Effective mass of a radiating charged particle in Einstein’s universe

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Abstract

The effective gravitational mass as well as the energy and momentum distributions of a radiating charged particle in Einstein’s universe are evaluated. The Møller’s energy-momentum complex is employed for this computation. The spacetime under study is a generalization of Bonnor and Vaidya spacetime in the sense that the metric is described in the cosmological background of Einstein’s universe in lieu of the flat background. Several spacetimes are limiting cases of the one considered here. Particularly for the Reissner-Nordstrøm black hole background, our results are exactly the same with those derived by Cohen and Gautreau using Whittaker’s theorem and by Cohen and de Felice using Komar’s mass. Furthermore, the power output for the spacetime under consideration is obtained.

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Introduction

Energy-momentum localization has been one of the most interesting but also thorny problems for the General Theory of Relativity. A plethora of different attempts to solve this problem have led to inconclusive results till now. Energy-momentum complexes introduced first by Einstein [1], were the foremost endeavor to solve this problem. After that a large number of different expressions for the energy-momentum complexes were proposed [2]. A drawback of this attempt was that energy-momentum complexes had to be computed in quasi-Cartesian coordinates. Møller [3] proposed a new expression for an energy-momentum complex which could be utilized to any coordinate system. However, the idea of the energy-momentum complex was severely criticized for a number of reasons [4]. Considerable attempts to deal with this problematic issue are also the quasi-local [5] and the superenergetic quantities [6] 2.

Virbhadra and collaborators enlivened anew the concept of energy-momentum complexes [8]. Since then, numerous works have been performed on evaluating the energy and momentum distributions of several gravitational backgrounds using the energy-momentum complexes [9]. In support of the importance of the concept of energy-momentum complexes, Chang, Nester and Chen [10] proved that every energy-momentum complex is associated with a Hamiltonian boundary term. Thus, the energy-momentum complexes are quasi-local and acceptable.

In this paper we evaluate the energy and momentum density distributions of a radiating charged particle in Einstein’s universe. Additionally, the total energy and the power output are computed. The total energy is actually the effective gravitational mass whose gravitational field a neutral particle experiences. The prescription that is used in the present analysis, is the one introduced by Møller. The reasons for presenting here the Møller’s description are: (a) the argument that it is not restricted to quasi-Cartesian coordinates and (b) a work of Lessner [11] who argues that the Møller’s energy-momentum complex is a powerful concept of energy and momentum in General Theory of Relativity. The remainder of the paper is organized as follows. In Section 1 we consider the concept of energy-momentum complexes in the framework of General Theory of Relativity. In Section 2 the Møller’s energy-momentum complex is presented. In Section 3 we give the metric that describes a radiating charged particle in Einstein’s universe. In Section 4 we utilize Møller’s prescription and we calculate the energy and momentum density distributions of the afore-mentioned spacetime. We also compute explicitly the effective

2 There is some interest in employing the energy-momentum complexes in the framework of teleparallel equivalent of General Relativity, i.e. teleparallel gravity [7].
gravitational mass and the power output for the specific background. Furthermore, we evaluate the above-mentioned quantities for several spacetimes which are limiting cases of the one presented in Section 3. We compare the results derived in the present analysis with the ones that already exist in the literature. Finally, Section 5 is devoted to a brief summary of results and concluding remarks.

1 Energy-Momentum Complexes

The conservation laws of energy and momentum for an isolated (closed), i.e. no external force acting on the system, physical system in the Special Theory of Relativity are expressed by a set of differential equations. Defining $T_{\mu}^{\nu}$ as the symmetric energy-momentum tensor of matter and all non-gravitational fields the conservation laws are given by

$$T_{\nu, \mu} = \frac{\partial T_{\nu}^{\mu}}{\partial x^{\mu}} = 0$$  (1)

where

$$\rho = T_{t}^{t}, \quad j^{i} = T_{t}^{i}, \quad p_{i} = -T_{i}^{t}$$  (2)

are the energy density, the energy current density, the momentum density, respectively, and Greek indices run over the spacetime labels while Latin indices run over the spatial coordinate values.

