Superconformal operators in $\mathcal{N} = 4$ super-Yang-Mills theory

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Abstract

We construct, in the framework of the $\mathcal{N} = 4$ SYM theory, a supermultiplet of twist-two conformal operators and study their renormalization properties. The components of the supermultiplet have the same anomalous dimension and enter as building blocks into multi-particle quasipartonic operators. The latter are determined by the condition that their twist equals the number of elementary constituent fields from which they are built. A unique feature of the $\mathcal{N} = 4$ SYM is that all quasipartonic operators with different $SU(4)$ quantum numbers fall into a single supermultiplet. Among them there is a subsector of the operators of maximal helicity, which has been known to be integrable in the multi-color limit in QCD, independent of the presence of supersymmetry. In the $\mathcal{N} = 4$ SYM theory, this symmetry is extended to the whole supermultiplet of quasipartonic operators and the one-loop dilatation operator coincides with a Hamiltonian of integrable $SL(2|4)$ Heisenberg spin chain.

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1 Introduction

Quantum chromodynamics—the theory of strong interactions—enjoys a number of space-time symmetries at the classical level: it is invariant under the $SO(4,2)$ group of conformal transformations in four dimensions [1]. When the theory is quantized, five of the charges, the dilatation and four conformal boosts cease to conserve and the conformal group is reduced down to its Poincaré subgroup. Still, the conformal symmetry has important consequences in QCD and provides powerful tools in various applications (see the review [2] and references therein).

It turns out that the conformal symmetry is not the only symmetry of QCD. A few years ago it has been found that evolution equations for scattering amplitudes in high-energy QCD possess a hidden symmetry in the multi-color limit [3, 4]. Namely, the partial waves of the scattering amplitudes satisfy a Schrödinger-like equation which has a large number of conserved charges and is completely integrable. The underlying integrable structure has been identified as corresponding to the celebrated Heisenberg spin magnet. Later, similar integrable structures have been discovered in studies of renormalization group equations (or the dilatation operator, phrased differently [5, 6]) for multi-particle Wilson operators in QCD [7, 8, 9, 10, 11]. The Wilson operators $W_{N}^{\mu_{1}...\mu_{j}}(0)$ are gauge-invariant local composite operators built from $N$ fundamental fields $X_{i}$, carrying in general Lorentz indices, and covariant derivatives $D_{\mu} = \partial_{\mu} - igA_{\mu}$ acting on them

$$W_{N}^{\mu_{1}...\mu_{j}} = \text{tr} \left\{ (D_{\mu_{1}}...D_{\mu_{k}}X_{1}(0)) (D_{\mu_{k+1}}...D_{\mu_{n}}X_{2}(0)) ... (D_{\mu_{m+1}}...D_{\mu_{j}}X_{N}(0)) \right\} .$$

According to their tensor structure these operators can be decomposed into irreducible components with respect to the Lorentz group. The component with the maximal Lorentz spin has a symmetric traceless Lorentz structure. It is projected from $W_{N}^{\mu_{1}...\mu_{j}}(0)$ via

$$O_{N}^{\mu_{1}...\mu_{j}} = S_{\mu_{1}...\mu_{j}} W_{N}^{\mu_{1}...\mu_{j}},$$

where the operation $S$ symmetrizes corresponding indices and subtracts traces, e.g., $S_{\mu\nu}T^{\mu\nu} = \frac{1}{2}(T^{\mu\nu}+T^{\nu\mu}) - \frac{1}{2}g^{\mu\nu}T_{\sigma\sigma}$. A distinguished property of the maximal-spin operators is that their twist, i.e., dimension minus spin, equals to the sum of twists of the individual fields $X_{i}$. The operators [2] mix under renormalization and the properties of their mixing matrix depend on the choice of fundamental constituent $X_{i}$. As was shown in the above-mentioned papers, for specific helicities of $N$ elementary fields entering the composite operator, the corresponding one-loop dilatation operator admits, in multi-color limit, $N-2$ extra nontrivial conserved charges in addition to the conformal Casimir and its third projection. Thus, the evolution equations for such operators are completely integrable in QCD.

It has been found that the one-loop dilatation operator in QCD gives rise to two essentially different integrable structures: For operators with aligned helicities, like three-quark baryons of
helicity-3/2 or spin-three “glueballs,” the dilatation operator is equivalent to a Hamiltonian of a closed non-compact $SL(2)$ Heisenberg spin chain. The second structure has emerged in the multi-color limit from the renormalization of operators built from the quark-antiquark pair and gluon strength tensors. The corresponding dilatation operator is integrable as well and is also related to a Heisenberg spin magnet. Since the QCD quarks belong to the fundamental representation of the gauge group, the interaction between the quark and antiquark fields is suppressed for $N_c \to \infty$. As a consequence, the arising Hamiltonian turns out to be of an inhomogeneous open Heisenberg spin chain. In both cases, the spin operators acting on the sites of the chain are the generators of the $SL(2)$ group. The latter corresponds, on the QCD side, to the so-called collinear subgroup of the full conformal group.

We would like to stress that in one-loop approximation the non-conformal nature of QCD is irrelevant and the one-loop dilatation operator ought to be conformally invariant. At the same time, integrability emerges as yet another symmetry of the dilatation operator in four-dimensional Yang-Mills theory which has immediate consequences for renormalization of composite operators in QCD and its supersymmetric extensions. To understand its origin one can consider the dilatation operator in maximally supersymmetric extension of QCD — the $\mathcal{N} = 4$ SYM theory. As compared to QCD, this model exhibits a number of exceptional properties which simplify its structure enormously. The $\mathcal{N} = 4$ SYM is an example of a full-fledged interacting conformal field theory in four dimensions invariant under superconformal $PSU(2, 2|4)$ transformations. More recently it was proposed that in the multi-color limit the $\mathcal{N} = 4$ SYM theory admits a dual description in terms of a superstring theory on $AdS_5 \times S^5$ background. This suggests that integrability of Yang-Mills theory is in one-to-one correspondence with symmetries of the string theory.

The Wilson operators in the $\mathcal{N} = 4$ SYM theory have the same form with the only difference being that they carry the additional $SO(6) \sim SU(4)$ charge. Among them there is a subclass of the Berenstein-Maldacena-Nastase (BMN) operators which, in the simplest form, do not involve covariant derivatives and have a near-to-extremal charge with respect to an $SO(2)$ subgroup of the $R$-symmetry group $SO(6)$. These operators do not have their counter-partners in QCD and reflect the supersymmetric nature of the $\mathcal{N} = 4$ model.

Recently it has been found that the one-loop dilatation operator of the $\mathcal{N} = 4$ model possesses integrable structures analogous to the those observed in QCD in two different sectors of Wilson operators. Namely, renormalization of the BMN operators is driven by an evolution operator which can be identified as a Hamiltonian of a Heisenberg spin magnet with spins belonging to the $SO(6)$ group. As we outlined above, the one-loop dilatation operator in QCD is integrable in the $SL(2)$ sector and gives rise to the $SL(2)$ Heisenberg spin magnet. The same $SL(2)$ integrable
sector arises in the \( \mathcal{N} = 4 \) super-Yang-Mills theory [23]. The subsequent studies of renormalization properties of “mixed” operators involving covariant derivatives and possessing a large \( R \)-charge have allowed to the authors of Refs. [24, 21] to conclude that for an arbitrary Wilson operator the one-loop dilatation operator in the \( \mathcal{N} = 4 \) super-Yang-Mills theory is equivalent to the Hamiltonian of the \( PSU(2,2|4) \) Heisenberg spin chain. These intriguing results indicate the potential, if extended to all orders, as was pointed out in Refs. [25, 26], of complete integrability of the \( \mathcal{N} = 4 \) super-Yang-Mills theory. Additional support for this conjecture comes from the gauge/string duality: recently is has been found that the classical world-sheet sigma model on the \( AdS_5 \times S^5 \) admits an infinite set of conserved charges [27, 28, 29, 30]. In Ref. [31] it was found that the Yangian structure of the superstring sigma model maps into symmetries of the dilatation operator in the Yang-Mills theory. Several successful matchings, starting from Refs. [32, 33], of the gauge-theory results to the rotating string solutions serve as a very nontrivial verification of the AdS/CFT correspondence [34].

Let us elaborate more on the Wilson operators (1). As we already mentioned, their maximal-spin component corresponds to a symmetric, traceless tensor and has the lowest twist in each \( N \)-particle sector. Below we discuss only this specific subclass of operators. To simplify the analysis even further, one projects out the Lorentz tensor onto the light-cone. This is accomplished by contracting the indices with a light-like vector \( n^\mu \) \((n^2 = 0)\), which has the virtue of automatic symmetrization and trace subtraction. Thus the resulting tensor transforms in an irreducible representation of the Lorentz group. This makes the covariance manifest without the need to deal with open indices. Thus, the dynamics is effectively projected on the light cone and the full superconformal group is reduced to its collinear subgroup. This results into the following single-trace multi-particle operators:

\[
O_{i_1, \ldots, i_N}(\xi_1, \ldots, \xi_N) = \text{tr} \, X_{i_1}(n\xi_1) \ldots X_{i_k}(n\xi_k)X_{i_{k+1}}(n\xi_{k+1}) \ldots X_{i_N}(n\xi_N),
\]

(3)

which is built from the “good” field components of the \( \mathcal{N} = 4 \) model \( X = \{F^\perp_{\mu}, \lambda^A_{\alpha}, \tilde{\lambda}^A_{\dot{\alpha}}, \phi^{AB}\} \), defined in the next section. By making such a choice we automatically restrict ourselves to the so-called quasipartonic operators. The Taylor expansion of the non-local operators (3) with respect to the field separations on the light-cone produces sets of local gauge-invariant operators (2). The gauge invariance of (3) is restored by inserting the gauge links \([s_{k+1}, s_k] = P \exp \left( ig \int_{\xi_k}^{\xi_{k+1}} d\xi' n \cdot A(n\xi') \right)\) between the fields. We do not display them in (3) since later on we will be using the light-like gauge \( A^+ \equiv (n \cdot A) = 0 \). In this gauge, the above projections of fields represent on-shell partons; hence, the name quasipartonic operators. These operators form a closed set under the action of the dilatation operator and have the following unique properties [35]: They carry a definite twist which is equal to the number of constituents. Since the twist is conserved under renormalization, the dilatation operator does not change the number of partons.
In the present paper we will be dealing with two-particle quasipartonic operators. The latter serve as building blocks for the multi-particle operators discussed above and, correspondingly, to the one-loop dilatation operator acting on this space: only pair-wise interactions are relevant to this order. Moreover, we restrict ourselves to the discussion of the color-singlet two-particle blocks because the gauge invariance of the Wilson operators allows one to reduce the action of the pair-wise dilatation generator in the color space in the multi-color limit to the quadratic Casimirs, \( t^a_n \otimes t^a_{n+1} \rightarrow -\frac{1}{2} N_c \) for \( N_c \rightarrow \infty \). Here an \( SU(N_c) \) generator \( t^a_n \) is assumed to act on a constituent placed on the \( n \)-th site of an \( N \)-particle colorless operator. Two-particle quasipartonic operators have obviously twist-two, and, according to their quantum numbers, they can be separated into different sectors. The operators belonging to each sector mix only among themselves and their renormalization is driven by an evolution kernel, or equivalently a dilatation operator. As was already mentioned, the one-loop dilatation operator is completely integrable in QCD in the sector of operators with maximal helicity. However, this sector is autonomous in QCD and there exists no relation between it and the dilatation operators acting in other sectors. To obtain such a relation one has to employ supersymmetric extensions of QCD.

Supersymmetry connects Wilson operators belonging to different sectors and, in addition, imposes severe constraints on their mixing matrix. These constraints tighten up as we go from non-supersymmetric theories all the way up to the maximal \( \mathcal{N} = 4 \) supersymmetry. Ultimately this allows one to resolve efficiently the operator mixing since the supersymmetry regroups all operators into supermultiplets whose components evolve autonomously under scale transformations and possess, at the same time, identical anomalous dimensions. The number of supermultiplets and their structure depend on how much supersymmetry a theory possesses. In the present paper we construct a supermultiplet of twist-two operators for the \( \mathcal{N} = 4 \) model by a direct calculation and identify its anomalous dimension. We demonstrate that the supermultiplet is unique; that is, its components span all twist-two operators transforming in different irreducible representations of the Lorentz and internal symmetry groups.\(^2\)

The crucial difference between \( \mathcal{N} = 4 \) and other Yang-Mills theories is that the former allows one to relate the sector of the maximal helicity operators to all other sectors. This is a mere consequence of the span in helicities of one-particle massless states which enter a given multiplet in supersymmetric models. In \( \mathcal{N} = 1 \) supersymmetry, the gauge multiplet is formed by two CPT-conjugated sets of states with the helicities \( \lambda = 1, \frac{1}{2} \) and \( \lambda = -1, -\frac{1}{2} \), respectively. Supersymmetric transformations change the helicity by a half, so that for a composite operator \( O^{(\lambda, \lambda')}(\xi_1, \xi_2) \sim \)

\(^2\)A subset of \( SU(4) \) flavor-singlet twist-two operators was identified in Ref. \[36\] by a diagonalization of the anomalous dimension matrix deduced from calculations of Feynman diagrams.
$X^{(\lambda)}(\xi_1)X^{(\lambda')}(\xi_2)$ built from fields of helicities $\lambda$ and $\lambda'$ one has

$$[Q, O^{(\lambda,\lambda')}] \sim O^{(\lambda+1/2,\lambda')} + O^{(\lambda',\lambda+1/2)}. \quad (4)$$

This is not sufficient to link the aligned-helicity two-fermion operators to the counter-aligned ones since for that one needs to flip the helicity ($\lambda$ or $\lambda'$) by one unit. The $\mathcal{N} = 4$ super-Yang-Mills, having the maximal possible number of global supersymmetries, is CPT self-conjugate and spans all possible helicities of particle states $\lambda = \pm 1, \pm \frac{1}{2}, 0$. Thus, to connect the above two sectors it is enough to apply the supersymmetric transformations twice

$$[Q^A, [Q^B, O^{(\lambda,\lambda')}]] \sim O^{(\lambda+1,\lambda')} + O^{(\lambda+1/2,\lambda'+1/2)} + O^{(\lambda',\lambda+1)}. \quad (5)$$

This explains why in the minimal supersymmetric extension of QCD — $\mathcal{N} = 1$ super-Yang-Mills — there are two independent supermultiplets of conformal operators: one of aligned helicities [35, 37] and another one of opposite helicities [38, 35, 39, 40]. The former supermultiplet inherits integrability of the one-loop dilatation operator in QCD. The exceptional place of the maximally supersymmetric gauge theory in a row of Yang-Mills theories is that both types of the operators enter the same supermultiplet. Therefore the one-loop dilatation operator is integrable in the $\mathcal{N} = 4$ model [21] and anomalous dimension of the supermultiplet is identical at one-loop order to the anomalous dimension of the maximal-helicity quark [7, 8] and gluon [10] QCD conformal operators.

Our presentation is organized as follows. In the next section, after recalling the definition of conformal operators in a generic case, we turn to the classification of all two-particle conformal blocks in maximally supersymmetric gauge theory. Subsequently in section 3 we discuss the projection of the $\mathcal{N} = 4$ supersymmetry algebra on the light-cone: one needs only this subalgebra for discussion of twist-two conformal blocks. Then we construct a supermultiplet which embeds all of them. The anomalous dimension of the supermultiplet is fixed in section 4 to two-loop order by computing one of its components and using the result of Ref. [41]. Section 5 is devoted to discussions and conclusions. A few appendices contain some details of the calculations.

