Solutions of the Schrödinger equation for the time-dependent linear potential*

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(October 31, 2003)

By making use of the Lewis-Riesenfeld invariant theory, the solution of the Schrödinger equation for the time-dependent linear potential corresponding to the quadratic-form Lewis-Riesenfeld invariant \( I_2(t) \) is obtained in the present paper. It is emphasized that in order to obtain the general solutions of the time-dependent Schrödinger equation, one should first find the complete set of Lewis-Riesenfeld invariants. For the present quantum system with a time-dependent linear potential, the linear \( I_1(t) \) and quadratic \( I_2(t) \) (where the latter \( I_2(t) \) cannot be written as the squared of the former \( I_1(t) \), i.e., the relation \( I_2(t) = cI_1^2(t) \) does not hold true always) will form a complete set of Lewis-Riesenfeld invariants. It is also shown that the solution obtained by Bekkar et al. is the one corresponding to the linear \( I_1(t) \), e. one of the invariants that form the complete set. In addition, we discuss some related topics regarding the comment [Phys. Rev. A 68, 016101 (2003)] of Bekkar et al. on Guedes’s work [Phys. Rev. A 63, 034101 (2001)] and Guedes’s corresponding reply [Phys. Rev. A 68, 016102 (2003)].

PACS: 03.65.Fd, 03.65.Ge

Keywords: exact solutions, Lewis-Riesenfeld invariant formulation, unitary transformation

I. INTRODUCTION

Recently, Guedes used the Lewis-Riesenfeld invariant formulation [1] and solved the one-dimensional Schrödinger equation with a time-dependent linear potential [2]. More recently, Bekkar et al. pointed out that [3] the result obtained by Guedes is merely the particular solution (that corresponds to the null eigenvalue of the linear Lewis-Riesenfeld invariant) rather than a general one. In the comment [3], Bekkar et al. stated that they correctly used the invariant method [1] and gave the general solutions of the time-dependent Schrödinger equation with a time-dependent linear potential [3]. However, in the present paper, I will show that although the solutions of Bekkar et al. is more general than that of Guedes [2], what they finally achieved in their comment [3] is still not the general solutions, either. On the contrary, I think that their result [3] also belongs to the particular one. The reason for this may be as follows: according to the Lewis-Riesenfeld invariant method [1], the solutions of the time-dependent Schrödinger equation can be constructed in terms of the eigenstates of the Lewis-Riesenfeld (L-R) invariants. It is known that both the squared of a L-R invariant (denoted by \( I(t) \)) and the product of two L-R invariants are also the invariants, which agree with the Liouville-Von Neumann equation \( \frac{\partial}{\partial t} I(t) + \frac{i}{\hbar} [I(t), H(t)] = 0 \), and that if \( I_a \) and \( I_b \) are the two L-R invariants of a certain time-dependent quantum system and \( \langle \psi(t) \rangle \) is the solution of the time-dependent Schrödinger equation (corresponding to one of the invariants, say, \( I_a \)), then \( I_b \langle \psi(t) \rangle \) is another solution of this quantum system. So, in an attempt to obtain the general solutions of a time-dependent system, one should first analyze the complete set of all L-R invariants of the system under consideration. Historically, in order to obtain the complete set of invariants, Gao et al. suggested the concept of basic invariants which can generate the complete set of invariants [4], as stated in Ref. [4], the basic invariants can be called invariant generators. As far as Bekkar et al.’s result [3] is concerned, the obtained solutions are the ones corresponding only to the linear invariant (i.e., \( I_1(t) = A(t)p + B(t)q + C(t) \)) that is simply one of the L-R invariants, which form a complete set. It is apparently seen that the quadratic form, \( I_2(t) = D(t)p^2 + E(t)(pq + qp) + F(t)q^2 + A'(t)p + B'(t)q + C'(t) \), is also the one that can satisfy the Liouville-Von

*I think that this paper will be a supplement to the recent comment [Phys. Rev. A 68, 016101 (2003)] of Bekkar et al. on Guedes’s work [Phys. Rev. A 63, 034102 (2001)] and Guedes’s reply to Bekkar et al.’s comment. It will be submitted nowhere else for publication, just uploaded at the e-print archives.