Making the transition from the Special to General Theory of Relativity one adopts a simplicity principle which is called principle of minimal gravitational coupling. As a result of this, the conservation equation is now written as

$$T_{\nu, \mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left( \sqrt{-g} T_{\nu}^{\mu} \right) - \Gamma_{\nu\lambda}^{\kappa} T_{\lambda}^{\kappa} = 0$$  (3)

where $g$ is the determinant of the metric tensor $g_{\mu\nu}(x)$. The conservation equation may also be written as

$$\frac{\partial}{\partial x^{\mu}} \left( \sqrt{-g} T_{\nu}^{\mu} \right) = \xi_{\nu}$$  (4)

where

$$\xi_{\nu} = \Gamma_{\nu\lambda}^{\kappa} T_{\lambda}^{\kappa}$$  (5)

is a non-tensorial object. For $\nu = t$ this means that the matter energy is not a conserved quantity for the physical system\(^3\). From a physical point of view this lack of energy conservation can be understood as the possibility of transforming matter energy into

\(^3\)It is possible to restore the conservation law by introducing a local inertial system for which at a specific spacetime point $\xi_{\nu} = 0$ but this equality by no means holds in general.
gravitational energy and vice versa. However, this remains a problem and it is widely believed that in order to be solved one has to take into account the gravitational energy [2].

By a well-known procedure, the non-tensorial object $\xi_\nu$ can be written as

$$\xi_\nu = -\frac{\partial}{\partial x^\mu} \left( \sqrt{-g} \vartheta^\mu_\nu \right)$$

where $\vartheta^\mu_\nu$ are certain functions of the metric tensor and its first order derivatives. Therefore, the energy-momentum tensor of matter $T^\mu_\nu$ is replaced by the expression

$$\theta^\mu_\nu = \sqrt{-g} \left( T^\mu_\nu + \vartheta^\mu_\nu \right)$$

which is called energy-momentum complex since it is a combination of the tensor $T^\mu_\nu$ and a pseudotensor $\vartheta^\mu_\nu$ which describes the energy and momentum of the gravitational field. The energy-momentum complex satisfies a conservation law in the ordinary sense, i.e.

$$\theta^\mu_\nu, \mu = 0$$

and it can be written as

$$\theta^\mu_\nu = \chi^\mu_\nu, \lambda$$

where $\chi^\mu_\nu, \lambda$ are called superpotentials and are functions of the metric tensor and its first order derivatives.

It is obvious that the energy-momentum complex is not uniquely determined by the condition that is usual divergence is zero since it can always been added to the energy-momentum complex a quantity with an identically vanishing divergence.

## 2 Møller’s Prescription

The energy-momentum complex of Møller in a four-dimensional background is given as [3]

$$J^\mu_\nu = \frac{1}{8\pi} \xi^\mu_\nu, \lambda$$

where the Møller’s superpotential $\xi^\mu_\nu, \lambda$ is of the form

$$\xi^\mu_\nu, \lambda = \sqrt{-g} \left( \frac{\partial g_{\nu\sigma}}{\partial x^\kappa} - \frac{\partial g_{\nu\kappa}}{\partial x^\sigma} \right) g^{\mu\kappa} g^{\lambda\sigma}$$

with the antisymmetric property

$$\xi^\mu_\nu, \lambda = -\xi^{\lambda\mu}_\nu.$$
It is easily seen that the Møller’s energy-momentum complex satisfies the local conservation equation
\[ \frac{\partial J_\mu}{\partial x^\mu} = 0 \quad (13) \]
where \( J_0^0 \) is the energy density and \( J_i^0 \) are the momentum density components.

Thus, the energy and momentum in Møller’s prescription for a four-dimensional background are given by
\[ P_\mu = \int \int \int J_0^0 dx^1 dx^2 dx^3 \quad (14) \]
and specifically the energy of the physical system in a four-dimensional background is
\[ E = \int \int \int J_0^0 dx^1 dx^2 dx^3 . \quad (15) \]
It should be noted that the calculations are not anymore restricted to quasi-Cartesian coordinates but they can be utilized in any coordinate system.

### 3 A radiating charged particle in Einstein’s universe

In 1970 Bonnor and Vaidya [12] presented a solution describing a radiating charged particle in flat spacetime. The corresponding metric is of the form
\[ ds^2 = 2 dudr + \left( 1 - \frac{2M(u)}{r} + \frac{4\pi Q^2(u)}{r^2} \right) du^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (16) \]
where \( u \) is the retarded null coordinate, i.e. \( u = t - r \), and \( M(u) \) and \( Q(u) \) are respectively the mass and charge of the particle. The mass function \( M(u) \) is an arbitrary nonincreasing function of the retarded null coordinate \( u \). The particle lives in a flat background and this is easily seen by letting the radial coordinate go to infinity, i.e. \( r \to \infty \).

