The problem of local distinguishability of quantum states shared by distant parties has attracting considerable attentions recently. A number of interesting and often counterintuitive results have been obtained. It should be clear that orthogonal quantum states can be distinguished, while non-orthogonal states can only be distinguished probabilistically if there are no restrictions to measurements. If the quantum states are shared by two distant parties, say Alice and Bob, and only LOCC is allowed, the possibility of distinguishing these quantum states will decrease. Interestingly, Walgate et al showed that any two orthogonal pure states shared by Alice and Bob can be distinguished by LOCC [1,2]. On the other hand, there are a set of orthogonal bipartite pure product states cannot be distinguished with certainty by LOCC [3,4].

In this Letter, we will show two main results in the following. First, we will show that a set of maximally entangled states in the standard form can be discriminated by local projective measurements and classical communications. Secondly, using the property of entanglement breaking channel, we will show a set of quantum states are locally indistinguishable.

Let’s first introduce some notations. We consider the dimension of the Hilbert space is \(d\) which is prime. \(U_{m,n} = X^m Z^n, m = 0, \ldots, d - 1\) are generalized Pauli matrices constituting an basis of unitary operators, and \(X|j\rangle = |j + 1\mod d\rangle, Z|j\rangle = \omega^j|j\rangle, \omega = e^{2\pi i/d}\) is an orthonormal basis. \(|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_j |jj\rangle, |\Phi_{m,n}\rangle = (U_{m,n} \otimes I)|\Phi^+\rangle\) is a basis of maximally entangled states.

Walgate et al once showed that two Bell states can be distinguished by LOCC (their result is for the case of arbitrary two orthogonal states) [1]. On the other hand three Bell states are locally distinguishable probabilistically, and four Bell states are locally indistinguishable no matter whether the protocol is deterministic or probabilistic [5–7]. One straightforward question is what is the maximal set of quantum states which are locally distinguishable. In particular, we are interested in the following problem: suppose \((U_{m,n} \otimes I)|\Psi^{AB}\rangle_{m,n=0}^{d-1}\) is a complete set of maximally entangled states in \(d \otimes d\) system, are any \(d\) maximally entangled states from this set locally distinguishable? This set is the best-known complete set of maximally entangled states. It is obvious that if we let \(|\Phi^{AB}\rangle = |\Phi^+\rangle^{AB} = \sum_j |jj\rangle\), here we omit a normalized factor, then for arbitrary \(n_i, d\) maximally entangled states \((X^i Z^{n_i} \otimes I)|\Psi^{AB}_{i=0}^{d-1}\) are locally distinguishable by simply projecting measurements in the computational basis on both sides, and subsequently by a classical communication. For the general case, we do not yet have a complete answer to this question. However, we can obtain a rather general result:

**Theorem:** Any \(l\) maximally entangled states from the set \((U_{m,n} \otimes I)|\Psi^{AB}\rangle_{m,n=0}^{d-1}\) can be distinguished by LOCC if \(l(l - 1)/2 \leq d\).

For example, if \(d = 2\), then any two \((l = 2)\) Bell states are locally distinguishable. If \(d = 3\) or \(d = 5\), then any three maximally entangled states from this set are locally distinguishable. An arbitrary maximally entangled state can always be transformed to \(|\Phi^+\rangle^{AB} = \sum_j |jj\rangle\) by a local unitary operation on one side (A or B, the difference between the unitary operations on A and B is a transposition). So, we just need to prove our claim in the case \(|\Psi^{AB}\rangle = |\Phi^+\rangle^{AB}\).

Let’s suppose these \(l\) maximally entangled states take the form \((X^i Z^{n_i} \otimes I)|\Phi^+\rangle^{AB}_{i=0}^{l-1}\). To locally distinguish these states, we first let A and B do unitary operations \(U\) and \(V^t\), respectively, where \(t\) is a transposition. This operation is equivalent to the transformation \(U(X^m Z^n)\) on A side. We next will show that we can find these unitary operators which can transform these \(l\) maximally entangled states to the set \((X^{m_i} Z^{n_i} \otimes I)|\Phi^+\rangle^{AB}_{i=0}^{l-1}\) where there are no equal \(m_i\). As we mentioned that this set can be simply distinguished by LOCC. Thus we can prove our previous claim. We remark that unitary operations \(U\) and \(V^t\) on A,B sides followed by a projective measurements in the computational basis is equivalent to projective measurements on A,B sides in two basis corresponding to \(U\) and \(V^t\).