## 2 Conformal operators in $\mathcal{N} = 4$ super-Yang-Mills

As we outlined in the introduction, the nonlocal light-cone operators [3] serve as generating functions for the Wilson operators of the lowest twist. Indeed, expanding Eq. (3) in powers of the “light-cone distances” $\xi_j - \xi_k$, one can obtain an infinite set of local composite operators (2) with covariant derivatives projected with the light-like vector\(^3\) $n^\mu$. The main advantage of dealing with

\(^3\)In addition to $n^\mu$, one introduces also an orthogonal light-cone vector $n^{*\mu}$, $n^\mu n^{*\mu} = 0$ normalized such that $n^{*\mu} n^\mu = 1$. Then, an arbitrary vector can be decomposed as $v^\mu = v^+ n^\mu + v^- n^{*\mu} + v^\perp$, where the “plus” and
the nonlocal light-cone operators \( \mathcal{L}_\pm \) is that they have simple transformation properties under a (super)conformal group.

Since the fields entering Eq. \( \mathcal{L}_\pm \) “live” on the light-cone, defined by the four-vector \( n^\mu \), we can significantly simplify our considerations by restricting the full \( SO(4,2) \) conformal symmetry group down to its collinear conformal subgroup \( SO(2,1) \sim SL(2) \), for a review (see Ref. [2]). The latter is a group of projective transformations on a line:

\[
\xi \rightarrow a\xi + b
c\xi + d, \quad ad - bc = 1,
\]

with \( \xi \) defining the position of a field operator on the light-cone, \( X(\xi n^\mu) \equiv X(\xi) \). Its algebra is formed by projections of the generators of translations \( \mathcal{L}^+ = i\mathcal{P}^+ \), conformal boosts \( \mathcal{L}^- = \frac{i}{2}\mathcal{K}^- \), and a combination of Lorentz transformations and dilatations \( \mathcal{L}^0 = \frac{i}{2}(\mathcal{D} + \mathcal{M}^-) \). Yet another combination of the latter two determines the operator of the twist, \( \mathcal{E} = i(\mathcal{D} - \mathcal{M}^-) \), which commutes with \( \mathcal{L}^0, \mathcal{L}^+ \) and \( \mathcal{L}^- \). Primary fields are transformed under the \( SL(2) \) transformations \( \mathcal{L}_\pm \) as

\[
X(\xi) \rightarrow X'(\xi) = (c\xi + d)^{-2j}X\left(\frac{a\xi + b}{c\xi + d}\right), \quad j = \frac{1}{2}(d + s)
\]

where \( j \) is the conformal spin of the field. It is given by half the sum of the canonical dimension of the field, \( d \), and its spin projection on the light-cone \( \Sigma_\pm X(\xi) = sX(\xi) \). The operator \( \mathcal{E} \) counts the twist \( t = (d - s) \) of the field \( X \), i.e.,

\[
[\mathcal{E}, X(\xi)] = (d - s)X(\xi).
\]

Obviously, the twist of the multi-particle operator \( \mathcal{L}_\pm \) is equal to the sum of the twists of elementary fields. In what follows, as we pointed out above, we shall consider the operators of the minimal twist.

Let us examine the transformation properties of the elementary fields in the \( \mathcal{N} = 4 \) theory. For scalars one has \( d_{sc} = 1 \) and \( s_{sc} = 0 \) leading to \( j_{sc} = 1/2 \) and \( t_{sc} = 1 \). For vectors and fermions, one has to separate the corresponding fields into components with different values of the spin projection \( s_q = \pm 1/2 \) and \( s_g = \pm 1, 0 \), respectively. As before, the minimal twist corresponds to the component with the maximal spin. Let us discuss the fermion and gauge fields separately.

The four-component fermion field is decomposed into two components with \( s = \pm 1/2 \) by means of the projection operators

\[
\psi = \psi_+ + \psi_-, \quad \psi_\pm \equiv \Pi^\pm \psi, \quad \Pi^\pm \equiv \frac{1}{2} \gamma^\pm \gamma^\pm,
\]

so that \( \psi_\pm \) have the following spins:

\[
\Sigma^+ \psi_\pm = \pm \frac{1}{2} \psi_\pm,
\]
where we have used the light-cone projected spin tensor $\Sigma^{\mu\nu} \equiv \frac{1}{4} [\gamma^{\mu}, \gamma^{\nu}]$. Since the canonical dimension of the fermion field is $d_q = 3/2$, one finds that the $\psi_+$-component has the conformal spin $j_q = 1$ and the twist $t_q = 1$. Similarly, for the $\psi_-$-component one finds $j_q = 1/2$ and $t_q = 2$. Therefore, only the $\psi_+$ component enters a nonlocal operator of the minimal twist. Notice that Eq. (9) is identical to the decomposition used in the light-cone quantization [42] and we will use the latter formalism momentarily to point out important modifications when the $\psi_-$ enters the composite operator. So far we did not refer to the specifics of the maximally supersymmetric Yang-Mills theory. Its Lagrangian is defined in appendix A in terms of the Weyl spinors in order to preserve the $SU(4)$ covariance. Going from the four-component notations to the Weyl gaugino fields, the projection of the “good” components converts into\(^4\)

\[
\lambda_+^{+\alpha} \equiv \frac{1}{2} \bar{\sigma}^{-}_{\alpha\beta} \sigma^+_{\beta\gamma} \lambda^{+\gamma}, \quad \bar{\lambda}^+_{\dot{\alpha}} \equiv \frac{1}{2} \bar{\sigma}^{-}_{\dot{\alpha}\dot{\beta}} \sigma^+_{\dot{\beta}\dot{\gamma}} \bar{\lambda}^{+\dot{\gamma}}.
\]

Each Weyl spinor $\lambda_+^{+\alpha}$ and $\bar{\lambda}^+_{\dot{\alpha}}$ has only one nonvanishing component which describes a state with a definite helicity, $+1$ and $-1$, respectively. For brevity, we suppress in what follows the subscript “plus” on the fermion field designating its “good” light-cone projections, i.e., $\lambda_+ \rightarrow \lambda$.

For the gauge field, one has to project the Lorentz indices of the strength tensor $F^{\mu\nu}$ onto the longitudinal light-cone directions and the transverse space with the two-dimensional metric $g_{\mu\nu} = g_{\mu\nu} - n_{\mu} n^*_{\nu} - n_{\nu} n^*_{\mu}$. The spin assignments for different projections of $F^{\mu\nu}$ are as follows

\[
\Sigma^{-+} F^{\pm\mu}_\perp = \pm F^{\pm\mu}_\perp, \quad \Sigma^{-+} F^{+-} = \Sigma^{-+} F^{\mu\nu}_{\perp\perp} = 0,
\]

where $\Sigma^{\mu\nu} F^{\rho\sigma} = g^{\mu\rho} F^{\nu\sigma} - g^{\nu\rho} F^{\mu\sigma} - (\rho \leftrightarrow \sigma)$. Since the canonical dimension of the strength tensor is $d_g = 2$, the $F^{+-}_\perp$-component has the conformal spin $j_g = 3/2$ and twist $t_g = 1$. For $F^{-+}_\perp$ and $F^{\mu\nu}_{\perp\perp}$-components one gets $j_g = 1$ and $t_g = 2$, while for the $F^{-\mu}_\perp$-component one has $j_g = 1/2$ and $t_g = 3$. Thus, the minimal twist is associated with the $F^{+-}_\perp$-component only.

To summarize, the leading twist multi-particle operators are constructed solely from the “good” fields $\psi_+$, $F^{+-}_\perp$ and scalars $\phi$ living on the light cone. Let us reiterate that the twist of such nonlocal operators equals the number of primary fields involved and they are known in QCD as quasipartonic operators [35]. The “good” fields are transformed as conformal primaries with respect to the collinear conformal subgroup and carry the following conformal spins:

\[
\begin{align*}
  j_q &= 1, & j_g &= \frac{3}{2}, & j_s &= \frac{1}{2},
\end{align*}
\]

for gauginos, gluons and scalars, respectively, as we established earlier. A unique feature of quasipartonic operators is that they form a closed set with respect to the action of the dilatation operator. Since the twist is preserved under dilatations, the quasipartonic operators can only

\(^4\)The conventions for four-dimensional Clifford algebra is fixed in appendix B.
be transformed into operators of the same twist. Therefore, the total number of constituents is conserved. In other words, the dilatation operator acts “elastically” on the space of quasipartonic operators — there is no annihilation/production of the “good” fields — and, therefore, it can be represented as a quantum-mechanical Hamiltonian.

As was already mentioned above, the nonlocal light-cone quasipartonic operators are generating functions of the Wilson operators (2) of the lowest twist or, equivalently, the maximal Lorentz spin. For obvious reasons, the nonlocal light-cone operators corresponding to non-maximal spin components of the Wilson operators (1) and/or containing “bad” components of fields belong to the class of non-quasipartonic operators. Their twist is higher compared to the twist of quasipartonic operators built from the same number of constituent fields. The transformation of non-quasipartonic operators under dilatations takes a more complicated form. One of the reasons for this is that the operators containing “bad” components are not dynamically independent and can be re-expressed nonlocally in terms of “good” fields integrating the former out in the functional integral in the light-like gauge $A^+ = 0$. This effectively results into the use of equations of motion, e.g.,

$$\psi_-(\xi) = \frac{i}{2} (\partial^+)^{-1} \mathcal{P}_{\perp} \gamma^+ \psi_+(\xi) + \ldots = \frac{i}{2} \int \frac{d\xi'}{2\pi} \int \frac{d\nu}{\nu} e^{-i\nu(\xi-\xi')} \mathcal{P}_{\perp} \gamma^+ \psi_+(\xi') + \ldots,$$

(14)

where the ellipses stand for contributions from scalars. This relation implies that the “bad” component $\psi_-(\xi)$ can be treated as a (multi-particle) composite state built from the “good” components $\psi_+(\xi')$ and $A_{\perp}^\mu(\xi') = \int d\xi'' F_{\perp +}^\mu(\xi' + \xi'')$ smeared along the light-cone. This implies that, in distinction with quasipartonic operators, the total set of nonquasipartonic operators of a given twist is overcomplete. Construction of the basis of operators in this sector remains an open problem.

### 2.1 Two-particle conformal primaries

In general, the Wilson operators (2) do not transform covariantly under the action of the $SO(4, 2)$ conformal group and, as a consequence, they do not have an autonomous renormalization under dilatations. To identify the operators which are eigenfunctions of the dilatation operator, one makes use of the collinear $SL(2)$ subgroup of the conformal group to define the so-called conformal operators. The conformal operator $O_J$ is determined by requiring that it is transformed under the $SL(2)$ transformations as a primary field of the conformal spin $J$, Eqs. (6) and (7). The same condition can be expressed as

$$[\mathcal{L}^-, O_J(0)] = 0, \quad [\mathcal{L}^2, O_J(0)] = J(J - 1)O_J(0),$$

(15)

where $\mathcal{L}^2$ is the quadratic Casimir of the $SL(2)$ subgroup.
To construct a two-particle conformal operator, one considers the product of two fields on the light-cone $X_1(0)X_2(\xi)$. It is transformed in accordance with the direct product of two $SL(2)$ representations labelled by the conformal spins of the fields, $j_1$ and $j_2$. Decomposing this product into a sum of irreducible components, $[j_1] \otimes [j_2] = \sum_{n \geq 0} [n + j_1 + j_2]$, one identifies the conformal operator $O_j(0)$ with $j = n + j_1 + j_2 - 2$ as the highest weight of the spin-$(j + 2)$ component. In this way, one obtains the explicit expression for $O_j(0)$ in terms of the Jacobi polynomials

$$O_j(0) = X_1(0)(i\partial_\perp)^nP_n^{(2j_1-1,2j_2-1)}(\frac{\nabla}{\partial_\perp})X_2(0),$$

with $J = j + 2 = n + j_1 + j_2$ being its conformal spin defined by Eq. (16). Here we have used the notations

$$\partial \equiv \partial_\perp + \partial, \quad \nabla \equiv \nabla_\perp - \nabla.$$

for the total and left-right derivatives, respectively. The spin-$(j+2)$ representation space is spanned by the conformal operator $O_j(0)$ and its descendants, generated with the step-up operator $L^+$,

$$O_{jl}(0) = (i\partial_\perp)^{l-n}[L^+,\ldots,[L^+,O_j(0)]\ldots] = (i\partial_\perp)^{l-n}O_j(0), \quad (l \geq n),$$

from the vacuum state $O_{j0}(0) \equiv O_j(0)$. Then, the product $X_1(0)X_2(\xi)$ can be expanded over the conformal operators as follows (we recall that we are using the $A^+ = 0$ gauge):

$$X_1(0)X_2(\xi) = \sum_{n=0}^{\infty} C_n(j_1,j_2)(-i\xi)^n \int_0^1 du u^{n+2j_1-1}(1-u)^{n+2j_2-1}O_{j=n+j_1+j_2-2}(u\xi),$$

where we introduced the “reduced” Clebsch-Gordan coefficients

$$C_n(j_1,j_2) = \frac{(2n+2j_1+2j_2-1)\Gamma(n+2j_1+2j_2-1)}{\Gamma(n+2j_1)\Gamma(n+2j_2)}.$$

As we can see, the definition of conformal operators does not rely on supersymmetry and makes use of the invariance of the underlying Yang-Mills theory under the conformal $SO(4,2)$ transformations. In addition, the conformal symmetry protects mixing between operators of different conformal spin. In QCD this property is valid to one-loop order only due to the conformal anomaly, while in $\mathcal{N} = 4$ super-Yang-Mills the conformal symmetry holds to all orders provided that the theory is regularized in the manner that preserves the symmetry of the classical Lagrangian. We will return to this issue later in section 4 where we discuss subtleties of the dimension regularization of loop corrections.

In general, there exists a set of conformal operators possessing the same conformal spin $j$ and other quantum numbers with respect to the Lorentz and internal flavor groups. Combining them
together, one can define a vector \( \mathbf{O}_{jl} \) which obeys a matrix renormalization group equation\(^5\)

\[
\frac{d}{d \ln \mu} \mathbf{O}^R_{jl} = -\gamma_j(g^2) \mathbf{O}^R_{jl},
\]

(19)

where the anomalous dimension matrix \( \gamma_j(g^2) \) has an infinite series expansion in perturbation theory;

\[
\gamma_j(g^2) = \sum_{n=0}^{\infty} \left( \frac{g^2}{8\pi^2} \right)^{n+1} \gamma_j^{(n)}.
\]

(20)

Conformal symmetry implies that \( \gamma_j(g^2) \) depends only on the conformal spin \( j \), but it does not fix its functional form. Additional constraints on the mixing matrix are imposed by supersymmetry. As we will show below, it allows one to determine the eigenstates of the matrix \( \gamma_j(g^2) \) in \( \mathcal{N} = 4 \) theory to all orders in the coupling constant without actual calculations of Feynman diagrams.

The rest of this section is devoted to the counting and construction of all twist-two operators in \( \mathcal{N} = 4 \) super-Yang-Mills by classifying them with respect to irreducible representations of the Lorentz and flavor group. The generic expression is given by Eq. (16) with \( X_i \) being one of the “good” fields \( X = \{ F^\pm_\mu, \lambda^{A}_+\alpha, \bar{\lambda}^\dagger_+\dot{\alpha} + A, \phi^{AB} \} \). For the sake of convenience, we split all bilinears into bosonic and fermionic operators. The first set involves gaugino-gauginos, gluon-gluon, scalar-scalar and gluon-scalar ones, while the second contains gaugino-gluon and gaugino-scalar bilinears. Later using the supersymmetry algebra we construct a supermultiplet which embraces all of them.

