A Vector Hysteresis Model for Superconducting Filament Magnetization in Accelerator Magnets

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Abstract—Magnetic field changes in the coils of superconducting magnets are shielded from the filaments' core by so-called persistent currents which can be modeled by means of the critical state model. This paper presents a semianalytical two-dimensional model of the filament magnetization due to persistent currents for changes of the magnitude of the magnetic induction and its direction while taking the field dependence of the critical current density into account. The model is combined with numerical field computation for the calculation of field errors in superconducting magnets. The filament magnetization and the field errors in a nested orbit corrector magnet for the large hadron collider project at CERN have been calculated as an example.

Index Terms—Critical state model, hysteresis, magnetization, persistent currents, superconductivity.

I. INTRODUCTION

THE large hadron collider (LHC) will consist of about 8400 superconducting (SC) magnet units of different types. The magnetic field (which is ramped between injection and nominal level for the accelerator) induces currents in the superconducting filaments of the coils. These so-called persistent currents screen the external field change. The filaments (in the range of 6–7 μm in diameter) are made of a NbTi alloy embedded in a copper matrix. Strong pinning that makes type II (hard) superconductors useful in high field magnet applications is responsible for magnetic hysteresis and thus for field-dependent multipole errors of the magnetic field. Although the pinning centers prevent flux flow in hard superconductors, some small *flux creep* effect remains, which reveals an exponentially varying resistivity when the current approaches its critical value

$$\vec{E}(\vec{J}) = E_c \left(\frac{|\vec{J}|}{J_c} \right)^{n-1} \frac{\vec{J}}{J_c}. \quad (1)$$

The critical current density is usually defined as the point where the electrical field is below 1 $\mu\text{V}/\text{cm}$. Note that the critical current density and the n -value is a function of the applied magnetic field. From (1) and expressing the conductivity in terms of the electric field, a diffusion equation can be derived and numerically solved in particular for low n -values, i.e., strong flux creep in high-temperature superconductors. In [4], an integral equation method is used; [10] presents a time-stepping finite-difference scheme which is also used in [6] considering also an anisotropic and field-dependent critical current density. Mayergoyz presents in [7] the method and experimental testing

of a vector-Preisach hysteresis model for superconductors. All methods require, however, the meshing of the field domain down to the superconducting filament level which is prohibitive for the LHC superconducting magnets with filament sizes in the micrometer range. We therefore propose a semianalytical model based on the *critical state model* (CSM) [2] which can be combined with the coupling method of finite elements and boundary elements (BEM–FEM) [5], which does not require the meshing of the coil, air, and austenitic steel areas in the magnet.

As the resistivity of hard superconductors is almost a step function (1), the CSM postulates that the current density in the superconductor is either zero or equals the critical current density. A hard superconductor tries to expel a varying external field by generating a bipolar current distribution of the critical density J_c , which depends on the local field level and the temperature. Macroscopically, these currents are the source of a magnetization $\vec{M}(\vec{B})$ in the strands. The limitations of the CSM stem from ignoring the Meissner phase (very small in type II superconductors), the idealization of the electrical field versus current $\vec{E}(\vec{J})$ relation as a step function (justified in case of the LHC wires with typical n -values of 50) and from the fact that the explicit solution of the Maxwell equations is only possible for simple shapes of the superconductor (round filaments in our case). In particular, it cannot explain the logarithmic time dependence (called decay) that is observed in the magnetization effects.

Although the critical current density decreases with field in all real superconductors, the original CSM model by Bean [2] assumes a field-independent critical current density to simplify the mathematical treatment of the magnetization problem. Our model differs from the vector-hysteresis model presented in [8] as it takes the field dependence of the critical current density into account and it allows the calculation of the feedback of the persistent currents on the field distribution within the coil. As a consequence, also low field effects such as the peak-shifting (asymmetry in the magnetization curve for vanishing external field), magnetizations due to minor loops and rotating external fields are reproduced by the model. The one-dimensional (1-D) model introduced in [1] (where the field remained at constant direction yielding first-order reversal curves) has been extended to account for filament magnetizations due to magnetic fields changing in magnitude and direction and thus constitutes an isotropic vector hysteresis model.