As we analyzed, the problem of local distinguishability now becomes whether we can find two unitary operations \(U\) and \(V\) which transform \((X^m Z^n)_{i=0}^{l-1}\) to the set \((X^{m_i} Z^{n_i})_{i=0}^{l-1}\) in which no \(m_i\) are equal. We next will give these
unitary operations. The case of \( d = 2 \) is trivial, with the help of the Hadamard transformation \( H_0 \), we can always discriminate two Bell states by \( Z \) basis measurements on both sides. In the following, we suppose \( d \neq 2 \). We define \( d \) unitary operators \( H_\alpha \), \((\alpha = 0, 1, \cdots, d-1)\) like this, the entries of matrices \( H_\alpha \) take the form

\[
(H_\alpha)_{jk} = \omega^{-jk} \omega^{-\alpha s_k}, \quad j, k = 0, \cdots, d-1, \quad s_k = k + (k+1) + \cdots + (d-1).
\]

By using \( H_\alpha \), we have the following relations

\[
\begin{align*}
H_\alpha X H_\alpha^\dagger &= Z^{-1} X^\alpha, \\
H_\alpha Z H_\alpha^\dagger &= X.
\end{align*}
\]

Thus \( H_\alpha \) can transforms \( U_{m, n} \) as follows,

\[
H_\alpha X^{m_i} Z^{n_i} H_\alpha^\dagger = X^{m_i, \alpha+n_i} Z^{-m_i}
\]

up to a whole phase. Given \( l \) maximally entangled states corresponding to \( \{X^{m_i} Z^{n_i}\}_{i=0}^{l-1} \), we can always transform them to the case where the powers of \( X \) are different by identity (do nothing) or \( H_\alpha, \alpha = 0, \cdots, d-1 \). If that means for each transformation at least two powers of \( X \) are equal. So we have at least \( d+1 \) equations altogether. But different combinations between \( l \) elements \( \{(m_i, n_i)\}_{i=0}^{l-1} \) is \( \binom{l}{2} = l(l-1)/2 \) which is less than or equal to \( d \).

That means two pairs, for example, \((m_0, n_0)\) and \((m_1, n_1)\) without loss of generality will appear twice in two different transformations, say \( \alpha_0 \) and \( \alpha_1 \). Thus we should have the following relations,

\[
\begin{align*}
\alpha_0 m_0 + n_0 &= \alpha_0 m_1 + n_1, \quad (\text{mod} \, d) \\
\alpha_1 m_0 + n_0 &= \alpha_1 m_1 + n_1, \quad (\text{mod} \, d)
\end{align*}
\]

That means \((m_0, n_0) = (m_1, n_1)\) which contradicts with our assumption that these \( l \) maximally entangled states are orthogonal. This completes our proof.

We next clarify our proof in the case \( d = 3 \). Explicitly, the three operators \( H_\alpha \) take the following form

\[
H_0 = \begin{pmatrix} 1 & 1 & 1 \\
1 & \omega^2 & 1 \\
1 & \omega & \omega^2 \end{pmatrix},
H_1 = \begin{pmatrix} 1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & 1 \end{pmatrix},
H_2 = \begin{pmatrix} 1 & 1 & \omega^2 \\
1 & 1 & \omega \\
1 & \omega & \omega \end{pmatrix}.
\]

Given three maximally entangled states corresponding to \( \{X^{m_i} Z^{n_i}\}_{i=0}^2 \), if \( \{m_i\}_i^2 = \{0, 1, 2\} \), it is obvious that they are distinguishable by LOCC. If \( \{n_i\}_i^2 = \{0, 1, 2\} \), by transformation \( H_0 \), they can be distinguished locally. The left unsolved cases have the form \( \{(m_0, n_0), (m_0, n_1), (m_2, n_0)\} \) where \( n_0 \neq n_1, m_0 \neq m_2 \). This form can neither be locally distinguished by direct measurements in the computational basis nor can be distinguished by \( H_0 \) followed by measurements in the computational basis. But \( H_1 \) or \( H_2 \) will transfer the power of \( X \) to the set \( \{0, 1, 2\} \). If not that means

\[
\begin{align*}
m_0 + n_1 &= m_2 + n_0, \quad \text{mod} \, 3.
2m_0 + n_1 &= 2m_2 + n_0, \quad \text{mod} \, 3.
\end{align*}
\]

