl}] G(z, z') = \delta(z - z').
\]

We first study the Green’s function \( G(\zeta, \zeta') \) on the Riemann sphere \( S^2 \) of radius \( R = 1 \), on which the Laplacian (6.28) is of the form

\[
L_0 = \frac{1}{4}(1 + \zeta \bar{\zeta})^2 \frac{\partial}{\partial \zeta} \frac{\partial}{\partial \bar{\zeta}}.
\]

Define the point-pair invariant

\[
u = - \frac{(\zeta - \zeta')(\bar{\zeta} - \bar{\zeta}')}{{(1 + \zeta \bar{\zeta})(1 + \zeta' \bar{\zeta}')}}.
\]

Then the explicit solution for the Green’s function is

\[
G = -\frac{1}{2\pi} \frac{(2u + 1) \ln(u + 1) - 1}{u - 1},
\]

with the desired asymptotic behavior. Note that here \( u \in (-1, 0) \).

Using the map (6.41) to obtain \( G(z, z') \) on \( X \) from \( G(\zeta, \zeta') \), then using (6.40), a straightforward calculation reveals that for \( \beta = 1 \), at the tree level,

\[
\langle \langle T(z)T(w)X \rangle \rangle_{A_0} = \frac{6}{\gamma^2} \left[ \frac{1}{(z-w)^4} - \frac{1}{2(z-w)^2w^2} + \frac{1}{2(z-w)w^3} \right],
\]

which combined with (6.17), (6.18) and (6.34), gives rise to the expression

\[
\langle T(z)T(w)X \rangle_{A_0} = \frac{c_{cl}/2}{(z-w)^4} \langle X \rangle_{A_0} + \left\{ \frac{2}{(z-w)^2} + \frac{1}{z-w} \frac{\partial}{\partial w} + \sum_{i=1}^{2} \frac{\Delta}{(z-z_i)^2} \right\} \langle T(w)X \rangle_{A_0},
\]

where \( c_{cl} = 12/\gamma^2 \) is the classical central charge, and \( \Delta = -3/2\gamma^2 = \Delta/\gamma^2 \) is the conformal weight of the vertex operator as \( \gamma \to 0 \).
In the standard conformal Ward identity obtained from the operator product expansion (OPE) [58],
\[
\langle T(z)T(w)X \rangle = \frac{c/2}{(z-w)^4} \langle X \rangle + \left\{ \frac{2}{(z-w)^2} + \frac{1}{z-w} \right. \\
+ \left. \frac{\partial}{\partial w} + \sum_{i=1}^{n} \left( \frac{\Delta}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \right\} \langle T(w)X \rangle,
\]
(6.48)
where the terms after the summation sign come from the OPE of the vertex operators, while the terms before that are due to the OPE of the energy-momentum tensor itself. Comparing with (6.47), we see that the contributions from the OPE of the \(T\)'s remain the same; all difference comes in from the fact that
\[
\langle T(z)X \rangle_{A_0} = \langle \langle T(z)X \rangle_{A_0} \langle X \rangle_{A_0} = \sum_{i=1}^{2} \frac{\Delta}{(z-z_i)^2} \langle X \rangle_{A_0} \]
(6.49)
is different from the standard form
\[
\langle T(z)X \rangle = \sum_{i=1}^{n} \left( \frac{\Delta}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \langle X \rangle.
\]
(6.50)
This difference, in turn, is the consequence of the difference between (6.34) and (6.36).

In Section 3.4, we saw that the \(SL(2,C)\)-invariant vacuum does not exist in Liouville field theory. Thus it is not surprising to have a Ward identity that is different from the standard form, since we do not have the usual operator-state correspondence; instead we treat each Hartle-Hawking state as a ground state, and use the OPE of energy-momentum tensor to define \(\langle \prod_{i=1}^{k} T(z_i)X \rangle_{A_0}\) from \(\langle T(z)X \rangle_{A_0}\).