II. VECTOR HYSTERESIS MODEL

The CSM states that field changes on the surface of SC filaments induce layers of shielding currents of critical density $J_c(B)$ (B is the magnitude of the penetrated magnetic induction

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inside the filament). As in [1], the induction shielded by such layers is modeled by the integral over intersecting circles, each carrying currents of critical density $J_c(B)$ of opposite direction (cf. Fig. 2). It can be proved that intersecting circles of opposite constant current density create a homogeneous field inside their overlap area. The relative penetration q parameterizes linearly the filament cross section, assuming values between zero (on the outer radius) and one in the center of the filament. Thus, the magnitude of the shielding induction vector $\Delta\vec{T}$, generated by a layer between q_1 and q_2 , is given by

$$|\Delta\vec{T}| = \frac{\mu_0 r}{2} \int_{q_1}^{q_2} J_c(B(q)) dq \quad (2)$$

where r is the filament radius.

Let $\vec{B}_{\text{out}}$ denote the external magnetic field at a given excitation step, and $\vec{B}(q) = \vec{B}_{\text{out}} + \vec{T}(q)$ the course of the magnetic induction inside the filament. With $|\vec{T}| = T$ and $q_1 = 0$, the differential of the shielding reads

$$dT(q) = \frac{\mu_0 r}{2} J_c(B(q)) dq \quad (3)$$

derived from (2) for $(q_2 - q_1) \rightarrow 0$. The mathematical treatment of the problem consists of the following.

- 1) The calculation of the shielding induction $\vec{T}(q)$ for a given history of excitations at the n th excitation step.
- 2) The determination of the penetration depth q^* of a new shielding current layer, from the continuity requirements

$$\vec{B}(0) = \vec{B}_{\text{new}} + \underbrace{\vec{T}_{\text{new}}(0)}_0 = \vec{B}_{\text{new}} \quad (4)$$

$$\vec{B}(q^*) = \vec{B}_{\text{new}} + \vec{T}_{\text{new}}(q^*) = \vec{B}_{\text{old}} + \vec{T}_{\text{old}}(q^*) \quad (5)$$

where the subscript n is the index of the excitation step.

- 3) Calculation of the resulting magnetization from a set of shielding current layers.

For the determination of the penetration depth, J_c needs to be approximated around the working point. As described in detail in [1], for small magnetic inductions $B(q) \rightarrow 0$, where the persistent currents have the biggest impact, the critical current density approaches infinity with $J_c(B(q)) \propto B(q)^{-0.43} \simeq 1/\sqrt{B(q)}$. For the computation of the induction inside the filament, J_c is thus approximated around the working point $B_{\text{out}} = |\vec{B}_{\text{out}}|$ with the following function (the temperature is constant):

$$J_c(B(q)) \simeq J_c(B_{\text{out}}) \frac{\sqrt{B_{\text{out}}}}{\sqrt{B(q)}} \equiv \frac{\mathcal{F}(B_{\text{out}})}{\sqrt{B(q)}} \quad (6)$$

where $J_c(B_{\text{out}})$ is the evaluation of the J_c fit function [3] in the working point. With this approximation, the differential equation (3) for $T(q)$ reads

$$dT \sqrt{B_{\text{out}}^2 - 2B_{\text{out}}T(q)\cos(\alpha) + T(q)^2} = \mu_0 \mathcal{F}(B_{\text{out}}) \mathcal{H} dq \quad (7)$$

where the geometrical factor $\mathcal{H} = 2r(1 - \ln 2) = 0.614r$ is explained in [1] and the angle α between the external field and

Fig. 1. SC filament exposed to a 1-D field change. In both cases, the field change was preceded by an up-ramp of magnetic induction on the outside. The varying current density in the cross section is indicated by the shading of the individual layers. For the continuous model, the thickness of these layers approaches zero.

Fig. 2. As Fig. 1 but with a 2-D field change applied to the SC filament.

the shielding vector is illustrated in Fig. 2. A solution is found for the inverse relation [11]

$$q(B_{\text{out}}, T, \alpha) = \frac{1}{3\mu_0 \mathcal{F}(B_{\text{out}}) \mathcal{H}} \left[B_{\text{out}} \cos(\alpha) \left(2\sqrt{B_{\text{out}}} + \sqrt{B_{\text{out}} |\sin(\alpha)|} {}_2F_1 \left(\frac{1}{2}, \frac{3}{4}, \frac{3}{2}, -\cot^2 \alpha \right) \right) + (T - B_{\text{out}} \cos \alpha) \left(2(B_{\text{out}}^2 + T^2 - 2B_{\text{out}}T \cos \alpha)^{1/4} + \sqrt{B_{\text{out}} |\sin \alpha|} {}_2F_1 \left(\frac{1}{2}, \frac{3}{4}, \frac{3}{2}, -\frac{(T - B_{\text{out}} \cos \alpha)^2}{\sin^2 \alpha B_{\text{out}}^2} \right) \right) \right] \quad (8)$$

where ${}_2F_1$ denotes the Gauss' hypergeometric function. The function for $T(B_{\text{out}}, \alpha, q)$ is obtained by a Newton algorithm.