Then we know \( m_0 = m_2, n_0 = n_1 \) which contradict with our assumption. So, any 3 maximally entangled states from the set \( \{(U_{m,n} \otimes I)|\Psi\rangle^{AB}\}_{m,n=0}^2 \) can be distinguished by LOCC.

We give some examples to show our local discrimination method. We have three maximally entangled states \(|00\rangle + \omega|11\rangle + \omega^2|22\rangle, |00\rangle + \omega^2|11\rangle + \omega|22\rangle \) and \(|10\rangle + |21\rangle + |02\rangle \), corresponding to set \( \{Z, Z^2, X\} \). They cannot be discriminated directly by measurements in computational basis. By transformation \( H_0 \), these three maximally entangled states can be transformed as \( \{X, X^2, Z^2\} \) corresponding to \( |10\rangle + |21\rangle + |02\rangle, |20\rangle + |01\rangle + |12\rangle, |00\rangle + \omega^2|11\rangle + \omega|22\rangle \) which
can be discriminated by projective measurements in the computational basis followed by a classical communication. More explicit, if the powers of $Z$ are different in the given set, we can discriminate them by transformation $H_0$ which is essentially Hadamard transformation. Suppose the given set is $\{X, Z, XZ\}$ corresponding to $|10\rangle + |21\rangle + |02\rangle, |00\rangle + \omega|11\rangle + \omega^2|22\rangle, |10\rangle + \omega|21\rangle + \omega^3|02\rangle$ which cannot be discriminated by transformations $H_0$ and $H_1$. By transformation $H_2$, we have $\{XZ^2, X^2, Z^2\}$ corresponding to $|10\rangle + \omega^2|21\rangle + \omega|02\rangle, |20\rangle + |01\rangle + |12\rangle$ and $|00\rangle + \omega^2|11\rangle + \omega|22\rangle$ which can be simply discriminated locally. The whole procedure is like the following, A and B do unitary transformations $H_2$ and $H_2$ so that the three maximally entangled states corresponding to $\{XZ^2, X^2, Z^2\}$. Then A and B do measurements in the computational basis and subsequently by a classical communication can discriminate these three maximally entangled states.

In general $d$ case, $d$ independent transformations $H_\alpha$ is not enough to locally distinguish arbitrary $d$ maximally entangled states in the set $\{(X^{m_i}Z^{n_i}I)|\Phi^{+}\rangle^{AB}\}_{i=0}^{d-1}$. So, we need to find other transformations. Here we remark that any transformation which changes the power of $X$ to $jm_i + kn_i, j, k = 0, \cdots, d-1$ cannot provide new transformations different from identity and these $d$ transformations $H_\alpha$ which change the power of $X$ as $\alpha m_i + n_i$ as shown in (3).

Combine the result in this Letter for 3-dimension and the fact that two Bell states can be distinguished locally, we can generalize our result to $(2 \otimes 2)^{\otimes M} \otimes (3 \otimes 3)^{\otimes N}$ case. Suppose $|\Psi\rangle^{AB}$ is a maximally entangled state in $2^M3^N \otimes 2^M3^N$ system, then $2^M3^N$ maximally entangled states from the set $\{(U_{m,n} \otimes I)|\Psi\rangle^{AB}\}$ can be distinguished by LOCC, where $U_{m,n}$ are tensor product of identity and Pauli matrices in 2 and 3-dimensions. Certainly, similar result based on our theorem for general $\prod_{i=0}^{d-1}(d_i \otimes d_i)$ case can also be presented.