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Neumann equation, since it is readily verified that the Lie algebraic generators of $I_q(t)$ form a Lie algebra, which possesses the following commutators

\[
[q^2, p^2] = 2i(pq + qp), \quad [pq + qp, q^2] = -4iq^2, \quad [pq + qp, p^2] = 4ip^2, \\
\]

(1.1)

However, for the cubic-form invariant, it is easily seen that there exists no such closed Lie algebra. This point holds true also for the algebraic generators in any high-order L-R invariants $I^n_l$. So, it is concluded that for the driven oscillator, only the linear $I_1(t)$ and quadratic $I_2(t)$ will form a complete set of L-R invariants. Note that here $I_q(t)$ should not be the squared of $I_l(t)$, i.e., $I_q(t) \neq cI^2_l(t)$, where $c$ is an arbitrary c-number. It is emphasized here that Bekkar et al.’s solution is the one constructed only in terms of the eigenstates of the linear invariant $I_1(t)$. Even though only for the linear invariant $I_1(t)$ Bekkar et al.’s result [3] can truly be viewed as the complete set of solutions, it still cannot be considered general one of the Schrödinger equation, since the latter should contain those corresponding to the quadratic invariant $I_2(t)$. In brief, Bekkar et al.’s solution and my solution, which will be found in what follows, together constitute the complete set of solutions of the Schrödinger equation involving a time-dependent linear potential.

II. ON THE COMPLETE SET OF L-R INVARIANTS

According to the L-R invariant theory [1], if the eigenstate of the linear invariant $I_1(t)$ corresponding to $\lambda_n$, i.e., one of the eigenvalues of $I_1(t)$, is $|\lambda_n, t\rangle$, then the solution of the Schrödinger equation can be written in the form

\[
|\Psi(t)\rangle_{Schr} = \sum_n c_n \exp \left[ \frac{i}{\hbar} \phi_n(t) \right] |\lambda_n, t\rangle
\]

(2.1)

with $c_n$’s and $\phi_n(t)$’s being the time-independent coefficients and time-dependent phases [1.4], respectively. This, therefore, means that the solutions of the time-dependent Schrödinger equation can be constructed in terms of the complete set of eigenbasis set, $\{|\lambda_n, t\rangle\}$, of $I_1(t)$. Moreover, one can readily verify that the squared, $I_2^2$, of the linear invariant is the one satisfying the Liouville-Von Neumann equation, and that $I_1(t)|\Psi(t)\rangle_{Schr}$ is also a solution (but not another new general one) of the same time-dependent Schrödinger equation, since it is readily verified that $I_1(t)|\Psi(t)\rangle_{Schr}$ can also be the linear combination of the eigenstate basis set $\{|\lambda_n, t\rangle\}$ of $I_1(t)$, i.e.,

\[
I_1(t)|\Psi(t)\rangle_{Schr} = \sum_n b_n \exp \left[ \frac{i}{\hbar} \phi_n(t) \right] |\lambda_n, t\rangle,
\]

(2.2)

where the time-independent coefficients $b_n$’s are taken $b_n = \lambda_n c_n$, which is obtained via the comparison of the expression (2.2) with (2.1).

Thus, the above discussion shows that the linear invariant $I_1$ and its squared $I_2^2$ have the same eigenstate basis set and therefore $I_1$ and $I_2^2$ cannot form a complete set of L-R invariants. In contrast, if for any c-number $c$, the quadratic $I_4$ cannot be written as the squared of linear $I_1$ with various integral constants $A_0$, $B_0$ and $C_0$ (for the definition of $A_0$, $B_0$ and $C_0$, see, for example, in Ref. [3]), namely, the relation $I_4 = cI_2^2$ is always not true, then $\{I_1, I_4\}$ is the complete set of L-R invariants, which enables us to obtain the general solutions (complete set of solutions) of the time-dependent Schrödinger equation.