Patel and Akabari [13] realized that it would be more interesting to have the particle in a cosmological background. Therefore, they considered the space surrounding the radiating charged particle to be occupied by a spherical symmetric matter distribution of nonzero density \( \rho \) and pressure \( p \). Finally, they derived the following metric
\[ ds^2 = 2 dudr + \left( 1 - \frac{2M(u)}{R} \cot \left( \frac{r}{R} \right) + \frac{4\pi Q^2(u)}{R^2} \left[ \cot^2 \left( \frac{r}{R} \right) - 1 \right] \right) du^2 \\
- R^2 \sin^2 \left( \frac{r}{R} \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (17) \]
where \( R \) is a constant and by setting the mass and the charge of the particle equal to zero, namely \( M(u) = Q(u) = 0 \), the metric (17) reduces to the metric of Einstein’s universe
\[ ds^2 = 2 dudr + du^2 - R^2 \sin^2 \left( \frac{r}{R} \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) . \quad (18) \]
It is noteworthy that several spacetimes are limiting cases of the gravitational background under study.

4 The effective mass

The aim of this section is to evaluate the effective gravitational mass of the radiating charged particle in Einstein’s universe using the Møller’s energy-momentum complex. We first have to evaluate the superpotentials in the context of Møller’s prescription. There are twelve nonzero superpotentials

\[
\begin{align*}
\xi_{21}^2 &= -\xi_{12}^2 = -2 \left[ M(u) - \frac{4\pi Q^2(u)}{R} \cot \left( \frac{r}{R} \right) \right] \sin \theta \\
\xi_{31}^3 &= -\xi_{13}^3 = -2R \sin \left( \frac{r}{R} \right) \cos \left( \frac{r}{R} \right) \sin \theta \\
\xi_{41}^4 &= -\xi_{14}^4 = -2R \sin \left( \frac{r}{R} \right) \cos \left( \frac{r}{R} \right) \sin \theta \\
\xi_{33}^3 &= 2 \left( 2M(u) \cos^2 \left( \frac{r}{R} \right) - R \sin \left( \frac{r}{R} \right) \cos \left( \frac{r}{R} \right) \right) - \frac{4\pi Q^2(u)}{R} \cot \left( \frac{r}{R} \right) \left[ 2 \cot^2 \left( \frac{r}{R} \right) - 1 \right] \sin \theta \\
\xi_{33}^2 &= -\xi_{33}^3 \\
\xi_{44}^3 &= -\xi_{44}^3 = -2 \cos \theta \\
\xi_{44}^4 &= -\xi_{44}^2 = \xi_{23}^3 \\
\end{align*}
\]

By substituting the Møller’s superpotentials, as given by (19), into equation (10), one gets the energy density distribution

\[
\mathcal{J}_0^0 = \frac{Q^2(u)}{R^2 \sin^2 \left( \frac{r}{R} \right)} \sin \theta
\]

(20)

while the momentum density distributions take the form

\[
\begin{align*}
\mathcal{J}_1^0 &= 0 \\
\mathcal{J}_2^0 &= \frac{1}{4\pi} \sin \left( \frac{r}{R} \right) \cos \left( \frac{r}{R} \right) \cos \theta \\
\mathcal{J}_3^0 &= 0 .
\end{align*}
\]

(21) (22) (23)

Therefore, if we substitute equation (20) into equation (15), we get the energy of the radiating charged particle in Einstein’s universe that is contained in a “sphere” of radius \( r \)

\[
E(r) = M(u) - \frac{4\pi Q^2(u)}{R} \cot \left( \frac{r}{R} \right)
\]

(24)
which is also the energy (mass) of the gravitational field that a neutral particle experiences at a finite distance \( r \). Thus, the energy given by equation (24) is in addition called effective gravitational mass \( (E = M_{\text{eff}}) \) of the spacetime under study. It is obvious that the repulsive effect of the electric charge does not depend on its sign and that the particle experiences a negative mass which acts repulsively when

\[
\frac{R}{\cot \left( \frac{R}{2} \right)} < \frac{4\pi Q^2(u)}{M(u)}.
\]

(25)

Additionally, if we replace equations (21-23) into equation (14) we get the momentum components which are given by

\[ P_1 = P_2 = P_3 = 0. \]

(26)

Furthermore, we are interested in evaluating the power output, i.e. the luminosity, for an observer at rest. The power output is given by the expression

\[ L = -\frac{dM_{\text{eff}}}{du} \]

(27)

and hence for the spacetime under study, it takes the form

\[ L = -M_u(u) + \frac{8\pi Q(u)Q_u(u)}{R} \cot \left( \frac{r}{R} \right) \]

(28)

where the subscript \( u \) denotes the derivative with respect to the retarded null coordinate \( u \). One should not worry about the positivity of the power output due to the subtractive mass term since as we have already mentioned, the mass function \( M(u) \) is a nonincreasing function of the retarded null coordinate \( u \).

As it was pointed out in the previous section several spacetimes are limiting cases of gravitational background described by the metric (17). Therefore, it would be interesting to derive the effective mass and the total output for these spacetimes by taking the appropriate limits. Some interesting spacetimes that are reduced as limiting cases of the metric (17) are described below.