Horodecki et al showed that an arbitrary complete set of orthogonal states of any bipartite system is locally indistinguishable if at least one of the vectors is entangled [7]. Next we will show the following result: An ensemble of linearly independent quantum states $\{\rho_{m,n}\}_{m,n=0}^{d-1}$ cannot be discriminated deterministically or probabilistically by LOCC, where $\rho_{m,n} = (U_{m,n} \otimes I)\rho^{AB}(U_{m,n} \otimes I)$. We remark that the quantum states of this ensemble are generally mixed states. And this set may includes both orthogonal and non-orthogonal quantum states.

We say a quantum channel $\Lambda$ is entanglement breaking if for all input states, the output states of the channel $\Lambda \otimes I$ are separable states. We define a quantum channel $\Lambda^{AC}$ as follows

$$\Lambda^{AC}(\rho^{AC}) = \sum_{m,n} U_{m,n} \otimes U_{m,n} (\rho^{AC} U_{m,n}^\dagger \otimes U_{m,n}^\dagger). \tag{7}$$

Next, we will prove that this quantum channel is entanglement breaking. To prove that a quantum channel $\Lambda$ is entanglement breaking, it is enough to show that $\Lambda \otimes I$ maps a maximally entanglement state into a separable state [8–10]. Considering that the quantum state $|\Phi^{AB}\rangle \otimes |\Phi^{CD}\rangle$ of four systems $H_A \otimes H_B \otimes H_C \otimes H_D$ is a maximally entangled state across $AC : BD$ cut, we should show that

$$\Lambda^{AC} \otimes I^{BD}(|\Phi^{AB}\rangle \otimes |\Phi^{CD}\rangle) = \frac{1}{d^2} \sum_{m,n} |\Phi^{AB}_{m,n}\rangle \otimes |\Phi^{CD}_{m,n}\rangle |\Phi^{CD}_{m,n}\rangle \langle |\Phi^{CD}_{m,n}\rangle |$$

is a separable state. Actually, we have the following symmetry,

$$\frac{1}{d^2} \sum_{m,n} |\Phi^{AB}_{k,l}\rangle \otimes |\Phi^{CD}_{k,l}\rangle \langle |\Phi^{CD}_{k,l}\rangle |$$

is a separable state. Actually, we have the following symmetry,

$$\frac{1}{d^2} \sum_{m,n} |\Phi^{AC}_{k,l}\rangle \otimes |\Phi^{BD}_{k,l}\rangle \langle |\Phi^{BD}_{k,l}\rangle |$$

It is obvious that this is a separable state across $AC : BD$ cut. Thus we show that $\Lambda^{AC}$ defined in Eq.(7) is an entanglement breaking channel. Eq.(9) can be proved like the following, we substitute the following relation,

$$|\Phi_{0,0}\rangle \otimes |\Phi_{0,0}\rangle = \frac{1}{d} \sum_{m,n} |\Phi_{m,n}^{AC}\rangle \otimes |\Phi_{m,n}^{BD}\rangle. \tag{10}$$

into $\Lambda^{AC} \otimes I^{BD}(|\Phi_{0,0}^{AB}\rangle \otimes |\Phi_{0,0}^{CD}\rangle)$. With the help of the relation $U_{m,n}U_{k,l} = \omega^{nk-ml}U_{k,l}U_{m,n}$, and also we know $|\Phi_{0,0}\rangle$ is invariant under the action of $U_{m,n} \otimes U_{m,n}$, one can readily show Eq.(9). We remark that the quantum state (9) is the so-called unlockable bound entangled state in $d$-dimension [11].

Now we are ready for our result of local indistinguishability. Given the set of linearly independent states $\{\rho_{m,n}\}_{m,n=0}^{d-1}$ to be discriminated, we can construct a quantum state

$\omega$
\[ \rho = \frac{1}{d^2} \sum_{m,n} \rho_{m,n}^{AB} \otimes |\Phi^{CD}_{m,-n}\rangle \langle \Phi^{CD}_{m,-n}|, \]
\[ = \Lambda^{AC} \otimes I^{BD}(\rho^{AB} \otimes \Phi^{CD}). \] (11)