### 2.2 Two-gauginos operators

Let us start with the two-gauginos operators. The gaugino fields \( \lambda^A_+\alpha \) and \( \bar{\lambda}^{\dagger}_+\dot{\alpha} + A \) carry the conformal spin \( j_q = 1 \) and the corresponding conformal operator is given by Eq. (16) with the Jacobi polynomial being reduced to the Gegenbauer polynomial \( C_j^{3/2} \). One can assign a definite spatial parity to the two-gauginos operators. In the Majorana notations, the difference between the even and odd parity bilinears is encoded in the chirality matrix \( \gamma_5 \), so that \( \bar{\psi}\gamma^+\psi \) and \( \bar{\psi}\gamma^+\gamma_5\psi \) are the vector and the axial-vector, respectively. When expressed in terms of two-component Weyl spinors, \( \lambda^A_+\alpha \) and \( \bar{\lambda}^{\dagger}_+\dot{\alpha} + A \), the bilinears will be accompanied by the signature factors

\[
\sigma_j \equiv 1 - (-1)^j,
\]

(21)

which trace back to the vanishing of the corresponding operators with even/odd spatial parity.

Next, since the gaugino transforms in the \( 4 \) of \( SU(4) \), the bilinear built from it and its complex conjugate is decomposed into two irreducible representations \( 4 \otimes \bar{4} = 1 \oplus 15 \). The latter is

\(^5\)The superscript \( R \) stands for a subtracted operator \( \mathbf{O}^R = Z\mathbf{O} \). The latter generates finite Green functions with elementary field operators.
Therefore, in total we have the following two-fermion conformal operators:

\[ [P_{15}]_{BC}^{AD} = \delta_A^C \delta_B^D - \frac{1}{4} \delta_A^B \delta_D^C. \]  

(22)

The maximal-helicity gaugino operators, with fields having aligned helicities, have two flavor components \( 4 \otimes 4 = 6 \oplus 10 \), projected with

\[ [P_{10}]_{CD}^{AB} = [P_{10}]_{CD}^{AB} = \delta_C^A \delta_D^B + \delta_D^A \delta_C^B, \quad [P_6]_{CD}^{AB} = [P_6]_{CD}^{AB} = \delta_C^A \delta_D^B - \delta_D^A \delta_C^B. \]  

(23)

Therefore, in total we have the following two-fermion conformal operators:

- the singlet even- and odd-parity operators, respectively,
  
  \[ \mathcal{O}_{jl}^{qq} = \sigma_j \text{tr} \, \bar{\lambda}_{\alpha A} \sigma^+ \bar{\alpha} \beta (i \partial^+) \frac{1}{\sqrt{2}} \left( \bar{\lambda}^+ / \partial^+ \right) \lambda_{\beta}^A, \]
  \[ \mathcal{O}_{jl}^{\bar{q} q} = \sigma_{j+1} \text{tr} \, \bar{\lambda}_{\alpha A} \sigma^+ \bar{\alpha} \beta (i \partial^+) \frac{1}{\sqrt{2}} \left( \bar{\lambda}^+ / \partial^+ \right) \lambda_{\beta}^A. \]  

(24)

- the even- and odd-parity operators in \( 15 \)
  
  \[ [\mathcal{O}_{jl}^{qq,15}]_B^A = \sigma_{j+1} \text{tr} [P_{15}]_{BC}^{AD} \bar{\lambda}_{\alpha C} \sigma^+ \bar{\alpha} \beta (i \partial^+) \frac{1}{\sqrt{2}} \left( \bar{\lambda}^+ / \partial^+ \right) \lambda_{\beta}^D, \]
  \[ [\mathcal{O}_{jl}^{\bar{q} q,15}]_B^A = \sigma_j \text{tr} [P_{15}]_{BC}^{AD} \bar{\lambda}_{\alpha C} \sigma^+ \bar{\alpha} \beta (i \partial^+) \frac{1}{\sqrt{2}} \left( \bar{\lambda}^+ / \partial^+ \right) \lambda_{\beta}^D. \]  

(25)

- the maximal-helicity operators in antisymmetric \( n = 6 \), symmetric \( n = 10 \) and their complex-conjugated \( \bar{n} = \bar{6}, \bar{10} \) representations

  \[ [\tilde{T}_{jl}^{qq, \bar{n} \mu}]_{AB} = \text{tr} [P_n]_{CD}^{\alpha \beta} \lambda^C \sigma^+ \bar{\alpha} \beta (i \partial^+) \frac{1}{\sqrt{2}} \left( \bar{\lambda}^+ / \partial^+ \right) \lambda_{\gamma}^D, \]
  \[ [\tilde{T}_{jl}^{\bar{q} q \mu}]_{AB} = \text{tr} [P_{\bar{n}}]_{CD}^{\alpha \beta} \lambda^C \sigma^+ \bar{\alpha} \beta (i \partial^+) \frac{1}{\sqrt{2}} \left( \bar{\lambda}^+ / \partial^+ \right) \lambda_{\gamma}^D. \]  

(26)

We also introduce \( SU(4) \)-conjugated operators

\[ [T_{jl}^{qq, \bar{6} \mu}]_{AB} = \frac{1}{2} \varepsilon_{ABCD} [T_{jl}^{qq, \bar{6} \mu}]_{CD}, \quad [\bar{T}_{jl}^{qq, \bar{6} \mu}]_{AB} = \frac{1}{2} \varepsilon_{ABCD} [\bar{T}_{jl}^{qq, \bar{6} \mu}]_{CD}. \]  

(27)

As we will see in the next section, they naturally arise in supersymmetric variations of other operators which form the supermultiplet. Notice that in order to simplify our notations we have introduced a superscript of the flavor representation only for non-singlet cases. Analogously, below we will omit this label for the lowest-dimensional representation in a given set of operators with the same field content or when the \( SU(4) \) representation is obvious from the particle content.

In the above expressions, the signature factors \( \sigma_j (\sigma_{j+1}) \) imply that the corresponding operators are identically zero for even (odd) conformal spins. The Fermi statistics of gaugino fields has a consequence that \( [T_{jl}^{qq, \bar{6} \mu}]_{AB} \) and \( [T_{jl}^{qq, \bar{10} \mu}]_{AB} \) also vanish for odd and even \( j \)'s, respectively.
2.3 Two-gluon operators

Let us address now gluonic operators. The vector field \( F^{+\mu}_{\perp} \) has the conformal spin \( j_g = 3/2 \) and carries no charge with respect to the internal \( R \)-symmetry group. The two-gluon conformal operators are given by (16) for \( j_1 = j_2 = 3/2 \), so that \( P^{(2,2)}_j \sim C^{5/2}_j \). The product of two vectors, \( g_{\perp}^{\mu\nu} g^{\nu\beta}_\perp F^{+\alpha}_\perp F^{+\beta}_\perp \), can be decomposed into irreducible representations of the Lorentz group as \( (1/2, 1/2) \otimes (1/2, 1/2) = (0, 0) \oplus ((1, 0) \oplus (0, 1)) \oplus (1, 1) \), or, equivalently,

\[
\begin{align*}
g_{\perp}^{\mu\alpha} g_{\perp}^{\nu\beta} &= \frac{1}{2} g_{\perp}^{\mu\nu} g_{\perp}^{\nu\alpha} + \frac{1}{2} \varepsilon_{\perp}^{\mu\nu} \varepsilon_{\perp}^{\alpha\beta} + \tau_{\perp}^{\mu\nu;\rho\sigma} \tau_{\perp}^{\alpha\beta;\rho\sigma},
\end{align*}
\]

with \( \varepsilon_{\perp}^{\mu\nu} = \varepsilon^{\alpha\beta\rho\sigma} g_{\alpha\mu} g_{\beta\nu} n^\rho n_\sigma \) normalized as \( \varepsilon^{0123} = 1 \) and \( \tau_{\perp}^{\mu\nu;\rho\sigma} = \frac{1}{2} \left( g_{\perp}^{\mu\rho} g_{\perp}^{\nu\sigma} + g_{\perp}^{\mu\sigma} g_{\perp}^{\nu\rho} - g_{\perp}^{\mu\nu} g_{\perp}^{\rho\sigma} \right) \)

being a totally symmetric and traceless tensor in each pair of its indices. Thus, the two-gluon twist-two operators may have only three independent Lorentz structures.

For further convenience we introduce the gluon fields of positive- and negative-helicity \( F^{+\mu}_{\perp} + i \tilde{F}^{+\mu}_{\perp} \) and \( F^{+\mu}_{\perp} - i \tilde{F}^{+\mu}_{\perp} \), respectively. Here the dual gluon field strength is defined as \( \tilde{F}_{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu;\rho\sigma} F_{\rho\sigma} \) and \( \tilde{F}^{+\mu}_{\perp} = \varepsilon_{\perp}^{\mu\nu} F^{+\nu}_{\perp} \). From the positive- and negative-helicity operators we build the even- and odd-parity combinations

\[
\begin{align*}
\frac{1}{4} \left\{ \left( F^{+\mu}_{\perp} - i \tilde{F}^{+\mu}_{\perp} \right) \left( F^{+\nu}_{\perp} + i \tilde{F}^{+\nu}_{\perp} \right) \pm \left( F^{+\mu}_{\perp} + i \tilde{F}^{+\mu}_{\perp} \right) \left( F^{+\nu}_{\perp} - i \tilde{F}^{+\nu}_{\perp} \right) \right\} = - \left\{ \frac{g_{\mu\nu}}{i \varepsilon_{\mu\nu}} \right\} F^{+\mu}_{\perp} F^{+\nu}_{\perp},
\end{align*}
\]

as well as the maximal-helicity operator

\[
\begin{align*}
\frac{1}{4} \left\{ \left( F^{+\mu}_{\perp} + i \tilde{F}^{+\mu}_{\perp} \right) \left( F^{+\nu}_{\perp} + i \tilde{F}^{+\nu}_{\perp} \right) \pm \left( F^{+\mu}_{\perp} - i \tilde{F}^{+\mu}_{\perp} \right) \left( F^{+\nu}_{\perp} - i \tilde{F}^{+\nu}_{\perp} \right) \right\} = - \tau_{\mu\nu;\rho\sigma} \left\{ \frac{F^{+\mu}_{\rho} F^{+\nu}_{\sigma}}{i \tilde{F}_{\rho}^{\perp} F_{\sigma}^{\perp}} \right\},
\end{align*}
\]

(28)

where the sign assignment corresponds to the upper and lower component of the tensor structure on the right-hand side.

According to the above nomenclature, we introduce the following two-gluon conformal operators:

- the even-parity operator\(^6\)

\[
O^{gg}_{jl} = \frac{1}{2} \sigma_j \text{tr} F^{+\mu}_{\perp} g_{\mu\nu} (i \partial^+) \left( C_{j-1}^{5/2} \left( \frac{\partial^+}{\partial^+} \right) F^{\nu+}_{\perp} \right),
\]

(29)

- the odd-parity operator

\[
\tilde{O}^{gg}_{jl} = \frac{1}{2} \sigma_{j+1} \text{tr} F^{+\mu}_{\perp} i \varepsilon_{\mu\nu} (i \partial^+) \left( C_{j-1}^{5/2} \left( \frac{\partial^+}{\partial^+} \right) F^{\nu+}_{\perp} \right),
\]

(30)

\(^6\)Here, compared to Eq. (16), we shifted the argument of the Gegenbauer polynomial to ensure that the total conformal spin of the operator \( J = (j - 1) + 3 \) coincides with the conformal spin of the gaugino operators.
• the maximal-helicity operators

\[
[T_{jl}^{gg}]^{\mu \nu} = \text{tr} \left( F^{\mu+} + i iF^{\mu+} \right) (i\partial^+)^{l-1} C^{5/2}_{j-1} \left( D^+ / \partial^+ \right) \left( F^{\nu+} + i iF^{\nu+} \right), \\
[\tilde{T}_{jl}^{gg}]^{\mu \nu} = \text{tr} \left( F^{\mu+} - i iF^{\mu+} \right) (i\partial^+)^{l-1} C^{5/2}_{j-1} \left( D^+ / \partial^+ \right) \left( F^{\nu+} - i iF^{\nu+} \right). 
\]

Notice that we have split the maximal-helicity gluon operator \( T_{jl}^{gg} \) into negative and positive total helicity components, \( T_{jl}^{gg} \) and \( \tilde{T}_{jl}^{gg} \), respectively. The Bose statistics of the gluon fields implies that the conformal operators \( O_{jl}^{gg} / \tilde{O}_{jl}^{gg} \) vanish for even/odd conformal spins \( j \). To make these properties explicit we inserted the signature factors into their definition.

### 2.4 Two-scalar operators

We recall that the six real scalar fields of the \( N = 4 \) model are combined into a complex antisymmetric field \( \phi^{AB} \) which transforms with respect to 6 of the \( SU(4) \) flavor group and carries the conformal spin \( j_s = 1/2 \). Thus, the product of two representations for scalars gives \( 6 \otimes 6 = 1 \oplus 15 \oplus 20 \), so that the Clebsch-Gordan decomposition reads

\[
\phi_{AB} \phi^{CD} = \frac{1}{12} \left( \delta^C_A \delta^D_B - \delta^C_B \delta^D_A \right) \phi_{FG} \phi^{FG} \\
+ \frac{1}{2} \left( \delta^C_A [P_{15}]_{BF}^{DE} - \delta^D_A [P_{15}]_{BF}^{CE} + \delta^D_B [P_{15}]_{AF}^{CE} - \delta^C_B [P_{15}]_{AF}^{DE} \right) \phi_{EF} \phi^{FG} \\
+ \frac{1}{12} \left( [P_{20}]^{CD;EF}_{AB;GH} \phi_{EF} \phi^{GH} \right),
\]

where the projector \( [P_{15}]^{DE}_{BF} \) was defined in (22) and

\[
[P_{20}]^{CD;EF}_{AB;GH} \equiv \delta^C_G \delta^D_H \varepsilon_{ABJ} \varepsilon^{EFIJ} + \delta^D_G \delta^C_H \varepsilon_{ABJ} \varepsilon^{CEFJ} + \delta^C_G \delta^D_H \varepsilon_{ABJ} \varepsilon^{DEFJ} + \varepsilon_{ABGH} \varepsilon^{CDF}. \tag{32}
\]

Applying (16) for \( j_1 = j_2 = 1/2 \) and taking into account that \( P_j^{(0,0)} \sim C_j^{1/2} \), we introduce the following two-scalar conformal operators:

• the singlet \( SU(4) \) representation

\[
O_{jl}^{ss} = \frac{1}{2} \sigma_j \text{tr} \phi_{AB} (i\partial^+)^{l+1} C_{j+1}^{1/2} \left( D^+ / \partial^+ \right) \phi^{AB}, \tag{33}
\]

• the non-singlet 15 and 20 \( SU(4) \) representations

\[
[O_{jl}^{ss;15}]_{AB} = \frac{1}{2} \sigma_{j+1} \text{tr} \left( \delta^C_A \delta^D_B \phi_{CE} (i\partial^+)^{l+1} C_{j+1}^{1/2} \left( D^+ / \partial^+ \right) \phi^{DE} \right), \tag{34}
\]

\[
[O_{jl}^{ss;20}]^{CD}_{AB} = \frac{1}{2} \sigma_j \text{tr} \left( \delta^C_A \delta^D_B \phi_{EF} (i\partial^+)^{l+1} C_{j+1}^{1/2} \left( D^+ / \partial^+ \right) \phi^{GH} \right). \tag{35}
\]

The Bose statistics implies that the operators \( O_{jl}^{ss} \) and \( [O_{jl}^{ss;20}]_{AB} \) vanish for even \( j \) whereas \( [O_{jl}^{ss;15}]_{AB} \) vanishes for odd \( j \).
2.5 Mixed bosonic operators

The last in a row of bosonic operators are mixed gluon-scalar operators, which are

\[ [T_{jl}^{sg}]^{\mu AB} = \frac{i}{\sqrt{2}} \text{tr} \left( F_\bot^{\mu} - i \tilde{F}_\bot^{\mu} \right) (i \partial^+) l P_j^{(2,0)} \left( \mathcal{D}^+ / \partial^+ \right) \phi^{AB}, \]

\[ [T_{jl}^{sg}]_{\mu AB} = \frac{i}{\sqrt{2}} \text{tr} \left( F_\bot^{\mu} + i \tilde{F}_\bot^{\mu} \right) (i \partial^+) l P_j^{(2,0)} \left( \mathcal{D}^+ / \partial^+ \right) \bar{\phi}_{AB}. \] (36)

They do not possess a definite parity. The indices of the Jacobi polynomials are defined by the conformal spins of the operators, Eq. (16).