Perhaps someone will ask such question as, “Does there really exist such quadratic $I_4$ that can always not be written in the form $cI_2^2$?” or “Maybe any $I_q$ that satisfies the Liouville-Von Neumann equation can surely be written as the squared of certain $I_l$.” Really?” Now I will discuss these questions. Consider a given quadratic invariant $I_q$ that is written $I_q(t) = D(t)p^2 + E(t)(pq + qp) + F(t)q^2 + A'(t)p + B'(t)q + C'(t)$ whose time-dependent parameters are determined by the Liouville-Von Neumann equation, and a certain linear invariant $I_1(t) = A(t)p + B(t)q + C(t)$, the squared of which is $I_1^2 = A^2p^2 + AB(pq + qp) + B^2q^2 + 2C(Ap + Bq + \frac{C}{2})$. Since the functions $A$, $B$ and $C$ can also be determined by the Liouville-Von Neumann equation, the only retained parts left to us to determine is the integral constants $A_0$, $B_0$ and $C_0$. Choose the appropriate integral constants in $A$, $B$ and $C$, and let $I_q$ be the squared of $I_1$ (should case exist), and then we have

\[
D = cA^2, \quad E = cAB, \quad F = cB^2, \\
A' = 2cAC, \quad B' = 2cBC, \quad C' = cc^2.
\]

(2.3)
If a given $I_q$ can really be written as the squared of $I_1$, the above six equations are just used to determine the c-number $c$ and the suitable integral constants $A_0$, $B_0$ and $C_0$ in the functions $A$, $B$ and $C$. It is seen that there are only four numbers expected to be determined, and that, in contrast, we have six equations. So, it is possible that there exist potential parameters $c$ and $A_0$, $B_0$, $C_0$ which will not agree with Eqs.(2.3) always for a given parameter set $\{D, E, F, A', B', C'\}$, or, for a given parameter set $\{D, E, F, A', B', C'\}$ there are always no such parameters $c$ and $A_0$, $B_0$, $C_0$ which satisfy Eqs.(2.3). The existence of $I_q$ that cannot be written as the squared of any $I_1$ is thus demonstrated.

So, in the above we indicate that such two invariants $I_1$ and $I_q$ (which are independent) form a complete set of L-R invariants.

III. UNITARY TRANSFORMATION ASSOCIATED WITH L-R INVARIANTS

Now I will solve the time-dependent Schrödinger equation, of which the time-dependent Hamiltonian [2] is given

$$H(t) = \frac{p^2}{2m} + f(t)q,$$

by making use of the Lewis-Riesenfeld invariant theory [1]. The time-dependent L-R invariant used here takes the form

$$I_q(t) = D(t)p^2 + E(t)(pq + qp) + F(t)q^2 + A(t)p + B(t)q + C(t).$$

(3.2)

With the help of the Liouville-Von Neumann equation, one can arrive at

$$\dot{D} + \frac{2E}{m} = 0, \quad \dot{E} + \frac{F}{m} = 0, \quad \dot{F} = 0,$$
$$\dot{A} + \frac{B}{m} - 2Df = 0, \quad \dot{B} - 2Ef = 0, \quad \dot{C} - fA = 0$$

(3.3)

with dot denoting the derivative with respect to time $t$. The above six equations (referred to as the auxiliary equations [4]) can be used to determine all the time-dependent parameters $A(t)$, $B(t)$, $C(t)$ and $D(t)$, $E(t)$, $F(t)$.

In accordance with the L-R theory, solving the eigenstates of the invariant (3.2) will enable us to obtain the solutions of the time-dependent Schrödinger equation. But, unfortunately, it is not easy for us to immediately solve the eigenvalue equation of the time-dependent invariant (3.2), for the invariant (3.2) involves the time-dependent parameters. So, in the following we will use the invariant-related unitary transformation formulation [4], under which the time-dependent invariant in (3.2) can be transformed into a time-independent one $I_V$, and if the eigenstates of $I_V$ can be obtained conveniently, the eigenstates of $I_q(t)$ can then be easily achieved.