In the BPZ formulation [58], all correlation functions of secondary fields are given by differential operators acting on those of primary states. Here we define differential operators \(L_{-n}\), consistent with our semi-classical result
\[
\langle (L_{-n}V)(w)V(z) \rangle_{A} = \mathcal{L}_{-n} \langle V(w)V(z) \rangle_{A}, \quad (n \geq 1),
\]
(6.51)
with a modified definition
\[
\mathcal{L}_{-n} = \frac{(n-1)\Delta}{(w-z)^2}.
\]
(6.52)
In particular,
\[
\mathcal{L}_{-1} = 0, \quad \mathcal{L}_{-2} = \frac{\Delta}{(w-z)^2}.
\]
(6.53)
7 Decoupling states with nonzero norm

7.1 The norm of the singular states

We now proceed to define the norm of the singular states described in Section 5.2. Because of the nonstandard form of the Ward identity of two-point functions and the differential operators (6.51), we will show by an explicit example here that the norm of such a singular state is in fact not zero, unlike the familiar case of the minimal models.

A simple example of such a singular state is at level $l = 2$ of the Hartle-Hawking state $|\Delta\rangle$ constructed from the operator $V = e^{-\tilde{\gamma}\phi/2}$,

$$|\delta\rangle = [L_{-2} + \eta L_{-1}] |\Delta\rangle,$$

where

$$\eta = -\frac{3}{2(2\Delta + 1)}. \tag{7.2}$$

The norm of such a state can be evaluated by gluing together two disks with operator $\delta(z, \bar{z})$ at the center of each disk.

The norm of the singular state $\langle \delta | \delta \rangle$ is then related to the correlation function by

$$\langle \delta | \delta \rangle = \lim_{w, \bar{w} \to \infty} \frac{\eta^2 \Delta^2 \bar{w}^2 \Delta^2 w^2 \Delta}{\Gamma(\tilde{\gamma}) \Gamma(1 - \tilde{\gamma}/2)} \langle V(w) V(0) | V(\bar{w}) V(0) \rangle. \tag{7.3}$$

The two-point function are expressed in terms of gamma functions as [75,76]

$$\langle e^{\alpha \phi(w)} e^{\alpha \phi(0)} \rangle = \left[ \frac{4\pi}{\Gamma(1 - \tilde{\gamma}/2)} \right]^{2(\tilde{\gamma} - \alpha)/\tilde{\gamma}} \Gamma(\tilde{\gamma}\alpha - \tilde{\gamma}^2/2) \Gamma(2\alpha/\tilde{\gamma} - 2/\tilde{\gamma}^2 - 1) \frac{1}{\tilde{\gamma}} \frac{\Gamma(-\tilde{\gamma}/2) \Gamma(3 + 2/\tilde{\gamma}^2)}{\Gamma(1 + \tilde{\gamma}/2) \Gamma(3 + 2/\tilde{\gamma}^2)} (w\bar{w})^{2\Delta}. \tag{7.4}$$

Using the two-point function for $V = e^{-\tilde{\gamma}\phi/2}$, we obtain

$$\langle \delta | \delta \rangle = \eta^2 \Delta^2 \left[ \frac{4\pi}{\Gamma(1 - \tilde{\gamma}/2)} \right]^{2(\tilde{\gamma} - \alpha)/\tilde{\gamma}} \frac{\Gamma(-\tilde{\gamma}/2) \Gamma(3 + 2/\tilde{\gamma}^2)}{\tilde{\gamma}^2 \Gamma(1 + \tilde{\gamma}/2) \Gamma(3 + 2/\tilde{\gamma}^2)}. \tag{7.5}$$
7.2 Decoupling of the singular states

In minimal models, the representation of the Virasoro algebra $V_\Delta$ is irreducible unless the dimension $\Delta$ takes values in Kac table. The singular vector $|\delta\rangle$ is orthogonal to any state of $V_\Delta$ and has zero norm. Thus such a singular state is also called a null state, and all its descendants are also null states since their norms are proportional to $\langle \delta | \delta \rangle$.