The determination of the relative penetration depth q^* of the new shielding current layer from the continuity requirements (4)–(5) and from geometric considerations yields an equation system that is solved by a 3-D Newton algorithm and yields α_{new} , $T_{\text{new}}(q^*)$, and $T_{\text{old}}(q^*)$. With (7) and (8), the distribution

of the magnetic induction in a fully penetrated filament with two screening current layers yields

$$\vec{B}(q) = \begin{cases} \vec{B}_{\text{new}} + \vec{T}_{\text{new}}(B_{\text{new}}, \alpha_{\text{new}}, q), & 0 \leq q < q^* \\ \vec{B}_{\text{old}} + \vec{T}_{\text{old}}(B_{\text{old}}, \alpha_{\text{old}}, q), & q^* \leq q < 1. \end{cases} \quad (9)$$

The magnetization of *one* shielding current layer between the penetration depths q_i and q_{i+1} is given by

$$\vec{M}_i = \frac{4r}{\pi} \int_{q_i}^{q_{i+1}} J_c(B(q))(1-q)^2 \vec{e}_{T_i} dq \quad (10)$$

where $\vec{e}_{T_i}$ denotes the direction of the corresponding shielding vector $\vec{T}_i(q)$. The correction term $4/\pi$ accounts for the difference between two current leading half-circles and a filament filled by nested intersecting circles in the fully penetrated state. Fig. 2 illustrates the transition from a field change in only one direction, discussed in [1], to field changes in arbitrary directions.

III. SOFTWARE IMPLEMENTATION AND VALIDATION

Our hysteresis model requires the management of relevant data for the excitation loop. For every magnetization layer $B_{\text{out}}, \beta_{\text{out}}, \alpha$, and the layer limits q_1 and q_2 are required, compare Fig. 2. This data is stored in a first in/last out stack register.

To store data for a new layer, the following steps are carried out.

- 1) The input data of every new excitation step consists of B_{new} and β_{new} . Starting from the last, outermost current layer ($q_2 = 0$), the program searches for a penetration depth q^* by a Newton algorithm as described above. If q^* is found to fulfill $q_1 \geq q^* \geq q_2$, we call it a valid intersection. A new layer from q^* to the outer diameter is created and the old layer is restricted to $q_2 = q^*$. As for multiple roots, their mathematical investigation poses a problem due to the highly nonlinear nature of solution (8). A phenomenological approach based on the vectorial representation in Fig. 2 suggests that multiple valid intersections must be extremely rare. Moreover, the Newton algorithm searches for roots starting from low values $q^* \geq 0$, which is consistent with the fact that a new layer penetrates the filament core from the outside to the depth where the continuity requirements at the interlayer boundary are met. If no valid penetration depth is found in the outermost layer, the layer is erased and the next nested layer of shielding currents is investigated. This way, the penetrating field approaches the filament core.
- 2) Once a valid intersection point is found, the magnetization vectors of all layers are calculated from (10) and added up to a total filament magnetization vector. Only such layers are to be considered and kept in the history that yield a substantial contribution to the magnetization. For this purpose a minimum layer thickness is introduced, thus regulating the maximum number of individual layers.

As far as step sizes for the excitation loop are concerned, the assumptions of the CSM have an important consequence: The time constant of diffusion processes in the SC filament is zero and, thus, the shielding current layers build up instantaneously. Time

Fig. 3. In sequence (a)–(c), one initial state subscripted “old” is followed by excitation steps of three different sizes. In sequence (d)–(f), step (a) is followed by three successive small steps. Comparison of (c) and (f) demonstrates that letting the vector $\vec{T}_{\text{new}}$ point to the arrow head of $\vec{T}_{\text{old}}$ (in the case of total memory cancellation) yields the optimum linear approximation of a succession of small steps.