Here we will employ two time-dependent unitary transformation operators

$$V_1(t) = \exp[\eta(t)q + \beta(t)p], \quad V_2(t) = \exp[\alpha(t)p^2 + \rho(t)q^2]$$

(3.4)

to get a time-independent $I_V$. The time-dependent parameters $\eta$, $\beta$, $\alpha$ and $\rho$ in (3.4) are purely imaginary functions, which will be determined in the following subsections. Since the canonical variables (operators) $q$ and $p$ form a non-semisimple Lie algebra, here the first step is to transform $I_q(t)$ into $I_1(t)$, i.e., $I_1(t) = V_1^\dagger(t)I_q(t)V_1(t)$, which no longer involves the canonical variables $q$ and $p$, and the retained Lie algebraic generators in $I_1(t)$ are only $p^2$, $pq + qp$, $q^2$. Note that these three generators also form a Lie algebra (see the commutators (1.1)) . The second step is to obtain the time-independent $I_V$, which will be gained via the calculation of $I_V = V_2^\dagger(t)I_1(t)V_2(t)$. In this step, the obtained $I_V$ has no other generators (and time-dependent c-numbers) than $p^2$ and $q^2$, namely, $I_V$ may be written in the form $I_V = \zeta(p^2 + q^2)$ with $\zeta$ being a certain parameter independent of time. It is well known that the eigenvalue equation of $I_V$ is of the form $I_V[n,q] = (2n + 1)\zeta[n,q]$, where $|n,q\rangle$ stands for the familiar harmonic-oscillator wavefunction. Hence, the eigenstates of the time-dependent L-R invariant $I_q(t)$ in (3.2) can be achieved and the final result is $V_1(t)V_2(t)|n,q\rangle$ with the eigenvalue being $(2n + 1)\zeta$.

A. The calculation of $I_1(t) = V_1^\dagger(t)I_q(t)V_1(t)$

By the aid of the Glauber formula, one can arrive at
\[
I_1(t) = Dp^2 + E(pq + qp) + Fq^2 + [A + 2i(E\beta - D\eta)]p + [B + 2i(F\beta - E\eta)]q \\
+ C - [-i(B\beta - A\eta) + D\eta^2 + F\beta^2 - 2E\beta\eta].
\] (3.5)

If the two relations
\[
A + 2i(E\beta - D\eta) = 0, \quad B + 2i(F\beta - E\eta) = 0
\] (3.6)
are satisfied, then we can obtain\(^1\)
\[
I_1(t) = D(t)p^2 + E(t)(pq + qp) + F(t)q^2.
\] (3.7)

It follows from (3.6) that the time-dependent parameters in the unitary transformation \(V_1(t)\) are expressed by
\[
\eta = \frac{EB - FA}{2i(E^2 - DF)}, \quad \beta = \frac{DB - EA}{2i(E^2 - DF)}.
\] (3.8)

**B. The calculation of \(I_V = V_2^\dagger(t)I_1(t)V_2(t)\)**

By using the Glauber formula, one can arrive at
\[
I_V = V_2^\dagger(t)I_1(t)V_2(t) = Dp^2 + E(pq + qp) + Fq^2,
\] (3.9)
where \(D, E\) and \(F\) are of the form
\[
D = D + \frac{4iE\alpha}{(16\rho\alpha)^{\frac{1}{2}}} \sinh(16\rho\alpha)^{\frac{1}{2}} + \frac{-8(F\alpha - D\rho)\alpha}{16\rho\alpha} \left[ \cosh(16\rho\alpha)^{\frac{1}{2}} - 1 \right],
\]
\[
E = \frac{2i(F\alpha - D\rho)}{(16\rho\alpha)^{\frac{1}{2}}} \sinh(16\rho\alpha)^{\frac{1}{2}} + E \cosh(16\rho\alpha)^{\frac{1}{2}},
\]
\[
F = F + \frac{-4iE\rho}{(16\rho\alpha)^{\frac{1}{2}}} \sinh(16\rho\alpha)^{\frac{1}{2}} + \frac{8(F\alpha - D\rho)\rho}{16\rho\alpha} \left[ \cosh(16\rho\alpha)^{\frac{1}{2}} - 1 \right],
\] (3.10)
respectively. It follows that if the following two equations are satisfied,
\[
E = \zeta \sinh(16\rho\alpha)^{\frac{1}{2}}, \quad \frac{2i(F\alpha - D\rho)}{(16\rho\alpha)^{\frac{1}{2}}} = -\zeta \cosh(16\rho\alpha)^{\frac{1}{2}},
\] (3.11)
then the coefficients of \(pq + qp\) in \(I_V\) is vanishing. In order that we can analyze the above equations (3.11) conveniently, the time-dependent parameters \(\alpha, \rho\) (which are expected to be determined) and \(F, D\) are respectively parameterized to be
\[
\alpha = \frac{u\theta}{4}, \quad \rho = \frac{v\theta}{4}, \quad F = h \cosh(\sqrt{uv}\theta), \quad D = g \cosh(\sqrt{uv}\theta).
\] (3.12)
Substitution of the expressions (3.12) into (3.11) yields
\[
E = \zeta \sinh(\sqrt{uv}\theta), \quad \frac{i(hu - gv)}{2\sqrt{uv}} = -\zeta,
\] (3.13)
which can determine \(\zeta\) and \(\theta\) (expressed in terms of \(E, h, g\) and \(u, v\)). It is noted that if the functions \(u\) and \(v\) are finally determined, then the time-dependent parameters \(\alpha\) and \(\rho\) in the unitary transformation operator \(V_2(t)\) (3.4) can be obtained.