(a) Einstein’s universe

As it was mentioned to the previous section, metric (17) becomes the Einstein’s universe when the mass and the charge of the particle are set equal to zero, i.e.

\[ ds^2 = 2dudr + du^2 - R^2 \sin^2 \left( \frac{r}{R} \right) (d\theta^2 + \sin^2 \theta d\phi^2). \]

(29)

It is easily seen from equation (24) that the effective gravitational mass of Einstein’s universe is zero, i.e.

\[ M_{\text{eff}}^{\text{Einstein}} = 0 \]

(30)
which is exactly the same result that it was recently derived by Vargas [14] who employed the Einstein’s and Landau-Lifshitz’s energy-momentum complexes in the framework of teleparallel gravity. Additionally, the power output, as given by (28), of the Einstein’s universe is zero, i.e.

$$L^{Einstein} = 0 . \quad (31)$$

(b) Bonnor- Vaidya spacetime

The Bonnor-Vaidya spacetime metric (16) which describes a radiating charged particle in flat spacetime, can be obtained from the metric (17) when $R$ tends to infinity. It is evident from equation (24) that the effective gravitational mass of Bonnor-Vaidya spacetime is given by

$$M_{e_{eff}}^{B-V} = M(u) - \frac{4\pi Q^2(u)}{r} . \quad (32)$$

This is twice the effective mass computed by Chamorro and Virbhadra [15] who utilized the Einstein’s and Landau-Lifshitz’s energy-momentum complexes. Additionally, the power output, as given by (28), of the Bonnor-Vaidya spacetime is

$$L^{B-V} = -M_u(u) + \frac{8\pi Q(u)Q_u(u)}{r} . \quad (33)$$

(c) Vaidya spacetime

When the electric charge of the particle is zero, i.e. $Q(u) = 0$, and $R$ tends to infinity, metric (17) reduces to the Vaidya radiating-star metric, i.e.

$$ds^2 = 2dudr + \left( 1 - \frac{2M(u)}{r} \right) du^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) . \quad (34)$$

It is evident from equation (24) that the effective gravitational mass of Vaidya spacetime is given by

$$M_{e_{eff}}^{Vaidya} = M(u) . \quad (35)$$

This is exactly the same result computed by Lindquist, Schwartz and Misner [17] who utilized the Landau-Lifshitz’s energy-momentum complex.

The power output, as given by (28), of the Vaidya spacetime takes the form

$$L^{Vaidya} = -M_u(u) . \quad (36)$$

(d) Reissner-Nordström black hole solution

When the mass and the electric charge of the particle are constants, and $R$ tends to infinity, metric (17) reduces to the Reissner-Nordström black hole metric, i.e.

$$ds^2 = 2dudr + \left( 1 - \frac{2M}{r} + \frac{4\pi Q^2}{r^2} \right) du^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) . \quad (37)$$

\[^4\text{For a short discussion on this discrepancy (the anomalous factor 2) see [16].}\]
It is clear from equation (24) that the effective gravitational mass of Reissner-Nordström black hole is given by

\[ M_{\text{eff}}^{R-N} = M - \frac{4\pi Q^2}{r} . \] (38)

This result agrees with the results computed by de la Cruz and Israel [18] who used junction conditions across thin shells, by Cohen and Gautreau [19] who implemented the Whittaker’s theorem, by Cohen and de Felice [20] who performed the evaluation using Komar’s integral for energy, and by others [21]. The power output, as given by (28), of the Reissner-Nordström black hole is

\[ L^{R-N} = 0 . \] (39)

Condition (25) that has to be fulfilled in order the particle to experience a repulsive effect of gravity, for the case of Reissner-Nordström black hole, takes the form

\[ r < \frac{4\pi Q^2}{M} . \] (40)

5 Conclusions

In this work, we explicitly calculate the energy and momentum density distributions associated with a metric that describes a radiating charged particle in Einstein’s universe. Additionally, the effective gravitational mass, i.e. the total energy, and the power output, i.e. the luminosity, of the specific gravitational background are explicitly evaluated. The concept of effective gravitational mass is related to the repulsive effects of gravitation. A condition is given which when satisfied a particle experiences a negative mass which acts repulsive. The luminosity and consequently the power output must be positive. Although the mass term in the expression for the power output is subtractive we have set the mass function to be a nonincreasing function of the retarded null coordinate and therefore to guarantee the positivity of power output. Since several interesting spacetimes are limiting cases of the gravitational background under study, we have derived the effective mass and power output for these spacetimes as limiting quantities. These results agree with the corresponding ones in the literature which were computed by using different prescriptions and/or methods. It is obvious that our results presented here provide evidence in support of Lessner statement for the significance of Möller’s prescription.
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References