That such a null state in minimal models is orthogonal to the whole Verma module translates, in the field language, into the vanishing of the correlator $\langle \delta X \rangle$, where $X \equiv \phi(z_1) \cdots \phi(z_N)$. This implies certain differential equation for $\langle \delta X \rangle$. In the example for level-2 null state the differential equation is

$$\{ L_{-2} + \eta L_{-1}^2 \} \langle \phi(z) X \rangle = 0. \quad (7.6)$$

Or more explicitly

$$\left\{ \sum_{i=1}^{N} \frac{1}{z - z_i} \frac{\partial}{\partial z_i} + \frac{\Delta_i}{(z - z_i)^2} \right\} \langle \phi(z) X \rangle = 0, \quad (7.7)$$

where $\Delta_i$ is the dimension of the primary field $\phi_i$.

Now look at the singular states in the Liouville field theory. We see that eqn. (7.7) remains valid for $N \geq 2$, since in this case the Ward identity is of the standard form. For $N = 1$ eqn. (7.6) took a different form because of (6.53). Yet because of the $\delta$-function in the two point function

$$\langle e^{\alpha \phi(w)} e^{\beta \phi(0)} \rangle \sim \frac{\delta(\alpha - \beta)}{(w \bar{w})^{2\Delta}}, \quad (7.8)$$

the singular state is still orthogonal to the whole Verma modules but itself - that is, it is a decoupling state with nonzero norm (7.5).

8 BTZ Black hole entropy

Now let us look at the states in Liouville field theory that are candidates for the BTZ black hole state counting.

As we have seen in Section 4 and 5, the deformation parameters of the quantum algebra $U_q(sl_2) \otimes U_q(sl_2)$ in Liouville field theory are

$$\begin{cases} q = \exp(i\pi \tilde{\gamma}^2/2), & p = 4/\tilde{\gamma}^2, \\ \hat{q} = \exp(i2\pi/\tilde{\gamma}^2), & \hat{p} = \tilde{\gamma}^2. \end{cases} \quad (8.1)$$

Let us consider the case in which $4/\tilde{\gamma}^2 = 2N + 1$, an odd integer, so

$$q = e^{i2\pi/(2N+1)}, \quad \hat{q} = e^{i(N+1/2)\pi}. \quad (8.1)$$
Accordingly the Hartle-Hawking states with negative conformal weight $\Delta_{J,J}$ given by (5.19) have spin

$$J = 1/2, 1, ..., 2/\gamma^2,$$

$$\hat{J} = 1/2, 1.$$  (8.2)

The conformal families built on these states are reducible, as we have discussed in Section 5.2. The decoupling singular states in these reducible Verma modules $\Delta_{J,J}$ have conformal weights

$$\Delta_{\delta} = \Delta_{J,J} + 4J\hat{J} = \frac{c-1}{24} + \frac{1}{4} [2\hat{J}\alpha_+ - 2J\alpha_-]$$

and are highest weight states themselves. Yet unlike in minimal models, the Verma modules built on these decoupling states are irreducible without further decoupling, since the $rs$ term in eqn. (5.18) is now negative.

In particular, for states with $\omega = \pm 1$ and spins

$$\hat{J} = \frac{1}{2}, \quad J = \frac{1}{2}, 1, ..., \frac{2}{\gamma^2},$$

and

$$\hat{J} = 1, \quad J = \frac{1}{\gamma^2}, \frac{1}{4}, ..., \frac{2}{\gamma^2},$$

$\Delta_{\delta}$ takes value between 0 and $2/\gamma^2$. Altogether there are $12/\gamma^2$ such states. For other spin values $\Delta_{\delta}$ is negative.

As we showed in Section 7, unlike the null states of minimal models, these decoupling states have nonzero norms. Furthermore, they can have positive definite norm for certain values of $N$. Consider for example, the decoupling state at level $l = 2$ of the Hartle-Hawking state constructed from the operator $V = e^{-\tilde{\gamma}\phi/2}$. Its norm was given in eqn. (7.5), and the sign of the norm is determined by

$$\Gamma(-\tilde{\gamma}^2)\Gamma(-2/\gamma^2 - 2) = (-1)^{N+3} \frac{2^{N+3}\sqrt{\pi}}{(2N + 5)!!} \Gamma(-\tilde{\gamma}^2).$$  (8.5)

We can see from the properties of gamma function that for large odd $N$, the norm (7.5) is positive definite and finite.

Furthermore, the Verma modules built on these decoupling states are unitary representations of the Virasoro algebra, since these are representations with $c > 1$, $\Delta_{\delta} > 0$. The proof follows directly from the Kac determinant; see for example [77].