\(^1\)In general, for the case of three-generator Hamiltonian (the generators of which form a non-semisimple algebra), the time-dependent c-number \(C(t) = [-i(B\beta - A\eta) + D\eta^2 + F\beta^2 - 2E\beta\eta] \) in \(I_1(t)\) are vanishing. See, for example, in Ref. [5], which is a special case of the present problem.
In what follows we will determine $u$ and $v$ via setting $D = \mathcal{F} = \varsigma$ with $\varsigma$ being constant (i.e., time-independent). Insertion of (3.12) into (3.10) will yield
\[
D + \frac{hu - gv}{2v} \left[ \cosh(\sqrt{uv} \theta) - 1 \right] = \varsigma, \quad F - \frac{hu - gv}{2u} \left[ \cosh(\sqrt{uv} \theta) - 1 \right] = \varsigma.
\] (3.14)

Eq. (3.14) can determine the functions $u$ and $v$, although the problem is very complicated. Here it should be noted that $\theta$ which has been determined by (3.13) is also the function of $u$ and $v$. Thus, in principle, we can obtain the time-dependent functions $\alpha$ and $\rho$ in the second unitary transformation operator $V_2(t) = \exp(\alpha(t)p^2 + \rho(t)q^2)$.

Now under the unitary transformation $V_1(t)V_2(t)$ the time-dependent invariant $I_q(t)$ is changed into a time-independent one, i.e.,
\[
I_V \equiv [V_1(t)V_2(t)]^\dagger I_q(t)[V_1(t)V_2(t)] = \varsigma (p^2 + q^2)
\] (3.15)

whose eigenvalue is $(2n + 1)\varsigma$ and the corresponding eigenstate is $|n, q\rangle$ that is the familiar stationary harmonic-oscillator wavefunction, and the eigenvalue equation of the time-dependent invariant $I_q(t)$ is thus given as follows
\[
I_q(t)V_1(t)V_2(t)|n, q\rangle = (2n + 1)\varsigma V_1(t)V_2(t)|n, q\rangle.
\] (3.16)

C. The solutions of the time-dependent Schrödinger equation

According to the L-R invariant theory, the particular solution $|n, t\rangle_{\text{Schr}}$ of the time-dependent Schrödinger equation is different from the eigenfunction of the invariant $I_q(t)$ only by a phase factor $\exp \left[ \frac{1}{\tau} \phi_n(t) \right]$, the time-dependent phase of which is written as
\[
\phi_n(t) = \int_0^t \langle n, q| [V_1(t')V_2(t')]^\dagger [H(t') - i \partial / \partial t'] [V_1(t')V_2(t')] |n, q \rangle dt'.
\] (3.17)

This phase $\phi_n(t)$ can be calculated with the help of the Glauber formula and the Baker-Campbell-Hausdorff formula [6,7].

The particular solution $|n, t\rangle_{\text{Schr}}$ of the time-dependent Schrödinger equation corresponding to the invariant eigenvalue $(2n + 1)\varsigma$ is thus of the form
\[
|n, t\rangle_{\text{Schr}} = \exp \left[ \frac{1}{\tau} \phi_n(t) \right] V_1(t)V_2(t)|n, q\rangle.
\] (3.18)

Hence the general solution of the Schrödinger equation can be written in the form
\[
|\Psi(q, t)\rangle_{\text{Schr}} = \sum_n c_n |n, t\rangle_{\text{Schr}},
\] (3.19)

where the time-independent c-number $c_n$’s are determined by the initial conditions, i.e., $c_n =_{\text{Schr}} \langle n, t = 0 | \Psi(q, t = 0) \rangle_{\text{Schr}}$.