2.6 Fermionic operators

Finally, we define operators with fermionic quantum numbers. There are obviously two types: constructed from the gaugino and the field strength and from the gaugino and the scalar. In the latter case, the product of gaugino and scalar fields is decomposed into \( 6 \otimes \bar{4} = 4 \oplus \bar{20} \). The projection onto irreducible components is accomplished with the Clebsch-Gordan decomposition

\[ \phi^{AB} \lambda_C = \frac{1}{3} \left( \delta^B_C \delta^A_D - \delta^A_C \delta^B_D \right) \phi^{DE} \bar{\lambda}_E + \frac{1}{6} [P_{20}]^{AB;F}_{C;DE} \phi^{DE} \bar{\lambda}_F, \]

where

\[ [P_{20}]^{AB;F}_{C;DE} = (\delta^F_C \delta^I_D + \delta^F_D \delta^I_C) \varepsilon^{ABGH} \varepsilon^{DEIH}. \] (37)

Analogously, one decomposes the product \( \bar{\phi}_{AB} \lambda^C \) as \( 6 \otimes 4 = \bar{4} \oplus 20 \) with the projector

\[ [P_{20}]^{C;DE}_{AB;F} = (\delta^C_E \delta^G_I + \delta^G_E \delta^C_I) \varepsilon^{ABGH} \varepsilon^{DEIH}. \] (38)

To complete the basis we introduce the operators built from the gaugino and the gluon. Thus we have

- the gaugino-scalar \( 4 \) and \( \bar{4} \)

\[ [\tilde{Q}^{eq}_{jl}]^{\alpha A} = i \sqrt{2} \text{tr} \phi^{AB} (i \partial^+) l P_j^{(0,1)} \left( \mathcal{D}^+ / \partial^+ \right) \bar{\lambda}^B, \]

\[ [Q^{eq}_{jl}]_{\alpha A} = i \sqrt{2} \text{tr} \phi_{AB} (i \partial^+) l P_j^{(0,1)} \left( \mathcal{D}^+ / \partial^+ \right) \lambda^B, \] (39)

- the gaugino-scalar \( \bar{20} \) and \( 20 \)

\[ [\tilde{Q}^{eq;20}_{jl}]^{\alpha AB} \equiv i \sqrt{2} \text{tr} [P_{20}]^{AB;F}_{C;DE} \phi^{DE} (i \partial^+) l P_j^{(0,1)} \left( \mathcal{D}^+ / \partial^+ \right) \bar{\lambda}^F, \]

\[ [Q^{eq;20}_{jl}]_{\alpha AB} \equiv i \sqrt{2} \text{tr} [P_{20}]^{C;DE}_{AB;F} \phi_{DE} (i \partial^+) l P_j^{(0,1)} \left( \mathcal{D}^+ / \partial^+ \right) \lambda^F, \] (40)
the gluon-gaugino operators

\[ \pm \bar{\Omega}^{\gamma q \mu} \equiv \frac{1}{2} \text{tr} \left( \bar{F}^{\mu \perp} \pm i \tilde{F}^{\mu \perp} \right) (i \partial^+)^j P_j^{(2,1)} \left( \mathcal{D}^+/\partial^+ \right) \lambda^A, \]

\[ \pm \bar{\Omega}^{\gamma q \mu} \hat{\alpha} \equiv \frac{1}{2} \text{tr} \left( F^{\mu \perp} \pm i \tilde{F}^{\mu \perp} \right) (i \partial^+)^j P_j^{(2,1)} \left( \mathcal{D}^+/\partial^+ \right) \bar{\lambda}^A. \]  

(41)

It is convenient to introduce into our consideration a projection of the latter four operators in Eq. (41) onto Pauli matrices. Namely, one defines

\[ [\bar{\Omega}^{\gamma q \mu}]_{\alpha A} \equiv \bar{\sigma} \lambda^A \gamma_{\alpha \beta} [\bar{\Omega}^{\gamma q \mu}]_{\beta}, \]

\[ [\bar{\Omega}^{\gamma q \mu}]_{\hat{\alpha} A} \equiv \sigma^A \gamma_{\alpha \beta} [\bar{\Omega}^{\gamma q \mu}]_{\beta}. \]  

(42)

The remaining two projections of the maximal-helicity fermionic operators vanish identically;

\[ \bar{\sigma} \frac{1}{2} [\bar{\Omega}^{\gamma q \mu}]_{\beta} = 0, \quad \sigma^A \frac{1}{2} [\bar{\Omega}^{\gamma q \mu}]_{\beta} = 0. \]  

(43)

To summarize, in this section we used the collinear \( SL(2) \) subgroup of the full \( SO(4, 2) \) conformal group together with the internal \( SU(4) \) symmetry to classify all possible twist-two operators with respect to irreducible representations of the two. The operators with the same \( SU(4) \) charge and conformal spin mix under renormalization and our goal is to diagonalize the corresponding mixing matrix making use of supersymmetry.

### 3 Building the supermultiplet

In the previous section, we have exploited the conformal invariance to construct the complete set of twist-two operators with a definite conformal spin \( j \). Going from QCD to its supersymmetric extensions, one can derive constraints on the properties of these operators. The reason for this is that supersymmetry transformations relate elementary primary fields of different conformal spins and, therefore, they lead to relations between various conformal operators. As was already explained, multiparticle operators of minimal twist are constructed solely from the “good” field components. Therefore, out of the full supersymmetry algebra of the \( \mathcal{N} = 4 \) super-Yang-Mills, described in appendix A, the analysis of such operators requires only a subalgebra for components projected on the light cone. This implies that for multiparticle operators of the lowest twist, the total superconformal group \( SU(2, 2|4) \) is reduced down to its subgroup \( SU(1, 1|4) \sim SL(2|4) \), which can be regarded as a supersymmetric extension of the collinear \( SL(2)-\)subgroup of the four-dimensional conformal group.

In the \( \mathcal{N} = 4 \) model, the supersymmetry transformations \([A.6]\) mix “good” and “bad” components of the primary fields. To project the transformation of the “good” components only, one has to impose the following constraint on the fermionic transformation parameter

\[ \xi_+ \equiv \frac{1}{2} \bar{\sigma}^{-\gamma} \sigma^+ \gamma^\gamma \xi_\gamma \equiv 0, \quad \bar{\xi}_+ = \frac{1}{2} \bar{\sigma}^{\hat{\gamma}} \sigma^+ \hat{\gamma} \xi_\hat{\gamma} = 0. \]  

(44)
which corresponds to $\frac{1}{2} \gamma^{-} \gamma^{+} \xi = 0$ in four-component notations so that $\xi = \xi_{-}$. From Eqs. (A.6) we find for the “good” components of fields, in the $SU(4)$ covariant form, the following rules

$$
\delta A_{\perp}^{+} = - i \xi^{A} A_{\perp}^{+} \sigma_{\perp}^{A} \lambda_{+}^{A} \lambda_{+}^{A},
\delta \phi^{AB} = - i \sqrt{2} \{ \xi^{A} \lambda_{+}^{B} - \xi^{A} \lambda_{+}^{A} - \varepsilon^{ABCD} \xi_{C} \lambda_{+}^{D} \},
\delta \lambda_{+}^{A} = - F^{+\perp}_{\mu} \sigma_{-}^{\mu} \lambda_{+}^{A} \lambda_{+}^{A} - \sqrt{2} (\partial^{+} \phi^{AB}) \sigma_{-}^{\beta} \lambda_{+}^{A},
\delta \lambda_{+}^{A} = - F^{+\perp}_{\mu} \sigma_{\perp}^{\beta} \lambda_{+}^{A} \lambda_{+}^{A} + \sqrt{2} (\partial^{+} \phi^{AB}) \sigma_{-}^{\alpha} \lambda_{+}^{A}.
$$

(45)

Note that $\delta A^{+} = 0$ due to Eq. (44) so that $\delta F^{+\perp}_{\mu} = D^{+} \delta A^{+}_{\mu}$. The following comments are in order.

In the light-cone gauge $A^{+} = 0$, the transformations (45) form the off-shell supersymmetry algebra, i.e., without gauge transformations, needed in a generic case (A.6) to bring the multiplet back to the Wess-Zumino gauge, and without the equations of motion. Namely,

$$
[\delta_{2}, \delta_{1}] X = 2 i \left\{ \xi^{A} A_{\perp}^{+} \sigma_{\alpha} \xi_{1}^{A} + \xi_{2}^{A} A_{\perp}^{+} \sigma_{\alpha} \xi_{1}^{A} \right\} \partial^{+} X,
$$

(46)

for $X = \{ A^{+}_{\perp}, \lambda_{+}^{A}, \lambda_{+}^{A}, \phi^{AB} \}$. In case the light-cone gauge is lifted, one has to replace $\partial^{+} A^{+}_{\perp} \rightarrow F^{+\perp}_{\perp}$ and $\partial^{+} \lambda_{+}^{A} \rightarrow D^{+} \lambda_{+}^{A}$ with identical substitution rules for $\lambda_{+}^{A}$ and $\phi^{AB}$.

Since the light-cone supersymmetry transformations are linear — they do not increase the number of fields — the set of quasipartonic operators is closed. Therefore, examining the transformation properties of such operators under the transformations (45) one can construct supermultiplets belonging to an irreducible representation of the light-cone super-algebra. A unique property of conformal operators entering the supermultiplets is that they diagonalize the dilatation operator and, therefore, have an autonomous scale dependence.

The procedure of constructing supermultiplets of conformal operators is straightforward [38, 35, 39, 40] once an appropriate component of the supermultiplet is chosen. The right choice would be any operator which renormalizes autonomously. Applying the light-cone supersymmetric transformations (45) to such an operator, one can reconstruct the remaining entries of the supermultiplet and, therefore, deduce automatically the combinations of conformal operators which are the eigenfunctions of the anomalous dimension matrix (see Ref. 13 for the BNM sector).

There are many candidates for this component. This can be judged on the grounds of unique quantum numbers either with respect to the Lorentz or isotopic groups. For definiteness we choose the conformal operator $O^{* s s}_{j k}$ defined in [35], but one could equally have taken instead any of the operators given in Eq. (51) or (53) below. We emphasize that the construction of the supermultiplet works to all orders in the coupling constant provided the superconformal symmetry is not broken on the quantum level. This is expected to be the case in $\mathcal{N} = 4$ super-Yang-Mills theory, however, potential complications with the explicit implementation of regularization procedures in perturbative computations will be postponed until section 4.
3.1 Components of the supermultiplet

The $\mathcal{N} = 4$ super-light-cone algebra is closed in the basis spanned by the bilinears introduced in section 2. As we just said, choosing $\mathcal{O}^{ss,20}_{jl}$ as a supersymmetric primary, the remaining components of the multiplet are deduced as its descendants. This is demonstrated by an explicit calculation in appendix C. The result of our analysis is represented by the diagram in Fig. 1, where the arrows indicate the super-variation of the corresponding conformal operator, and the operators within one step of the diagram arise as its supersymmetric descendants.

All components of the supermultiplet can be separated into bosonic and fermionic operators. Within each set, the operators are classified according to their $SU(4)$ charge. In addition, one has to consider separately the cases when the conformal spin $j$ is even/odd: therefore, in what follows the $[\pm]$ subscript denotes even (+) or odd (−) $j$’s.

- The bosonic components of the supermultiplet are the following:

\[
S_{jl}^1 = 6\mathcal{O}^{gg}_{jl} + \frac{j}{4}\mathcal{O}^{qq}_{jl} + \frac{j(j+1)}{4}\mathcal{O}^{ss}_{jl}, \\
S_{jl}^2 = 6\mathcal{O}^{gg}_{jl} - \frac{1}{4}\mathcal{O}^{qq}_{jl} - \frac{(j+1)(j+2)}{12}\mathcal{O}^{ss}_{jl}, \\
S_{jl}^3 = 6\mathcal{O}^{gg}_{jl} - \frac{j+3}{2}\mathcal{O}^{qq}_{jl} + \frac{(j+2)(j+3)}{4}\mathcal{O}^{ss}_{jl}, \\
P_{jl}^1 = 6\bar{\mathcal{O}}^{gg}_{jl} + \frac{j}{4}\bar{\mathcal{O}}^{qq}_{jl}, \\
P_{jl}^2 = 6\bar{\mathcal{O}}^{gg}_{jl} - \frac{j+3}{4}\bar{\mathcal{O}}^{qq}_{jl}, \quad (47)
\]

for the $SU(4)$ singlets, and

\[
[T_{[+jl]}^1]_{AB}^{\mu} = 2(j+1)[\bar{T}^{sg}_{jl}]_{AB}^{\mu} - [T^{qq,6}_{jl}]_{AB}^{\mu}, \\
[T_{[+jl]}^2]_{AB}^{\mu} = 2(j+2)[\bar{T}^{sg}_{jl}]_{AB}^{\mu} + [T^{qq,6}_{jl}]_{AB}^{\mu}, \\
[T_{[-jl]}^3]_{AB}^{\mu} = [T^{sg}_{jl}]_{AB}^{\mu}, \\
[T_{[+jl]}^1]_{AB}^{\mu} = 2(j+1)[T^{sg}_{jl}]_{AB}^{\mu} + [T^{qq,6}_{jl}]_{AB}^{\mu}, \\
[T_{[+jl]}^2]_{AB}^{\mu} = 2(j+2)[T^{sg}_{jl}]_{AB}^{\mu} - [T^{qq,6}_{jl}]_{AB}^{\mu}, \\
[T_{[-jl]}^3]_{AB}^{\mu} = [T^{sg}_{jl}]_{AB}^{\mu}, \quad (48)
\]

for the 6 and $\bar{6}$, and

\[
[S_{jl,15}^{1,15}]_{A}^{B} = (j+2)[\mathcal{O}_{jl,15}^{ss,15}]_{A}^{B} - [\mathcal{O}_{jl,15}^{qq,15}]_{A}^{B}, \\
[S_{jl,15}^{2,15}]_{A}^{B} = (j+1)[\mathcal{O}_{jl,15}^{ss,15}]_{A}^{B} + [\mathcal{O}_{jl,15}^{qq,15}]_{A}^{B}, \quad (49)
\]

\footnote{Notice that scalars do not contribute to the odd operators $P_{jl}^\mu$ since scalar operators have the opposite parity.}
Figure 1: A diagram representing the supermultiplet of twist-two conformal operators in $\mathcal{N} = 4$ super-Yang-Mills theory. The Lorentz, flavor and conformal-spin indices are omitted for brevity. For explicit transformation rules, see appendix C.5.