Thus we propose that the Hilbert space $\mathcal{H}$ contributing to black hole entropy is a tensor product of $12/\gamma^2$ unitary irreducible Verma modules $H$ built on the decoupling states described in (8.4) with conformal weights $0 < \Delta_{\delta} < 1/(2\gamma^2)$:

$$\mathcal{H} = H \otimes H \otimes ... \otimes H.$$  (8.6)

The structure is illustrated in Fig. 4.
The conformal weight $\Delta$ is the sum over these $12/\tilde{\gamma}^2$ sectors. Because these sectors decouple from each other, the asymptotic density of states contributed from each Verma module $H$ is equivalent to that of a theory with $12/\tilde{\gamma}^2$ scalar fields,

$$\ln \rho(\Delta) = 2\pi \sqrt{\frac{12 \Delta}{\tilde{\gamma}^2 \cdot 6}}. \quad (8.7)$$

For the semi-classical limit $\tilde{\gamma} \to 0$, this coincides with (1.2), the result from Cardy’s formula.

9 Conclusion and discussion

In this paper we have followed the canonical quantization approach of Liouville field theory due to Gervais and his collaborators, which shows an explicit quantum algebra structure $U_q(sl_2) \otimes U_q(sl_2)$ of the quantum Liouville theory. We considered the vertex operators that correspond to the graded tensor products of the irreducible representations of the quantum algebra with positive half-integer spins $j$, $\hat{j}$. The corresponding Hartle-Hawking states have negative conformal weights of Kac form, and the conformal families built on these highest-weight states are reducible. Yet unlike in minimal models, the decoupling states are not null states, due to the nonstandard form of the Ward identity for two-point functions. We showed that when the deformation parameter is a root of unity, more specifically, when $4/\tilde{\gamma}^2$ is an odd integer, there are natural cut-offs for spins $j$ and $\hat{j}$. The conformal families built on the decoupling states with positive conformal weights, which take value between 0 and $2/\gamma^2$, give rise to the correct Bekenstein-Hawking entropy for BTZ black hole.

This is a first step towards a thorough understanding of the microscopic states of the BTZ black hole. Many questions still remain to be answered, however.

- The derivation of the Ward identity for two-point functions is given here only for the tree-level calculation for “heavy” vertex operators. It would be interesting to have a nonperturbative derivation instead.

- We need to understand whether the result holds for general values of $\tilde{\gamma}$. As a first step, what happens if $4/\tilde{\gamma}^2$ is an even integer or a general rational number? For certain gauge groups, there are indications that although Chern-Simon theory is well-defined
for any coupling constant, but the physical Hilbert space becomes finite for rational
coupling constants and is different from the general case [78–80].

- When the deformation parameter is a root of unity, the fusion rules of the irreducible
representations of the quantum algebra $U_q(sl_2)$ necessarily include certain indecom-
posable representations, which may be related to the logarithmic operators in the
theory. It will be important to understand the role played by these indecomposable
representations.

- We must still understand the geometric meaning of these decoupling states.

- It will be interesting to understand the state counting when matter fields are coupled
to the gravitational field.

- Liouville theory may also be obtained near the horizon of an arbitrary black hole by
dimensionally reducing to the $r - t$ plane. Thus the calculation may be extended
towards the understanding of the gravitational degrees of freedom contributing to an
arbitrary black hole horizon.

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A Appendix

The following discussion basically follows [50].

Let $\mathbb{K}$ stand for a commutative ring with unit.

Definition 1. An (associative) algebra (with unity) is a vector space $\mathcal{A}$ over $\mathbb{K}$
together with two linear maps, $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, called the multiplication
or the product, and $\eta : \mathbb{K} \rightarrow \mathcal{A}$, called the unit, such that

$$m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m),$$

$$m \circ (\eta \otimes \text{id}) = \text{id} = m \circ (\text{id} \otimes \eta).$$

We now dualize this definition by reversing all arrows and replacing all mappings by the
premappings of the corresponding dual ones.