In the above we thus found the general solutions of the Schrödinger equation for the time-dependent linear potential, which corresponds only to the quadratic-form invariant (3.2). It is concluded here that the solutions obtained above do not form a complete set of solutions of this time-dependent Schrödinger equation, and that Bekkar et al.’s solution and my solution presented here will constitute together such complete set of solutions of the Schrödinger equation.

IV. DISCUSSIONS AND CONCLUSIONS

(i) In the present paper we show that since Bekkar et al.’s solution [3] has not yet contain those corresponding to the quadratic invariant, it is not the true general solution of the Schrödinger equation for the time-dependent linear potential. Instead, it is the solution corresponding only to the linear L-R invariant. The obtained solution here is the one that corresponds to the invariant (3.2), which is of the quadratic form. Since the linear and quadratic invariants
form a complete set of L-R invariants, Bekkar et al.’s solution [3] and my solution presented here constitute such complete set of solutions of the Schrödinger equation involving a time-dependent linear potential.

(ii) It is well known that in quantum optics there are three kinds of photonic quantum states, i.e., Fock state, coherent state and squeezed state. From my point of view, the calculation of the variations of creation and annihilation operators (a†, a) of photons under the translation (e.g., V1 of (3.4)) and squeezing transformation (e.g., V2 of (3.4)) operators shows that the variations of a† and a are exactly analogous to that of space-time coordinate variations under the translation, Lorentz rotation (boosts) and dilatation (scale) transformation [9] and thus these three quantum states (coherent, squeezed and Fock states) correspond to the above three conformal transformations, respectively. I think that this connection between them is of physical interest and deserves further consideration.

(iii) Guedes recently stated that in order to obtain the general solutions of the Schrödinger equation one must follow the L-R invariant theory step by step [8]. I don’t approve of this point of view, however. Personally speaking, in fact, the L-R method has only one step, namely, the particular solution of the time-dependent Schrödinger equation is different from the eigenfunction of the invariant only by a time-dependent phase factor. In Ref. [3] and [5], although we follow the L-R method step by step, what we obtained still cannot be viewed as the general solutions of the Schrödinger equation. For this reason, I think that “step by step” is not the essence of getting the general solutions of Schrödinger equation. Instead, the key point for the present subject is that one should first find the complete set of all L-R invariants of the time-dependent quantum systems under consideration. For some systems in the Hamiltonian there may exist no such closed Lie algebra as (1.1), the complete set of exact solutions can be found by working in a sub-Hilbert-space corresponding to a particular eigenvalue of one of the invariants, namely, only in the sub-algebra (quasi-algebra) corresponding to a particular eigenvalue of this invariant will such time-dependent quantum systems (which have no closed Lie algebra) be solvable [10]. For the time-dependent quantum systems, there are no other eigenvalue equations of Hamiltonian than that of the L-R invariants with time-dependent eigenvalues. The complete set of invariants, instead of the time-dependent Hamiltonian, can describe completely the time-dependent quantum systems. For this reason, it is essential to find the complete set of invariants for the time-dependent Hamiltonian of a given quantum system.

(iv) In the Ref. [2], the author says that to the best of his knowledge there was no publication reporting the solution of the Schrödinger equation for the system described by $H(t) = \frac{p^2}{2m} + f(t)q$ without considering approximate and/or numerical calculations. I think that this is, however, not the true case. In the literature, at least in the early of 1990’s, Gao et al. had reported their investigation of the driven generalized time-dependent harmonic oscillator which is described by the following Hamiltonian $H(t) = \frac{1}{2}[X(t)q^2 + Y(t)(pq + qp) + Z(t)p^2] + F(t)q$ [4]. It is believed that my solution presented here is only the special case of what they obtained [4].

Acknowledgements This project was supported partially by the National Natural Science Foundation of China under the project No. 90101024.