for the 15. Finally, the remaining bosonic operators, which have unique quantum numbers and, as a consequence, renormalize autonomously, are

$$ [T^{gg}_{jl}]^{\mu\nu}, \quad [\bar{T}^{gg}_{jl}]^{\mu\nu}, \quad [T^{qq,10}_{jl}]^{\mu AB}, \quad [\bar{T}^{qq,10}_{jl}]^{\mu AB}, \quad [\bar{\Omega}^{qq,15}_{jl}]^{AB}, \quad [\mathcal{O}^{ss,20}_{jl}]^{CD} . $$

(50)

- The fermionic components of the supermultiplet read as:

$$ [\Omega_{[+]jl}]^{A} = 3[\Omega^{gg}_{jl}]^{A} + [\Omega^{sq}_{jl}]^{A}, $$

$$ [\Omega_{[+]jl}]^{A} = (j + 3)[\Omega^{gg}_{jl}]^{A} - (j + 1)[\Omega^{sq}_{jl}]^{A}, $$

$$ [\Omega_{[-jl}]^{A} = 3(j + 3)[\Omega^{gg}_{jl}]^{A} + (j + 1)[\Omega^{sq}_{jl}]^{A}, $$

$$ [\bar{\Omega}_{[-jl}]^{A} = [\Omega^{gg}_{jl}]^{A} - [\Omega^{sq}_{jl}]^{A}, $$

(51)

and

$$ [\bar{\Omega}_{[+]jl}]^{A} = 3[\bar{\Omega}^{gg}_{jl}]^{A} - [\bar{\Omega}^{sq}_{jl}]^{A}, $$

$$ [\bar{\Omega}_{[+]jl}]^{A} = (j + 3)[\bar{\Omega}^{gg}_{jl}]^{A} + (j + 1)[\bar{\Omega}^{sq}_{jl}]^{A}, $$

$$ [\bar{\Omega}_{[-jl}]^{A} = 3(j + 3)[\bar{\Omega}^{gg}_{jl}]^{A} - (j + 1)[\bar{\Omega}^{sq}_{jl}]^{A}, $$

$$ [\bar{\Omega}^{2}_{[+]jl}]^{A} = [\bar{\Omega}^{gg}_{jl}]^{A} + [\bar{\Omega}^{sq}_{jl}]^{A}, $$

(52)

together with the remaining operators, which renormalize independently;

$$ [\bar{\Omega}^{gg}_{jl}]^{\mu A}, \quad -[\Omega^{gg}_{jl}]^{\hat{A} A}, \quad [\bar{\Omega}^{sq,20}_{jl}]^{\hat{A} AB}, \quad [\Omega^{sq,20}_{jl}]^{C}_{A AB}. $$

(53)
The following comments are in order: The result (47) for the operators $S_{jj}^k$ and $P_{jj}^k$ coincides, modulo multiplicative factors in front of $O_{jl}^{ss}$, with the ones of Ref. [36]. To perform the comparison, one has to take the “forward limit” of the Wilson operators. This amounts to neglecting contributions involving total derivatives and leads to

$$n_j^{-1}O_{jj}^{ss} \to \sigma_j \text{tr} \bar{\lambda}_A \sigma^+ \bar{\alpha} \beta (iD^+) j \lambda^A_j,$$

$$n_j^{-1}O_{jj}^{gg} \to \frac{j}{6} \text{tr} F^\perp \mu \nu (iD^+) j^{-1} F^\perp \nu,$$

$$n_j^{-1}O_{jj}^{ss} \to \frac{2}{j+1} \text{tr} \bar{\phi}_{AB} (iD^+) j+1 \phi^{AB},$$

where $n_j \equiv \Gamma(2j+2)/\Gamma^2(j+1)$. Analogous relations hold for the parity-odd sector.

The operators introduced in this subsection carry the same charges $j$ and $l$ which define their transformation properties with respect to the collinear $SL(2)$ subgroup. Note that for given $j$ and $l$ the bosonic and fermionic components of the supermultiplet have different conformal spins: $J = j + 2$ and $J = j + 5/2$, respectively. The same holds true for their canonical dimensions. For instance, the canonical dimension of the operators $O_{jl}^{ss,20}$ and $\Omega_{[-jl]}^{sq,20}$ is $j + 3$ and $j + 7/2$, respectively.

### 3.2 Moving along the multiplet

The relations between various twist-two conformal operators entering the supermultiplet in $\mathcal{N} = 4$ theory can be deduced from Fig. 1. As an example of a specific path which allows one to move along the multiplet from, say, the left upper corner of the graph to its right bottom corner, is achieved in four steps, in accordance with the number of the light-cone supersymmetries,

$$O_{jl}^{ss,20} \longrightarrow \Omega_{[-jl]}^{sq,20} \longrightarrow \mathcal{T}_{[+jl]}^2 \longrightarrow \Omega_{[+jl]}^2 \longrightarrow S_{jl}^1.$$  \hspace{1cm} (54)

The corresponding sequence of supersymmetric transformations looks like

$$\delta[O_{jl}^{ss,20}]_{CD}^{AB} = -\frac{1}{3} \frac{j+2}{2j+3} [P_{20}]_{CD;EF}^{AB;GH} \xi^\alpha [\Omega_{[-jl]}^{sq,20}]^H_{\alpha EF} + \ldots,$$

$$\delta[\Omega_{[-jl]}^{sq,20}]_{\alpha AB}^C = \frac{1}{4} \frac{j+2}{2j+3} [P_{20}]_{CD;EF}^{AB;GH} \xi^\alpha [\Omega_{[-jl]}^{sq,20}]^H_{\alpha EF} + \ldots,$$

$$\delta[\mathcal{T}_{[+jl]}^2]_{\mu AB} = -\frac{2}{2j+3} \xi^\alpha [\Omega_{[+jl]}^2]^\mu_{\alpha \beta} [\Omega_{[+jl]}^2]^\beta_{\alpha AB} + \ldots,$$

$$\delta[\Omega_{[+jl]}^2]_{\alpha A} = -\frac{1}{3} \frac{j+2}{2j+3} [S_{jl}^1]^\alpha_\beta \xi^\beta + \ldots.$$  \hspace{1cm} (55)

where the ellipses denote contributions of other conformal operators; see appendix C.5 for details.

Notice that the parameter of the supersymmetric transformations has the scaling dimension $(-1/2)$ and, therefore, the operators entering both sides of these relations have scaling dimensions which
differ by the same amount. The generators of supersymmetric transformations do not commute with the Casimir of the collinear subgroup $L^2$, Eq. (15), and, as a consequence, the conformal spin of the operators entering the right-hand side of (55) differ by $\pm 1/2$.

Examining the diagram shown in Fig. 1, one deduces the following remarkable property of the $\mathcal{N} = 4$ super-Yang-Mills theory. The fact that the diagram is simply connected implies that all two-particle (quasipartonic) conformal operators are unified into a single supermultiplet. This should be compared with the $\mathcal{N} = 1$ super-Yang-Mills theory \[35, 37, 38, 39, 40\]. In that case, a similar diagram has two disconnected components and, as a consequence, the two-particle conformal operators form two different supermultiplets. One of these supermultiplets comprises the operators with aligned helicities of particles \[35, 37\] and, therefore, it inherits integrability properties discovered in QCD \[8, 9, 10\]. The last two equations in (55) explicitly demonstrate a distinguished feature of the $\mathcal{N} = 4$ super-Yang-Mills. As was schematically illustrated in Eq. (5) of the introduction, the $\mathcal{N} = 4$ supersymmetry relates the operators of aligned helicities, $[\bar{T}^{a}_{+jl}]_{AB} \sim [T^{qq}_{jl}]_{AB}$, to the ones built from fields of opposite helicities, $S^{\dagger}_{jl} \sim O^{qq}_{jl}$. Since the operators entering the supermultiplet have the same properties with respect to the dilatation transformations, we conclude that going from QCD to $\mathcal{N} = 1$ and, finally, to $\mathcal{N} = 4$ theory, the integrability in the subsector of maximal-helicity operators gets extended to the entire sector of quasipartonic operators.

## 4 Anomalous dimension of the supermultiplet

If the superconformal symmetry is not broken by radiative corrections, then the components of the supermultiplet have an autonomous scale dependence and their anomalous dimensions are defined by a single function of the conformal spin $j$. To calculate this function, one has to regularize the theory in such a way that its symmetries are preserved. Unfortunately, a practical implementation of such perturbative regularization procedure does not exist. The most prominent candidate — the dimensional reduction \[44\] — preserves supersymmetry to rather high orders of perturbation theory, but breaks conformal boosts starting from two loops \[45, 39, 40\]. The reason for this is that the coupling constant acquires a nontrivial scaling dimension in the space-time with $d \neq 4$ leading to a modification of the beta-function $\beta_d(g) = \frac{1}{2}(d - 4)g + \beta(g)$. Thus, away from four dimensions, the conformal invariance is destroyed even when $\beta(g) = 0$. As a consequence, the subtraction of divergences via the conventional minimal subtraction procedure induces the mixing of conformal operators of different conformal spin starting from two loops, but still the conformal symmetry can be restored by performing a finite renormalization of conformal operators \[45\]. This transformation does not affect the eigenvalues of the mixing matrix, or equivalently, the anomalous
The dimension of the supermultiplet, but it induces a finite scheme transformation of its eigenstates.

The analysis of the present section relies on a tacit assumption that there exists a subtraction scheme which preserves the superconformal covariance. Then within this superconformal scheme the anomalous dimensions of components of the supermultiplet are equal to each other. Note that the supersymmetry transformation induces a shift ±1/2 in the conformal spin as we climb/descend the supersymmetric tower of states — here, conformal operators with different quantum numbers. As a consequence, the anomalous dimensions of the components are given by the same function with the argument shifted at most by four units of the conformal spin.

Let us denote the anomalous dimension of the operator $O_{ss}^{20}$ as $\gamma(j)$. Then, going through the diagram in Fig. 1 and inspecting the corresponding supersymmetric transformations presented in appendix C.5, one uniquely fixes the anomalous dimensions of all components of the supermultiplet (see Table 1).

### 4.1 Fixing the anomalous dimension of the supermultiplet

To determine the anomalous dimensions of the supermultiplet it suffices to calculate the anomalous dimension of one of its components. The simplest choice would be to consider either one of the operators displayed in Eqs. (50) and (53), or the operators $[T_{[-j]AB}^3]_{\mu}$ and $[\tilde{T}_{[-j]AB}^3]_{\mu}$ for odd $j$’s. Actually, for two such operators — the maximal-helicity gluonic or quark operators, Eqs. (31) and (26), respectively — one can immediately borrow corresponding one-loop QCD results to fix $\gamma(j)$. This would merely require an adjustment of color factors with no additional calculation of Feynman diagrams. Beyond leading order one has to add a few extra graphs involving scalars propagating in loops to the existing QCD calculations and transform them to the supersymmetry preserving dimensional reduction scheme.

However, in order to explicitly demonstrate that the components of the supermultiplet have an autonomous scale dependence, we will compute the one-loop anomalous dimension mixing matrix
for the operators $[T_{jl}^{qq}]_{AB}^{\mu}$ and $[\bar{T}_{jl}^{6}]_{AB}^{\mu}$. We will show that the eigenstates of this matrix give rise to the components $[\bar{T}_{jl}^{1}]_{AB}^{\mu}$ and $[\bar{T}_{jl}^{2}]_{AB}^{\mu}$ of the superconformal operator for even $j$, Eq. (18), while for odd $j$ we get the operator $[\bar{T}_{jl}^{3}]_{AB}^{\mu}$ which renormalizes autonomously.

Instead of dealing with an infinite tower of local conformal operators, we will work with non-local light-cone operators

\begin{align}
T_{\pm}^{\mu}(\xi_1, \xi_2) &\equiv \frac{i}{2\sqrt{2}} \text{tr} \left\{ \left( F^{\pm\mu} + i \tilde{F}^{\pm\mu} \right)(\xi_2) \phi_{AB}(\xi_1) \pm (\xi_1 \leftrightarrow \xi_2) \right\}, \\
T^{qq,6}(\xi_1, \xi_2) &\equiv \varepsilon_{ABCD} \text{tr} \left\{ \lambda^{\alpha C}(\xi_2) \sigma^+_{\alpha\beta} \sigma^+_{\beta\gamma} \lambda^D_{\gamma}(\xi_1) \right\},
\end{align}

where we have suppressed the Lorentz and isotopic indices on the left-hand side and have tacitly assumed the light-cone gauge $A^+ = 0$ and, therefore, neglected the light-like Wilson lines stretched between the elementary fields. These nonlocal operators serve as generating functions of local conformal operators $[T_{jl}^{qq,6}]_{AB}^{\mu}$ and $[\bar{T}_{jl}^{6}]_{AB}^{\mu}$ when expanded in the Taylor series (18).

The computation of the dilatation operator in the basis (56) and (57) has a number of advantages. Firstly, the renormalization acquires a very concise form and has a transparent meaning in the coordinate space; secondly, the conformal properties of the dilatation operator become manifest [2].

The non-local operators (56) and (57) possess additional singularities due to the light-light nature of separations between fields. To regularize them one uses the dimensional regularization and

Figure 3: One-loop diagrams for diagonal $T^{qq,6} \rightarrow T^{qq,6}$ transitions. The self-energy diagrams are multiplied by the combinatoric factor $1/2$. The solid line represent the gaugino.
Figure 4: One-loop transitions changing the particle content of operators and contributing to the off-diagonal elements of the mixing matrix.

Figure 4: One-loop transitions changing the particle content of operators and contributing to the off-diagonal elements of the mixing matrix.

subtracts divergences in the \( \overline{\text{MS}} \)-scheme. Due to the different parity under \( \xi_1 \leftrightarrow \xi_2 \), the operator \( T_{sg}^{\xi_1, \xi_2} \) cannot mix with the remaining two operators and, therefore, evolves independently. It is convenient to combine the parity-even operators \( \bar{T}_{sg}^{\xi_1, \xi_2} \) and \( T_{qq} \) into a two-vector

\[
\bar{T}_{[\pm]} \equiv \begin{pmatrix} T_{qq} \\ \bar{T}_{sg}^{\xi_1, \xi_2} \end{pmatrix}.
\]

Let us discuss both cases in turn.

• **Parity-even operators**: The one-loop mixing matrix for the operators is given by Feynman diagrams displayed in Figs. 2, 3 and 4. A computation, whose detailed account can be found in appendix D, leads to the following result for the \( \overline{\text{MS}} \)-subtracted operator

\[
\bar{T}_{[\pm]}(\xi_1, \xi_2) = \int_0^1 dy \int_0^1 dz \theta(1 - y - z) \delta(y) \delta(z) \left[ 1 + \frac{g^2 N_c}{8 \pi^2} S_{\varepsilon} K(y, z) \right] \bar{T}_{[\pm]}(\bar{y} \xi_1 + y \xi_2, z \xi_1 + \bar{z} \xi_2),
\]

with \( S_{\varepsilon} \equiv \exp(\varepsilon(\gamma - \ln 4\pi)) \). The integral kernel of the one-loop dilatation operator is given by

\[
K(y, z) = \begin{pmatrix} \bar{y} \left[ 1/y \right]_+ & 4 \bar{y} \\ 1/4 & (1 + y) \left[ 1/y \right]_+ \end{pmatrix} \delta(z) + \begin{pmatrix} \bar{z} \left[ 1/z \right]_+ & 4 \bar{z} \\ 1/4 & \bar{z}^2 \left[ 1/z \right]_+ \end{pmatrix} \delta(y),
\]

with \( \bar{y} \equiv 1 - y \) and analogously for other variables. Here and below the plus-regularization is defined conventionally as \( [1/y]_+ \equiv 1/y - \delta(y) \int_0^1 dy'/y' \).