Here the maximally entangled states \(\{|\Phi^{CD}_{m,-n}\rangle\}\) act as detectors. Since we know that the quantum channel \(\Lambda^{AC}\) is entanglement breaking, so this mixed state \(\rho\) is a separable state across \(AC\) : \(BD\) cut. Thus we can show that: A set of linearly independent quantum states \(\{\rho_{m,n}^{AB}\}_{m,n=0}^{d-1}\) cannot be distinguished deterministically or probabilistically by LOCC [12]. Because if they can be distinguished deterministically or probabilistically, one could distill non-zero entanglement by LOCC. This contradicts with the observation that \(\rho_{m,n}^{AB}\) is not separable state across \(AC\) : \(BD\) cut. Note that \(\rho_{m,n}^{AB}\) is in \(d \otimes d'\) system, and \(d, d'\) are not necessarily the same. We also should point out that if \(\rho^{AB} = |\Psi^{AB}\rangle \langle \Psi^{AB}|\) which is a pure state, \(\{\rho_{m,n}^{AB}\}_{m,n=0}^{d-1}\) are not necessarily orthogonal to each other, where we denote \(|\Psi_{m,n}\rangle = U_{m,n} \otimes I(\Psi)\). So, this case is not covered by the result in Ref. [6,7]. Certainly, distinguishability of non-orthogonal states is less than that of orthogonal states, but still they can be distinguished probabilistically by global measurements and for some cases by LOCC [13,14]. We will not discuss the case that \(\{\rho_{m,n}^{AB}\}_{m,n=0}^{d-1}\) are linearly dependent.

We next give three examples:

Example 1: According to our result, an ensemble of states \(|\Psi_{0,0}\rangle = \alpha|00\rangle + \beta|11\rangle\), \(|\Psi_{0,1}\rangle = \alpha|00\rangle - \beta|11\rangle\), \(|\Psi_{1,1}\rangle = \alpha|10\rangle + \beta|01\rangle\) and \(|\Psi_{1,0}\rangle = \alpha|10\rangle - \beta|01\rangle\) cannot be distinguished by LOCC [15]. Here we do not consider the special cases such as \(\alpha \beta = 0\) which lead to result that the quantum states of this ensemble are linearly dependent. One can find that \(|\Psi_{0,0}\rangle\) and \(|\Psi_{0,1}\rangle\) are generally non-orthogonal, while they are orthogonal with \(|\Psi_{1,0}\rangle, |\Psi_{1,1}\rangle\). So, this ensemble consists of both orthogonal and non-orthogonal states. And this case is not studied previously. As a special case, we can show that four Bell states cannot be distinguished by LOCC which has already been pointed out in Ref. [5].

Example 2: We can choose a quantum state in \(|\Psi^{AB}\rangle\) in \(2 \otimes d'\) system, say let \(d' = 4\). For example, let \(|\Psi^{AB}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |12\rangle + |13\rangle)\), and we have four orthogonal states \(\{U_{m,n} \otimes I)|\Psi^{AB}\rangle\}_{m,n=0}^{4}\). According to our criterion, they cannot be distinguished by LOCC. Horodecki et al. once showed that an arbitrary complete set of orthogonal states in bipartite system cannot be distinguished by LOCC if at least one of the states is entangled, deterministically or probabilistically [7]. For example, four Bell states cannot be distinguished by LOCC [6]. On the other hand, three Bell states which is incomplete can be distinguished probabilistically, and two Bell states can be distinguished deterministically [1]. An interesting question is whether there exist incomplete sets of orthogonal states which cannot be distinguished even probabilistically. Here we present an example to show that there exist an incomplete set of orthogonal states which cannot be distinguished by LOCC no matter whether the protocol is deterministic or probabilistic. Certainly, we can also give an example of non-orthogonal states with the same property. In Ref. [7], Horodecki et al. presented an example of incomplete set of orthogonal states which is indistinguishable by LOCC deterministically. However, it is still possible that this set is local indistinguishable probabilistically. One may point out that these four Bell states are essentially four Bell states. It’s true. But our conclusion is not trivial. In general for a bipartite system \(d \otimes d'\), there exist \(d^2\) orthogonal states which cannot be distinguished by LOCC [16], even probabilistically. These \(d^2\) orthogonal states are not a complete set if \(d \neq d'\).