Definition 2. A coalgebra is a vector space $\mathcal{A}$ over $\mathbb{K}$ equipped with two linear mappings,
$\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, called the comultiplication or the coproduct of $\mathcal{A}$, and $\varepsilon : \mathcal{A} \rightarrow \mathbb{K}$, called
the counit, such that
\((\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta,\)
\((\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta.\) \hfill (A.2)

**Definition 3.** A bialgebra is a vector space which is an algebra and a coalgebra, that is, the following two conditions are equivalent:

(i). \(\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}\) and \(\varepsilon : \mathcal{A} \rightarrow K\) are algebra homomorphisms.

(ii). \(m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}\) and \(\eta : K \rightarrow \mathcal{A}\) are coalgebra homomorphisms.

**Definition 4.** A bialgebra \(\mathcal{A}\) is called a Hopf algebra if there exists a linear mapping \(S : \mathcal{A} \rightarrow \mathcal{A}\), called the antipode or the coinverse of \(\mathcal{A}\), such that
\(m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\text{id} \otimes S) \circ \Delta.\) \hfill (A.3)

The elements of the ring \(\mathbb{C}[[h]]\) are formal power series \(f = \sum_{n=0}^{\infty} a_n h^n\) in an indeterminate \(h\) with complex coefficient. Let \(V\) and \(W\) be vector spaces over \(\mathbb{C}[[h]]\). The topology on \(V\) for which the set \(\{h^n v + v | n \in \mathbb{N}_0\}\) are a neighborhood base of \(v \in V\) is called the \(h\)-adic topology. Denote by \(\hat{V} \otimes W\) the completion of the tensor product space \(V \otimes \mathbb{C}[[h]] W\) in the \(h\)-adic topology.

**Definition 5.** An \(h\)-adic Hopf algebra \(\mathcal{A}\) is a vector space over \(\mathbb{C}[[h]]\) which is complete in the \(h\)-adic topology and endowed with \(\mathbb{C}[[h]]\)-linear mappings \(m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \eta : \mathcal{A} \rightarrow \mathbb{C}[[h]]\) and \(S : \mathcal{A} \rightarrow \mathcal{A}\) which satisfy the Hopf algebra axioms with \(\otimes\) replaced by \(\hat{\otimes}\).

Let \(\tau\) denote the flip operator given by \(\tau(a \otimes b) = b \otimes a\). Define the coopposite coproduct \(\Delta^{\text{cop}} \equiv \tau \circ \Delta\).

**Definition 6.** A bialgebra (resp. Hopf algebra) \(\mathcal{A}\) is called quasitriangular if there exists an invertible element \(R\) of \(\mathcal{A} \otimes \mathcal{A}\) such that
\(\Delta^{\text{cop}}(a) = R\Delta(a)R^{-1},\)
\((\Delta \otimes \text{id})(R) = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{13}R_{12}.\) \hfill (A.4)

where \(R_{12} = \sum_i x_i \otimes y_i \otimes 1,\)
\(R_{13} = \sum_i x_i \otimes 1 \otimes y_i,\)
\(R_{23} = \sum_i 1 \otimes x_i \otimes y_i,\)
for \(R = \sum_i x_i \otimes y_i\). An invertible element \(R \in \mathcal{A} \otimes \mathcal{A}\) is called a universal \(R\)-matrix of \(\mathcal{A}\). A quasitriangular bialgebra (resp. Hopf algebra) with universal \(R\)-matrix \(R\) is said to be triangular if \(R_{21} = R_{-1}\), where \(R_{21} = \tau(R) \equiv \sum_i y_i \otimes x_i\).

**Proposition** Let \(\mathcal{A}\) be a quasitriangular bialgebra with universal \(R\)-matrix \(R\), then we have
\(R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},\) \hfill (A.6)
\((\varepsilon \otimes \text{id})(R) = (\varepsilon \otimes \text{id})(R) = 1.\) \hfill (A.7)
If $A$ is a Hopf algebra, then we also have

$$(S \otimes \text{id})(R) = R^{-1}, \quad (\text{id} \otimes S)(R^{-1}) = R, \quad (S \otimes S)(R) = R.$$  \hspace{1cm} (A.8)

The relation (A.6) is called the *Quantum Yang-Baxter equation*.

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