Equation (59) has a simple physical meaning: under the renormalization group flow the fields are displaced on the light-cone in the direction toward each other. One can verify that (59) is invariant under the projective transformations on the light-cone (6) and, as a consequence, the operator \( K \) commutes with the generators of the collinear \( SL(2) \) subgroup. The above integral representation for the one-loop dilatation operator is diagonalized in the basis \( \bar{T}_{[\pm]}(\xi_1, \xi_2) \sim (\xi_1 - \xi_2)^j \). Going from the coordinate to the momentum representation, one can show (see Eq.

\[8\] E.g., the operator, which generates finite Green functions.
that the dilatation operator admits the conformal partial-wave expansion (D.22) making
the diagonalization property manifest.

Let us expand the non-local light-cone operator $\bar{T}_R^R(\xi_1, \xi_2)$ in the Taylor series over local conformal operators [18] and, then, substitute it into (59). Due to the positive parity, the expansion will involve only conformal operators with even conformal spins and each term of the expansion will diagonalize the operator $K$. In this way, one obtains the renormalization group equation

$$\frac{d}{d\ln \mu} [\bar{T}_R^R(\pm j)]_{AB}^{\mu} = -\frac{g^2 N_c}{8\pi^2} \gamma_j^{(0)} [\bar{T}_R^R(\pm j)]_{AB}^{\mu} + O(g^4),$$  \hspace{1cm} (61)

where we have extracted the color factor from the anomalous dimension matrix, cf. (20),

$$\gamma_j^{(0)} = \begin{pmatrix} 4\psi(j + 2) + 4\gamma_E & -8 \\ -\frac{2}{(j+1)(j+2)} & 4\psi(j + 2) + 4\gamma_E - \frac{4}{(j+1)(j+2)} \end{pmatrix}. \hspace{1cm} (62)$$

The eigenvectors of this matrix define two conformal operators $[\bar{T}_R^1]_{AB}^{\mu}$ and $[\bar{T}_R^2]_{AB}^{\mu}$, Eq. (48), and the corresponding eigenvalues determine their anomalous dimensions

$$[\bar{T}_R^1]_{AB}^{\mu} : \hspace{0.5cm} \gamma_j^{(0)} \equiv \gamma^{(0)}(j + 1) = 4\psi(j + 3) + 4\gamma_E,$$

$$[\bar{T}_R^2]_{AB}^{\mu} : \hspace{0.5cm} \gamma_j^{(0)} \equiv \gamma^{(0)}(j - 1) = 4\psi(j + 1) + 4\gamma_E,$$

respectively.

- **Parity-odd operators**: The non-local light-cone operator $[\bar{T}_{sg}^s]_{AB}^{\mu}(\xi_1, \xi_2)$ has an autonomous scale dependence. It satisfies an evolution equation analogous to (59) with the evolution kernel given by

$$\mathbb{K}(y, z) = \bar{y} [1/y]_+ \delta(z) + \bar{z}^2 [1/z]_+ \delta(y). \hspace{1cm} (64)$$

Analogous to the previous consideration, one expands $\bar{T}_{sg}^s(\xi_1, \xi_2)$ over local conformal operators and calculates their anomalous dimension with the result

$$[\bar{T}_{sg}^s]_{AB}^{\mu} : \hspace{0.5cm} \gamma_{j,III}^{(0)} \equiv \gamma^{(0)}(j) = 4\psi(j + 2) + 4\gamma_E. \hspace{1cm} (65)$$

Here we used the notation for the component of the multiplet [18]. The obtained eigensystem [18] and (65) is in perfect agreement with Table II.

To summarize, we demonstrated in this section that the anomalous dimensions of different components of the supermultiplet are determined by the same function of the conformal spin $j$. To one-loop accuracy, this function $\gamma^{(0)}(j)$ is expressed in terms of the Euler $\psi$-function, $\psi(j + 2) = \sum_{n=1}^{j+1} \frac{1}{n} - \gamma_E$ [46]. Making use of the recent two-loop calculation of the anomalous dimensions of $SU(4)$-singlet operators [36] one can fix the two-loop correction to the anomalous dimension of the supermultiplet

$$\gamma(j) \equiv \frac{g^2 N_c}{8\pi^2} \gamma^{(0)}(j) + \left(\frac{g^2 N_c}{8\pi^2}\right)^2 \gamma^{(1)}(j) + O(g^6), \hspace{1cm} (66)$$
where

\[ \gamma^{(0)}(j) = 4 \sum_{n=1}^{j+1} \frac{1}{n}, \]

\[ \gamma^{(1)}(j) = -4 \sum_{n=1}^{j+1} \left\{ \frac{\sigma_n}{n^3} + \frac{2}{n} \sum_{m=1}^{j+1} \frac{1}{m^2} \left( 1 + (-1)^m \theta_{n+1,m} \right) \right\}. \]  

(67)

Here \( \sigma_n \) stands the signature factor \(^{21}\) and the step-function is \( \theta_{n,m} = \{1, n > m; 0, n \leq m\} \).

5 Discussion and conclusions

In this paper, we have studied the constraints imposed by supersymmetry on renormalization properties of twist-two operators. These operators involve only “good” components of the fundamental fields and, as a consequence, they form a closed set with respect to the action of the collinear \( SL(2) \) subgroup of the full conformal group. In conformal theories, the operators that carry the same conformal spin mix under renormalization, and diagonalizing the corresponding mixing matrix one constructs their linear combinations which have an autonomous scale dependence. Being solutions to characteristic (polynomial) equations, the anomalous dimensions \( \gamma(j) \) are, in general, multi-valued functions of the conformal spin \( j \) in a Yang-Mills theory. These properties are generic and do not rely on supersymmetry.

Supersymmetry has the following remarkable consequences. Firstly, it allows one to classify all conformal operators with respect to the action of superconformal transformations of the collinear \( SL(2|4) \) subgroup of the full \( SU(2,2|4) \) superconformal group of the \( N = 4 \) SYM theory. As was shown in section 3.1, the coefficients defining relative weights of conformal operators in the expressions for the \( N = 4 \) superconformal operators have a very simple form. They can be interpreted as Clebsch-Gordan coefficients in the tensor product of two irreducible representations of \( SL(2|4) \). Secondly, the superconformal operators entering the same supermultiplet have the same scaling dimension \( \gamma(j) \). In distinction with a general case of conformal Yang-Mills theory, it is a single-valued, meromorphic function of the conformal spin \( j \). In the \( N = 4 \) theory, the one-loop anomalous dimension is given by the Euler \( \psi \)-function of the conformal Casimir while the two-loop corrections involve its generalizations, Eq. 66. Thirdly, we have demonstrated that a unique feature of the \( N = 4 \) SYM is that all quasipartonic operators with different \( SU(4) \) quantum numbers fall into a single supermultiplet. Among them there is a subsector of the operators of maximal helicity, Eqs. 26 and 31, which plays a special role. Independently on the presence of supersymmetry, the dilatation operator in conformal Yang-Mills theory exhibits a hidden symmetry in this sector in the multi-color limit — it is equivalent to a Hamiltonian of integrable \( SL(2) \) Heisenberg spin magnet 7, 8, 9, 10, 11. In the \( N = 4 \) SYM theory, this symmetry
is extended to the whole supermultiplet of quasipartonic operators with the only difference that the collinear $SL(2)$ group is replaced by its supersymmetric extension $SL(2|4)$, which is a subsector of the $PSU(2, 2|4)$-symmetric one-loop dilatation operator defined in Refs. [21, 21].

So far we have discussed twist-two conformal operators — two-particle blocks of multiparticle operators. The above consideration can be repeated to construct multi-particle superconformal operators involving “good” components but calculations become rather tedious. Our findings can be re-expressed in a concise form if one combines “good” components of fields into a light-cone superfield $\Phi(\xi, \theta^A)$. To obtain the supermultiplet of conformal operators constructed in section 3.1, one has to examine the operator expansion of the product of two superfields $\Phi(\xi, \theta^A)\Phi(0, 0)$ and expand it over irreducible components similar to (18). Each component of the operator product expansion evolves autonomously under dilatations and has the anomalous dimension $\gamma(j)$ with $j$ defined by quadratic Casimir of the $SL(2|4)$ group. As a consequence, the one-loop dilatation operator in the $\mathcal{N} = 4$ super-Yang-Mills acts on the product of two superfield as a two-particle Hamiltonian of the $SL(2|4)$ Heisenberg spin magnet. As the next step we consider the product of superfields $\prod_{k=1}^{L} \Phi(\xi_k, \theta_k^A)$. For this subclass of operators the particle creation and annihilation is absent, so that the number of sites $L$ is conserved. In the multi-color limit, the dilatation operator has a two-particle structure and defines a Hamiltonian of the integrable $SL(2|4)$ spin chain with $L$ sites. The multi-particle superconformal operators and their anomalous dimensions can be read from the energy spectrum of this spin chain.

Integrability of particle interactions implies stringent restrictions of the spectrum of anomalous dimensions in gauge theories at weak coupling. It is expected that integrability is a genuine property of the $\mathcal{N} = 4$ super-Yang-Mills and, therefore, it has also to reveal itself at strong coupling. Within the gauge/string correspondence, the operators with large conformal spin are dual to semiclassical stringy states with a large angular momentum on $AdS_5$. This allows to establish a correspondence between asymptotic behavior of anomalous dimensions of two-particle operators with large conformal spin at strong coupling and the energy of a long folded revolving string [32, 33, 34]. For multiparticle operators the spectrum of the anomalous dimensions at strong coupling was calculated in [23] using their intrinsic relation with a cusp anomaly of Wilson loops. This correspondence has been employed in earlier analyses of twist-two operators in Ref. [17]. It was conjectured in [23] that the string configuration, corresponding to higher-twist operators, possesses string junctions, e.g., a three-particle operator is expressed by a three-string, to match them to conserved charges of the gauge theory [23]. A more precise identification of string configurations still awaits deeper exploration.

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A \( \mathcal{N} = 4 \) super-Yang-Mills

In order to set our conventions, we perform a textbook exercise of reducing the ten-dimensional \( \mathcal{N} = 1 \) super-Yang-Mills down to the four-dimensional Minkowski space with the result of getting the \( \mathcal{N} = 4 \) supersymmetric theory \cite{12, 13}. We will adopt the \( SU(4) \) covariant form of Refs. \cite{48, 49} adjusted to the presently used nomenclature of the Dirac algebra in four dimensions which is discussed in the subsequent appendix \cite{13}. One starts with

\[
\mathcal{L}_{10} = \text{tr} \left\{ -\frac{1}{2} F_{MN} F^{MN} + i \bar{\Psi} \Gamma_M D^M \Psi \right\},
\]

where the field strength is \( F_{MN} = \partial_M A_N - \partial_N A_M - i g [A_M, A_N] \) and the covariant derivative correspondingly is \( D_M = \partial_M - i g [A_M, \cdot] \). All fields are matrix-valued in the \( SU(N) \) gauge group, i.e., \( \Phi \equiv \Phi^a t^a \) for any \( \Phi = \{ A_M, \bar{\Psi} \} \), with the generators normalized as \( \text{tr} t^a t^b = \frac{1}{2} \delta^{ab} \). Here \( \Psi \) is the Majorana-Weyl spinor, i.e., satisfying the conditions \( \Psi^T \Gamma^0 = \bar{\Psi} \) and \( \Gamma^{11} \Psi = \bar{\Psi} \), and the solution to them has the form

\[
\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \lambda^A_\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \bar{\lambda}^\alpha_A \end{pmatrix},
\]

(A.2)

with obviously \( (\lambda^A_\alpha)^* = \bar{\lambda}^\alpha_A \). The Lagrangian is a density with respect to supersymmetric transformation rules

\[
\delta \Psi = \frac{i}{2} F^{MN} \Gamma_{MN} \xi, \quad \delta A_M = -i \bar{\xi} \Gamma_M \Psi,
\]

(A.3)

i.e., \( \delta \mathcal{L} = \partial_M \Delta M \) with \( \Delta_M = \text{tr} \bar{\xi} \left\{ 2i F_{MN} \Gamma^N + \frac{1}{2} F_{N\Pi} \Gamma_{N\Pi} \Gamma_M \right\} \Psi \), where we introduced a matrix

\( \Gamma^{MN} \equiv \frac{1}{2} \left( \Gamma^M \Gamma^N - \Gamma^N \Gamma^M \right) \).

Reducing the ten-dimensional Lagrangian to four dimensions we get the standard maximally supersymmetric theory,

\[
\mathcal{L}_{N=4} = \text{tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \left( D_\mu \phi^{AB} \right) \left( D^\mu \bar{\phi}_{AB} \right) + \frac{1}{8} g^2 \left[ \phi^{AB}, \phi^{CD} \right] \left[ \bar{\phi}_{AB}, \bar{\phi}_{CD} \right] + 2i \bar{\lambda}_{\alpha A} \sigma_\mu^{\alpha\beta} D^\mu \lambda^B_\beta - \sqrt{2} g \lambda^{\alpha A} \left[ \bar{\phi}_{AB}, \lambda^B_\alpha \right] + \sqrt{2} g \bar{\lambda}_{\alpha A} \left[ \phi^{AB}, \bar{\lambda}^\alpha_A \right] \right\},
\]

(A.4)
where we have introduced the complex scalar field $\phi^{AB}$ related to the six real components of the ten-dimensional gauge field $A^a$ via

$$
\phi^{AB} = \frac{1}{\sqrt{2}} \Sigma^{aAB} A^a, \quad \bar{\phi}_{AB} = (\phi^{AB})^* = \frac{1}{2} \varepsilon_{ABCD} \phi^{CD} = \frac{1}{\sqrt{2}} \bar{\Sigma}^{aAB} A^a,
$$

where $\varepsilon_{ABCD} = \varepsilon^{ABCD}$. Here we have split the ten-dimensional index $M = \{\mu, a\}$ into four-dimensional $\mu = 0, 1, 2, 3$ and six-dimensional $a = 1, \ldots, 6$. The Lagrangian $\mathcal{L}_{N=4}$ is invariant under the transformation rules deduced from (A.3),

$$
\delta A^\mu = -i \xi^{\alpha} A_\alpha \overline{\lambda}_\beta - i \xi_{\dot{\alpha}} A^\alpha \lambda_\beta, \\
\delta \phi^{AB} = -i \sqrt{2} \left\{ \xi^{\alpha} A_\alpha B_{\beta} - \xi_{\dot{\alpha}} A^\alpha B_{\dot{\beta}} - \varepsilon^{ABCD} \xi_{\dot{\alpha}} C \lambda_D \right\}, \\
\delta \lambda_{\alpha} = i \frac{2}{F} F_{\mu \nu} \sigma^{\mu \nu} \lambda_{\alpha} - \sqrt{2} \left( D_\mu \phi^{AB} \right) \sigma_{\alpha \beta} \lambda_{\beta} + ig \left[ \phi^{AB}, \phi_{BC} \right] \xi_{\dot{\alpha}}, \\
\delta \lambda_{\dot{\alpha}} = i \frac{2}{F} F_{\mu \nu} \sigma^{\mu \dot{\nu}} \lambda_{\dot{\alpha}} + \sqrt{2} \left( D_\mu \phi^{AB} \right) \sigma_{\alpha \beta} \lambda_{\dot{\beta}} + ig \left[ \phi^{AB}, \phi^{BC} \right] \xi_{\dot{\alpha}}.
$$

(B.6)

B Dirac algebra in various dimensions

In this appendix we introduce representations of Clifford algebras in four, six and ten dimensions. Let us introduce them in turn.