Example 3: Our result is generally for mixed states. For qubits case, let \(\rho^{AB} = \frac{d-1}{4d^2} |\Phi^+\rangle \langle \Phi^+| + \frac{d-3}{4d^2} I\) be the Werner state. Then we know four different Werner states set \(\{U_{m,n} \otimes I)|\Phi^{AB}\rangle\}_{m,n=0}^{4}\) are locally indistinguishable, where \(p \neq 1/4\).

We can generalize the previous result to states in \(2^N \otimes 2^N\) case. It is straightforward to show that \(\{|\Psi_{m_1,n_1}\rangle \otimes |\Psi_{m_2,n_2}\rangle \otimes \cdots \otimes |\Psi_{m_N,n_N}\rangle\}_{m_1,n_1=0}^{1}\) are indistinguishable by LOCC across \(A_1 \cdots A_N\) : \(B_1 \cdots B_N\) cut irrespective the protocol is deterministic or probabilistic, where \(|\Psi_{m_1,n_1}\rangle = (U_{m_1,n_1} \otimes I)(\alpha_1|00\rangle + \beta_1|11\rangle)\), \(\alpha_1\beta_1 \neq 0\), and \(m_1,n_1 = 0,1\).

Similarly, we can study a more general case of \(\prod_{i=1}^{N}(d_i \otimes d_i')\) system. We define the quantum channel \(\Lambda^{AC}\), in Hilbert space \(\mathcal{H}_A \otimes \mathcal{H}_C\), where \(\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_N}\), similarly for \(\mathcal{H}_C\). We can find that the quantum channel defined as

\[ \Lambda^{AC}(\rho^{AC}) = \sum_{\bar{m},\bar{n}} U_{\bar{m},\bar{n}} \otimes U_{\bar{m},-\bar{n}} \otimes (\rho^{AC} U_{\bar{m},\bar{n}}^\dagger \otimes U_{\bar{m},-\bar{n}}^\dagger). \] (12)
Horodecki et al. also proposed a method to construct a pure quantum state by the superposition rather than the mixture [7]. Then by Jonathan-Plenio criterion [17] based on majorization scheme [18,19], one can check whether the given quantum states can be distinguished or not if only LOCC is allowed. Generally, this method relies on some numerical search which may be complicated. Chefles recently showed a necessary and sufficient condition for LOCC unambiguous state discrimination [20]. In this Letter, we develop the method of constructing a mixed state [21,5–7], then by the definition of entanglement breaking channel to show a family of states are indistinguishable by LOCC, deterministically or probabilistically.

We show that the quantum channel defined in (7) is an entanglement breaking channel, thus lead to some interesting results. The method to correspond the entanglement breaking channel with the indistinguishability by LOCC is a rather powerful method. Assume that the following quantum channel is entanglement breaking

$$
\Lambda^{AC}(\rho^{AC}) = \sum_i A_i \otimes C_i (\rho^{AC}) A_i^\dagger \otimes C_i^\dagger.
$$

And suppose that the set of quantum states \{(A_i \otimes I)\rho^{AB}(A_i^\dagger \otimes I)\} with normalization to be distinguished are linearly independent, and we assume that not all detectors \{(C_i \otimes I)\rho^{CD}(C_i^\dagger \otimes I)\} are separable states. Here the detector \rho^{CD} should be an entangled state but not necessarily a maximally entangled state. With the input state of the channel taking \rho^{AB} \otimes \rho^{CD}, we know the output state is a separable state across AC : BD cut. Thus we know this set of states \{(A_i \otimes I)\rho^{AB}(A_i^\dagger \otimes I)\} cannot be distinguished by LOCC [15].

In summary, we proposed a family of unitary transformations \{H_\alpha\}_{d-1}^{d-1} in (1). By projective measurements corresponding to these transformations, we can locally discriminate any l maximally entangled states choosed from the set \{(U_{m,n} \otimes I)|\Psi\rangle^{AB}\}_{d-1}^{d-1} if \(l(l-1) \leq 2d\). And from the property of entanglement breaking, we show that a family of quantum states are indistinguishable by LOCC.

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[12] In this Letter, we only consider the case that one copy of the quantum state is provided.
[15] We say that the quantum states cannot be distinguished by LOCC means that they cannot be distinguished by LOCC deterministically or probabilistically.
[16] Without lose of generality, we assume \(d' \geq d\).