steps can thus be chosen in order to give a sufficiently accurate stepwise linearization of the excitation function, without worrying about transient effects in the material. However, the question arises, what α angle is to be attributed to a new shielding current layer with a penetration such that all nested layers are erased. Fig. 3 illustrates the problem. In other words, if we assume an excitation step that yields a valid intersection with the innermost shielding current layer at $q^* = q_1 = 0$ and if this excitation step was replaced by a marginally larger one, the resulting magnetization is also to change only marginally. If the excitation step was larger, the resulting magnetization should yield the optimum approximation of a succession of small steps. These requirements impose that in the diagram the vector $\vec{T}_{\text{new}}$ points to the arrow head of $\vec{T}_{\text{old}}$. This approach also assures that the accuracy of the computed magnetization curve corresponds directly to the accuracy of the excitation stepping as illustrated in Fig. 4. The left-hand diagrams show a clockwise rotating excitation field on the strand level. As for the first excitation steps, the filament’s response (shown in the right diagram of Fig. 4) exhibits properties similar to a 1-D hysteresis loop: Up to point (a), the magnetization opposes the applied field. As the outer field further increases in magnitude, the magnetization decreases due to the field dependence of the critical current density. In (b), the filament is fully penetrated. The magnetization curve starts to follow the excitation loop lagging behind by a characteristic angle (c)–(e). In (f), the maximum of $|\vec{M}|$ is reached two steps after the excitation loop is completed at $|\vec{B}| = 0$, corresponding to the peak-shifting in the 1-D case [1]. Around point (g), the 2-D hysteresis loop closes.

IV. COMBINING THE MODEL WITH NUMERICAL FIELD CALCULATION

For the calculation of persistent current induced field errors in SC accelerator magnets with a ferromagnetic yoke, the magnetization model is combined with the BEM–FEM coupling method, which accounts for the local saturation effects in the yoke [5]. In order to calculate global shielding effects in the coil

Fig. 4. Rotational field excitation of one filament, scaled by the filling factor $\lambda(r = 3.5 \mu\text{m}, \lambda = 1/2.95, T = 1.9 \text{ K})$. Two complete turns have been computed. Left: The external field follows a circular, clockwise path that goes through the origin in the B_x/B_z -plane. Right: The filament's response in terms of magnetization; A virgin curve and a closed hysteresis slope can be identified. Up: 150 excitation steps per turn; Down: 20 excitation steps per turn.

Fig. 5. Left: MCBXA orbit corrector with multipole insert (display of field modulus in the iron yoke). Right: Amplitude of the sextupole field component (in tesla) as a function of the field direction in the aperture of the MCBXA magnet.

(in particular at low excitations), the feedback of the persistent currents on the excitational field is calculated by means of an $\vec{M}(\vec{B})$ -iteration on the strand level.

The MCBXA magnet is a combined beam orbit corrector, with a horizontal and a vertical dipole concentrically nested. In addition, a nested sextupole and dodecapole coil is inserted in the aperture of the dipoles; see Fig. 5 left. A somewhat academic excitational cycle is considered where first the outer layer coil is powered to a fifth of its nominal current; then the outer layer dipole follows a cosine and the inner layer dipole a sine-like excitation. This way, the resulting field rotates clockwise with about constant magnitude and most filaments are exposed to a rotating excitation field. This excitation cycle has been chosen for a later experimental validation of the model, as magnetization measurements of single strands for arbitrary transverse field and superfluid helium temperatures are difficult to obtain. The insert with the sextupole and dodecapole coil is not powered.

The field quality in the aperture of the magnet can be defined by the coefficients of the Fourier-series expansion of the radial

Fig. 6. The dipole coils of the MSCBXA are shown. Inside the aperture the excitation field is represented, whereas the arrows in the coil cross section denote the magnetization due to the preceding excitation cycle. Note that the magnetization does not oppose the exterior field but the change of the excitation field (clockwise rotation) in the position of the respective filaments.

field component at a given reference radius $r = r_0$ inside the aperture of the magnets, i.e.,

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi). \quad (11)$$

The magnet has been optimized geometrically so that the sextupole field component B_3 due to the transport currents is basically zero. Fig. 5 right shows the amplitude of the sextupole field component as a function of the field direction in the aperture. The amplitude is calculated by $C_3(r_0) = \sqrt{A_3^2 + B_3^2}$. The polar angle in the representation corresponds to the direction of the excitation field. Fig. 6 illustrates the behavior of SC filaments when exposed to a rotating field. The magnetization does not oppose the exterior field $\vec{B}$, as would be the result of a 1-D model, but it opposes the change of the excitation field $\Delta\vec{B}$.

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