- $D = 3 + 1$: The four-component spinor is composed from two Weyl spinors, transforming with respect to conjugated factors of the Lorentz group $L^1_+ = SO(3,1) = SO(4,\mathbb{C})_{1R} \approx (SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C}))_{1R}$ which are labeled by a pair $(j_1, j_2)$ of eigenvalues $j_1(j_1 + 1)$ of the $SL(2,\mathbb{C})$ Casimir operators $J^i_1$,

$$
\psi = \begin{pmatrix} \lambda_{\alpha} \\ \bar{\lambda}_{\dot{\alpha}} \end{pmatrix},
$$

(B.1)

with $\lambda_{\alpha} \sim (\frac{1}{2}, 0)$ and $\bar{\lambda}_{\dot{\alpha}} \sim (0, \frac{1}{2})$. The Dirac matrices admit the form

$$
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha \dot{\beta}} \\ \sigma^\mu_{\dot{\alpha} \beta} & 0 \end{pmatrix}, \quad \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

(B.2)

where $\sigma^\mu = (1, \sigma)$ and $\bar{\sigma}^\mu = (1, -\sigma)$ with the conventional vector of Pauli matrices $\sigma$. The Clifford algebra is $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu \nu}$, with the metric of signature $g^{\mu \nu} = \text{diag}(+,-,-,-)$ and reads for two-dimensional matrices

$$
\bar{\sigma}^\mu_{\alpha \gamma} \sigma^{\nu \gamma} \bar{\gamma}_\beta + \bar{\sigma}^{\nu \alpha \gamma} \sigma^{\mu \dot{\gamma}} \bar{\delta}_{\beta} = 2g^{\mu \nu} \delta^\beta_{\dot{\alpha}}, \quad \sigma^{\mu \dot{\alpha} \gamma} \bar{\sigma}^{\nu \gamma} \bar{\gamma}_\beta + \sigma^{\nu \gamma} \bar{\sigma}^\mu_{\dot{\alpha} \beta} = 2g^{\mu \nu} \delta^\beta_{\alpha}.
$$

(B.3)

The charge conjugation matrix is

$$
C_4 = i \gamma^2 \gamma^0 = \begin{pmatrix} -\varepsilon_{\alpha \beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha} \dot{\beta}} \end{pmatrix},
$$

(B.4)
where $\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta}$ and $\varepsilon^{12} = \varepsilon_{12} = -\varepsilon^i_2 = 1$. The raising and lowering of Weyl spinor indices is accomplished by the following set of rules

$$\lambda^\alpha = \varepsilon^{\alpha\beta} \lambda_\beta, \quad \bar{\lambda}_\dot{\alpha} = \varepsilon_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}}, \quad \lambda_\alpha = \lambda^\beta \varepsilon_{\beta\alpha}, \quad \bar{\lambda}^{\dot{\alpha}} = \bar{\lambda}_{\dot{\beta}} \varepsilon^{\dot{\beta}\alpha},$$

(B.5)
corresponding to conventions of Ref. [50]. While for the $\sigma$-matrices we have

$$\sigma^{\mu\dot{\alpha}\dot{\beta}} = \varepsilon^{\dot{\beta}\gamma} \sigma^{\mu\dot{\gamma}} \varepsilon^{\dot{\delta}\alpha}, \quad \bar{\sigma}_{\mu\alpha\dot{\beta}} = \varepsilon_{\beta\gamma} \sigma^{\mu\dot{\gamma}} \varepsilon_{\delta\alpha}.$$

(B.6)
The charge conjugation matrix obeys the relations $C_4^T = -C_4$, $C_4^2 = -1$ and $C_4 \gamma^\mu = - (\gamma^\mu)^T C_4$.

The Majorana condition for a four-component spinor $\psi^T C_4 = \psi^* \gamma^0 \equiv \bar{\psi}$ reads in terms of the Weyl components $(\bar{\lambda}^\alpha) = \lambda^\alpha$ and $(\lambda_\alpha)^* = \bar{\lambda}_\dot{\alpha}$. The sigma-matrices transform under the complex conjugation as follows:

$$\left(\sigma^{\mu\dot{\alpha}\dot{\beta}}\right)^* = \sigma^{\mu\dot{\beta}\alpha}, \quad \left(\bar{\sigma}_{\mu\alpha\dot{\beta}}\right)^* = \bar{\sigma}_{\mu\beta\dot{\alpha}}.$$

(B.7)

Note that $(\varepsilon_{\alpha\dot{\beta}})^* = -\varepsilon_{\alpha\beta}$. Any tensor can be expanded in the following basis of two-dimensional matrices $\{1, \sigma^\mu, \bar{\sigma}^{\mu}, \sigma^{\mu\nu}, \bar{\sigma}^{\mu\nu}\}$, where we introduced

$$\sigma^{\mu\nu\alpha\dot{\beta}} \equiv \frac{i}{2} \left[ \sigma^\mu_{\alpha\gamma} \sigma^{\nu\dot{\gamma}\dot{\beta}} - \sigma^\nu_{\alpha\gamma} \sigma^{\mu\dot{\gamma}\dot{\beta}} \right], \quad \bar{\sigma}^{\mu\alpha\dot{\beta}} \equiv \frac{i}{2} \left[ \sigma^\mu_{\dot{\gamma}\alpha} \bar{\sigma}^{\nu\dot{\gamma}} - \sigma^\nu_{\dot{\gamma}\alpha} \bar{\sigma}^{\mu\dot{\gamma}} \right],$$

(B.8)
which satisfy the following “self-duality” conditions

$$\frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma} = \sigma^{\mu\nu}, \quad \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\sigma}_{\rho\sigma} = -\bar{\sigma}^{\mu\nu}.$$

(B.9)

Note that $\sigma^{\mu\nu\alpha\dot{\beta}} = \varepsilon^{\beta\gamma} \sigma^{\mu\nu\gamma\dot{\delta}} \varepsilon_{\dot{\delta}\alpha}$ and analogousy for $\bar{\sigma}$ with undotted indices replaced by dotted. Then, the products of two Weyl spinors obey the following Fierz identities

$$\xi_\alpha \bar{\xi}^\beta = -\frac{1}{2} \delta_\alpha^\beta \left( \xi^\gamma \bar{\zeta}_\gamma \right) + \frac{1}{8} \sigma^{\mu\nu\alpha\beta} \left( \xi^\gamma \sigma_{\mu\nu\gamma\delta} \bar{\xi}_\delta \right),$$

$$\bar{\xi}^{\dot{\alpha}} \xi_\beta = -\frac{1}{2} \delta^{\dot{\alpha}}_{\dot{\beta}} \left( \bar{\xi}^\gamma \xi_\gamma \right) + \frac{1}{8} \bar{\sigma}^{\mu\nu\dot{\alpha}\dot{\beta}} \left( \bar{\xi}^\gamma \bar{\sigma}_{\mu\nu\gamma\delta} \bar{\xi}_\delta \right),$$

$$\bar{\xi}^{\dot{\alpha}} \bar{\xi}^{\dot{\beta}} = \frac{1}{2} \sigma^{\mu\dot{\alpha}\dot{\beta}} \left( \xi_\gamma \sigma_{\mu\gamma\delta} \bar{\xi}_\delta \right),$$

(B.10)
with the following traces used in the derivation

$$\frac{1}{2} \text{tr} \{ \bar{\sigma}^{\mu\nu} \} = g^{\mu\nu},$$

$$\frac{1}{2} \text{tr} \{ \bar{\sigma}^{\mu\nu} \sigma^{\rho\sigma} \} = g^{\mu\nu} g^{\rho\sigma} + g^{\rho\sigma} g^{\mu\nu} - g^{\mu\rho} g^{\nu\sigma} - i \varepsilon^{\mu\nu\rho\sigma},$$

(B.11)
$$\frac{1}{2} \text{tr} \{ \sigma^{\mu\nu} \sigma^{\rho\sigma} \} = g^{\mu\nu} g^{\rho\sigma} + g^{\rho\sigma} g^{\mu\nu} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\nu} g^{\rho\sigma},$$

(B.12)
where the normal position of spinor indices as in Eq. (B.2) is implied. The complex conjugation of products of Weyl spinors includes the reversal of ordering, e.g.,

$$(\xi_\alpha \zeta_\beta)^* \equiv (\zeta_\alpha)^* (\xi^\alpha)^* = \bar{\zeta}_\dot{\alpha} \bar{\xi}^{\dot{\alpha}}.$$

(B.13)
The following relations are useful to perform the Dirac algebra for the product of three matrices:

\[ \sigma_{\mu}^{\mu} \hat{\alpha} \gamma \left( \delta_{\gamma}^{\beta} - \sigma_{\gamma}^{+} \sigma_{\gamma}^{-} \delta_{\delta}^{\beta} \right) = i \varepsilon_{\mu \nu} \sigma_{\nu} \hat{\alpha} \beta, \quad \sigma_{\mu}^{\mu} \hat{\alpha} \gamma \left( \delta_{\beta}^{\gamma} - \sigma_{\gamma}^{+} \sigma_{\gamma}^{-} \delta_{\delta}^{\beta} \right) = -i \varepsilon_{\mu \nu} \sigma_{\nu} \hat{\alpha} \beta, \] (B.14)

for contraction of transverse Lorentz indices

\[ \sigma_{\mu}^{\mu} \hat{\alpha} \beta \gamma \left( \delta_{\beta}^{\gamma} \delta_{\gamma}^{\delta} \sigma_{\gamma}^{+} \sigma_{\gamma}^{-} \delta_{\delta}^{\beta} \right) = \bar{\sigma}_{\mu}^{\mu} \hat{\alpha} \beta \gamma \sigma_{\gamma}^{+} \sigma_{\gamma}^{-} \delta_{\delta}^{\beta} = 0, \] (B.15)

and for the antisymmetric tensors

\[ \sigma_{\mu \nu}^{\mu \nu} \hat{\alpha} \beta \gamma \left( \delta_{\gamma}^{\beta} \delta_{\beta}^{\gamma} \sigma_{\gamma}^{+} \sigma_{\gamma}^{-} \delta_{\delta}^{\beta} \right) = -i \varepsilon_{\mu \nu} \sigma_{\nu} \hat{\alpha} \beta \gamma \sigma_{\gamma}^{+} \sigma_{\gamma}^{-} \delta_{\delta}^{\beta}, \] (B.16)

• \( D = 6 \): The Dirac matrices in the six-dimensional Euclidean space are taken in the form

\[ \hat{\gamma}^{a} = \begin{pmatrix} 0 & \sum_{a}^{A} B \\ \sum_{a}^{A} B & 0 \end{pmatrix}, \quad \hat{\gamma}^{7} = i \hat{\gamma}^{1} \hat{\gamma}^{2} \hat{\gamma}^{3} \hat{\gamma}^{4} \hat{\gamma}^{5} \hat{\gamma}^{6} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] (B.17)

where \( \sum_{a}^{A} B = (\eta_{iA}, i \eta_{iA}) \) and \( \sum_{a}^{A} B = (\bar{\eta}_{iA}, -i \bar{\eta}_{iA}) \) are expressed in terms of 't Hooft symbols \[ \Box \]. They obey the Clifford algebra \( \{ \hat{\gamma}^{a}, \hat{\gamma}^{b} \} = -2 \delta^{ab} \), where \( \delta^{ab} = \text{diag}(+,...,+), \) with the normalization chosen with regards to the fact that it will be a part of the ten-dimensional Clifford algebra. 't Hooft symbols obey the following relations

\[ \eta_{iA} \equiv \varepsilon_{iA} + \delta_{iA} \delta_{iB} - \delta_{iB} \delta_{iA}, \quad \bar{\eta}_{iA} \equiv \varepsilon_{iA} - \delta_{iA} \delta_{iB} + \delta_{iB} \delta_{iA}, \] (B.18)

and \( \bar{\eta}_{iA} = (-1)^{\delta_{iA} + \delta_{iB}} \eta_{iA} \). They form a basis of anti-symmetric \( 4 \times 4 \) matrices and are (anti-) selfdual in vector indices (\( \varepsilon_{1234} = \varepsilon^{1234} = 1 \))

\[ \eta_{iA} \equiv \frac{1}{2} \varepsilon_{ABCD} \eta_{iCD}, \quad \bar{\eta}_{iA} = -\frac{1}{2} \varepsilon_{ABCD} \bar{\eta}_{iCD}. \] (B.19)

The \( \eta \)-symbols obey the following relations

\[ \varepsilon_{ijk} \eta_{iA} \eta_{kC} \eta_{jD} = \delta_{AC} \eta_{iB} + \delta_{BD} \eta_{iA} - \delta_{AD} \eta_{iB} \eta_{BC} - \delta_{BC} \eta_{iA}, \]

\[ \eta_{iA} \eta_{iB} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} + \varepsilon_{ABCD}, \]

\[ \eta_{iA} \eta_{iC} = \delta_{ij} \delta_{BC} + \varepsilon_{ijk} \eta_{kBC} \]

\[ \varepsilon_{ABCDE} \eta_{iD} \eta_{iE} = \delta_{DA} \eta_{iB} + \delta_{DC} \eta_{iA} - \delta_{DB} \eta_{iC}, \]

\[ \eta_{iA} \eta_{iB} = 12, \]

\[ \eta_{iA} \eta_{iB} = 4 \delta_{ij}, \]

\[ \eta_{iA} \eta_{iC} = 3 \delta_{BC}. \] (B.20)
The same holds for $\bar{\eta}$ except for
\[
\bar{\eta}_{AB} \bar{\eta}_{CD} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} - \varepsilon_{ABCD}.
\] (B.21)

Obviously $\eta_{AB} \bar{\eta}_{AB} = 0$ due to their different duality properties.

The six-dimensional charge conjugation matrix is
\[
C_6 = \hat{\gamma}^1 \hat{\gamma}^2 \hat{\gamma}^3 = \begin{pmatrix} 0 & \delta_A^B \\ \delta^A_B & 0 \end{pmatrix},
\] (B.22)
with properties $C_6^T = C_6$ and $C_6 \hat{\gamma}^a = - (\hat{\gamma}^a)^T C_6$.

- $D = 9 + 1$: We use the representations for the four- and six-dimensional Clifford algebra elaborated above in order to construct a representation for the ten-dimensional matrices. Namely,
\[
\Gamma^M = \left( \mathbb{1} \otimes \gamma^\mu, \hat{\gamma}^a \otimes \gamma^5 \right).
\] (B.23)

They obey the algebra $\{ \Gamma^M, \Gamma^N \} = 2 g^{MN}$ with the Minkowski metric $g^{MN} = \text{diag}(+, -, \ldots, -)$. The chiral and the charge conjugation matrices are defined as
\[
\Gamma^{11} = \hat{\gamma}^7 \otimes \gamma^5, \quad C_{10} = C_6 \otimes C_4,
\] (B.24)
respectively. Obviously, the latter satisfies the conditions $C_{10}^T = -C_{10}$ and $C_{10} \Gamma^M = - (\Gamma^M)^T C_{10}$.

## C Building the supermultiplet: transformation of ...

In this appendix we give the complete result of the supersymmetric transformations of conformal operators in $\mathcal{N} = 4$ Yang-Mills theory, which allows to construct their supermultiplet. In the final subsection C.5 we give the result for the transformation of the components of the multiplet. Throughout this section the symmetrization and antisymmetrization operations are defined as follows:

\[
T^{[AB]} \equiv T^{AB} - T^{BA}, \quad T^{\{AB\}} \equiv T^{AB} + T^{BA}.
\] (C.1)

### C.1 ... bosonic operators

$SU(4)$ singlet operators first: For the parity-even operators we get
\[
\delta \mathcal{O}^{qq}_{ji} = -\sigma_j \frac{j + 2}{2j + 3} \left\{ (j + 3) \left( \xi^{\alpha A}[\mathcal{O}^{qq}_{ji}]_\alpha_A + \bar{\xi}_{\dot{\alpha}A}[\bar{\mathcal{O}}^{qq}_{ji}]^{\dot{\alpha}A} \right) - (j + 1) \left( \xi^{\alpha A}[\mathcal{O}^{qq}_{j-1i}]_\alpha_A + \bar{\xi}_{\dot{\alpha}A}[\bar{\mathcal{O}}^{qq}_{j-1i}]^{\dot{\alpha}A} \right) \\
- (j + 1) \left( \xi^{\alpha A}[\mathcal{O}^{qq}_{ji}]_\alpha_A - \bar{\xi}_{\dot{\alpha}A}[\bar{\mathcal{O}}^{qq}_{ji}]^{\dot{\alpha}A} \right) + (j + 1) \left( \xi^{\alpha A}[\mathcal{O}^{qq}_{j-1i}]_\alpha_A - \bar{\xi}_{\dot{\alpha}A}[\bar{\mathcal{O}}^{qq}_{j-1i}]^{\dot{\alpha}A} \right) \right\},
\] (C.2)
\[ \delta \mathcal{O}_{jl}^{qq} = \sigma_j \frac{(j+2)(j+3)}{12(2j+3)} \times \left\{ j \left( \xi^{\alpha A} \mathcal{O}_{jl}^{qq} \right)_{\alpha A} + \bar{\xi}_{\dot{\alpha} A} \mathcal{O}_{jl}^{\dot{q} \dot{q}} \right\} + (j+1) \left( \xi^{\alpha A} \mathcal{O}_{j-1l}^{qq} \right)_{\alpha A} + \bar{\xi}_{\dot{\alpha} A} \mathcal{O}_{j-1l}^{\dot{q} \dot{q}} \right\}, \]  

\[ \delta \mathcal{O}_{jl}^{ss} = -\sigma_j \frac{2}{2j+3} \times \left\{ (j+2) \left( \xi^{\alpha A} \mathcal{O}_{jl}^{qq} \right)_{\alpha A} - \bar{\xi}_{\dot{\alpha} A} \mathcal{O}_{jl}^{\dot{q} \dot{q}} \right\} + (j+1) \left( \xi^{\alpha A} \mathcal{O}_{j-1l}^{qq} \right)_{\alpha A} - \bar{\xi}_{\dot{\alpha} A} \mathcal{O}_{j-1l}^{\dot{q} \dot{q}} \right\}. \]  

Analogously, for the parity-odd operators we get

\[ \delta \mathcal{O}_{jl}^{\dot{q} \dot{q}} = -\sigma_{j+1} \frac{j+2}{2j+3} \times \left\{ - (j+3) \left( \xi^{\alpha A} \mathcal{O}_{jl}^{qq} \right)_{\alpha A} - \bar{\xi}_{\dot{\alpha} A} \mathcal{O}_{jl}^{\dot{q} \dot{q}} \right\} + (j+1) \left( \xi^{\alpha A} \mathcal{O}_{j-1l}^{qq} \right)_{\alpha A} - \bar{\xi}_{\dot{\alpha} A} \mathcal{O}_{j-1l}^{\dot{q} \dot{q}} \right\}. \]  

Here we have used Eqs. (B.11) and relations between Jacobi polynomials of different order [52].

The variation of SU(4) non-singlet bosonic operators reads for scalar bilinears

\[ \delta [O_{jl}^{\dot{q} \dot{q},15}]_A^B = -\sigma_{j+1} \frac{2}{3(2j+3)} \left[ P_{15}^{BD} \right]_{ABCD} \left\{ (j+2) \left( \xi^{\alpha C} \mathcal{O}_{jl}^{\dot{q} \dot{q}} \right)_{\alpha D} + \bar{\xi}_{\dot{\alpha} D} \bar{\mathcal{O}}_{jl}^{\dot{q} \dot{q}} \right\} + (j+1) \left( \xi^{\alpha C} \mathcal{O}_{j-1l}^{\dot{q} \dot{q}} \right)_{\alpha D} + \bar{\xi}_{\dot{\alpha} D} \bar{\mathcal{O}}_{j-1l}^{\dot{q} \dot{q}} \right\} \]  

and quark operators

\[ \delta [O_{jl}^{\dot{q} \dot{q},15}]_A^B = \sigma_{j+1} \frac{j+2}{3(2j+3)} \left[ P_{15}^{BD} \right]_{AC} \left\{ 3(j+3) \left( \xi^{\alpha C} \mathcal{O}_{jl}^{\dot{q} \dot{q}} \right)_{\alpha D} - \bar{\xi}_{\dot{\alpha} D} \bar{\mathcal{O}}_{jl}^{\dot{q} \dot{q}} \right\} - 3(j+1) \left( \xi^{\alpha C} \mathcal{O}_{j-1l}^{\dot{q} \dot{q}} \right)_{\alpha D} - \bar{\xi}_{\dot{\alpha} D} \bar{\mathcal{O}}_{j-1l}^{\dot{q} \dot{q}} \right\} \]  

and quark operators

\[ \delta [O_{jl}^{\dot{q} \dot{q},15}]_A^B = -\sigma_{j} \frac{j+2}{3(2j+3)} \left[ P_{15}^{BD} \right]_{AC} \left\{ 3(j+3) \left( \xi^{\alpha C} \mathcal{O}_{jl}^{\dot{q} \dot{q}} \right)_{\alpha D} - \bar{\xi}_{\dot{\alpha} D} \bar{\mathcal{O}}_{jl}^{\dot{q} \dot{q}} \right\} - 3(j+1) \left( \xi^{\alpha C} \mathcal{O}_{j-1l}^{\dot{q} \dot{q}} \right)_{\alpha D} - \bar{\xi}_{\dot{\alpha} D} \bar{\mathcal{O}}_{j-1l}^{\dot{q} \dot{q}} \right\} \]  

32
\[ \begin{align*}
+(j + 1) \left( \xi^C [\Omega_{ij}^{sq}]_{\alpha D} - \bar{\xi}_{\dot{\alpha}} D [\Omega_{ij}^{sq}]_{\bar{\dot{\alpha}} C} \right) - (j + 1) \left( \xi^C [\Omega_{j-1l}^{sq}]_{\alpha D} - \bar{\xi}_{\dot{\alpha}} D [\Omega_{j-1l}^{sq}]_{\bar{\dot{\alpha}} C} \right)
\end{align*} \]

Finally, the 20 transforms via

\[ \delta [\Omega_{ij}^{sq,20}]_{AB} = -\frac{\sigma_j}{6(2j + 3)} \left[ P_{20}^{\text{CD,EF}} \right]_{\alpha BC} \left\{ (j + 2) \left( \xi^C [\Omega_{ij}^{sq,20}]_{\alpha EF} - \bar{\xi}_{\dot{\alpha}} E [\Omega_{ij}^{sq,20}]_{\bar{\dot{\alpha}} G} \right) + (j + 1) \left( \xi^C [\Omega_{j-1l}^{sq,20}]_{\alpha EF} - \bar{\xi}_{\dot{\alpha}} E [\Omega_{j-1l}^{sq,20}]_{\bar{\dot{\alpha}} G} \right) \right\}. \]

(C.10)

C.2 ... maximal-helicity bosonic operators

For antisymmetric representation

\[ \begin{align*}
\delta [T_{ij}^{sq,6}]_{\mu AB} &= -\sigma_{j+1} \frac{2(j + 2)}{2j + 3} \epsilon^{[A} \{ (j + 3)[+\Omega_{ij}^{sq,6}]_{\mu B]} - (j + 1)[+\bar{\Omega}_{j-1l}^{sq,6}]_{\mu B}] \\
+\sigma_{j+1} \frac{(j + 1)(j + 2)}{12(2j + 3)} \bar{\xi}_{\dot{\alpha}} C \sigma^\mu_{\dot{\alpha} \dot{\beta}} \left\{ 8\epsilon_{\text{ABC}} \left( [\Omega_{ij}^{sq}]_{\beta D} - [\Omega_{j-1l}^{sq}]_{\beta D} \right) - \epsilon_{\text{CDE}} \left( [\Omega^{sq,20}]_{\beta DE} - [\Omega^{sq,20}]_{\beta DE} \right) \right\}, \quad (C.11)
\end{align*} \]

and for symmetric representation

\[ \begin{align*}
\delta [T_{ij}^{sq,10}]_{\mu AB} &= -\sigma_j \frac{2(j + 2)}{2j + 3} \epsilon^{[A} \{ (j + 3)[+\Omega_{ij}^{sq,10}]_{\mu B]} - (j + 1)[+\bar{\Omega}_{j-1l}^{sq,10}]_{\mu B}] \\
-\sigma_j \frac{(j + 1)(j + 2)}{12(2j + 3)} \bar{\xi}_{\dot{\alpha}} C \sigma^\mu_{\dot{\alpha} \dot{\beta}} \epsilon_{\text{CDE}} \left( [\Omega^{sq,20}]_{\beta DE} - [\Omega^{sq,20}]_{\beta DE} \right), \quad (C.13)
\end{align*} \]

\[ \begin{align*}
\delta [T_{ij}^{sq,10}]_{\mu AB} &= -\sigma_j \frac{2(j + 2)}{2j + 3} \epsilon^{[A} \{ (j + 3)[+\Omega_{ij}^{sq,10}]_{\mu B]} - (j + 1)[+\bar{\Omega}_{j-1l}^{sq,10}]_{\mu B}] \\
+\sigma_j \frac{(j + 1)(j + 2)}{12(2j + 3)} \bar{\xi}_{\dot{\alpha}} C \sigma^\mu_{\dot{\alpha} \dot{\beta}} \epsilon_{\text{CDE}} \left( [\Omega^{sq,20}]_{\beta DE} - [\Omega^{sq,20}]_{\beta DE} \right), \quad (C.14)
\end{align*} \]

Here the square and curly brackets on indices denote antisymmetrization and symmetrization as defined in Eq. (C.1).

The maximal-helicity gluonic operators transform into the maximal-helicity gauge-gaugino operators via

\[ \delta [T_{ij}]_{\mu} = \frac{(j + 2)(j + 3)}{3(2j + 3)} \left\{ \bar{\xi}_{\dot{\alpha}} A \sigma^\mu_{\dot{\alpha} \dot{\beta}} \left( j [+\Omega_{ij}^{sq}]_{\beta A} + (j + 1)[+\bar{\Omega}_{j-1l}^{sq}]_{\mu A} \right) \right\}. \]
\[ -(-1)^j \xi_{\alpha A} \sigma^\mu_{\alpha \beta} \left( j[+\Omega^{gq}_{jl}]_{\beta} + (j + 1)[+\Omega^{gq}_{jl}]_{\beta} \right) \right\}, \quad (C.15) \]

\[ \delta [\bar{T}^{gq}_{jl}]^{\mu \nu} = \frac{(j + 2)(j + 3)}{3(2j + 3)} \left\{ \xi^{\alpha A} \sigma^\nu_{\alpha \beta} \left( j[-\Omega^{gq}_{jl}]_{A} + (j + 1)[-\Omega^{gq}_{jl}]_{A} \right) \right\} \]

\[ -(-1)^j \xi^{\alpha A} \sigma^\mu_{\alpha \beta} \left( j[-\Omega^{gq}_{jl}]_{A} + (j + 1)[-\Omega^{gq}_{jl}]_{A} \right) \right\} \right\}. \quad (C.16) \]

Notice that in the \( \mathcal{N} = 1 \) case, \( \mathcal{T} \)'s form a separate supermultiplet and do not enter the one with the chiral-even operators \([24], [29] \) and \([38] \).

C.3 ... mixed bosonic operators

\[ \delta [T^{sq}_{jl}]^{\mu AB} = \frac{1}{3(2j + 3)} \xi^{\alpha |A} \sigma^\mu_{\alpha \beta} \left\{ (3(j + 3)[\bar{T}^{sq}_{jl}]^{\hat{\alpha} B}] + (-1)^j (j + 1)[\bar{T}^{sq}_{jl}]^{\hat{\alpha} B}] \right\} \]

\[ + (j + 2) \left\{ 3[\bar{T}^{sq}_{jl}]^{\hat{\alpha} B}] + (-1)^j [\bar{T}^{sq}_{jl}]^{\hat{\alpha} B}] \right\} \right\} \]

\[ - \frac{(-1)^j}{6(2j + 3)} \xi^{\alpha C} \sigma^\mu_{\alpha \beta} \left\{ (j + 1)[\bar{T}^{sq}_{jl}]^{\hat{\alpha} AB} + (j + 2)[\bar{T}^{sq}_{jl}]^{\hat{\alpha} AB} \right\} \]

\[ - \frac{2}{2j + 3} \right\} \right\} \right\} \right\} \]

and

\[ \delta [\bar{T}^{sq}_{jl}]_{AB} = -\frac{1}{3(2j + 3)} \xi_{\hat{A} |A} \sigma^\mu_{\hat{A} \beta} \left\{ (3(j + 3)[\Omega^{sq}_{jl}]_{\alpha B}] - (-1)^j (j + 1)[\Omega^{sq}_{jl}]_{\alpha B}] \right\} \]

\[ + (j + 2) \left\{ 3[\Omega^{sq}_{jl}]_{\alpha B}] - (-1)^j [\Omega^{sq}_{jl}]_{\alpha B}] \right\} \right\} \]

\[ - \frac{(-1)^j}{6(2j + 3)} \xi_{\hat{A} C} \sigma^\mu_{\hat{A} \beta} \left\{ (j + 1)[\Omega^{sq}_{jl}]^{\hat{\alpha} AB} + (j + 2)[\Omega^{sq}_{jl}]^{\hat{\alpha} AB} \right\} \]

\[ + \frac{2}{2j + 3} \xi^{\alpha C} \left\{ (j + 3)[+\Omega^{sq}_{jl}]^{\mu D} + (j + 2)[+\Omega^{sq}_{jl}]^{\mu D} \right\} \]. \quad (C.17) \]

C.4 ... fermionic operators

Transformation of gluon-gaugino operators

\[ \delta [\Omega^{gq}_{jl}]_{\alpha A} = -\frac{1}{(j + 2)(j + 3)} \left\{ (6\Omega^{gq}_{jl+1} - \frac{1}{8}(j + 3)\Omega^{gq}_{jl+1}) - \left( 6\tilde{\Omega}^{gq}_{jl+1} - \frac{1}{8}(j + 3)\tilde{\Omega}^{gq}_{jl+1} \right) \right\} \]

\[ + \left( 6\tilde{\Omega}^{gq}_{jl+1} + \frac{1}{8}(j + 1)\Omega^{gq}_{jl+1} \right) - \left( 6\tilde{\Omega}^{gq}_{jl+1} + \frac{1}{8}(j + 1)\tilde{\Omega}^{gq}_{jl+1} \right) \right\} \sigma^\beta_{\alpha \beta} \xi_{\beta A} \]

\[ - \frac{1}{2(j + 2)(j + 3)} \left\{ (j + 1) \left( [\tilde{\Omega}^{gq,15}_{jl+1}]^{\mu B} - [\Omega^{gq,15}_{jl+1}]^{\mu B} \right) \right\} \sigma^\gamma_{\alpha \beta} \xi_{\beta B} \]

\[ - \frac{j + 1}{2(j + 2)} \left\{ [\bar{T}^{sq}_{jl+1}]^{\mu AB} + [\bar{T}^{sq}_{jl+1}]^{\mu AB} \right\} \sigma_{\mu \alpha \beta}^\gamma \gamma_{\gamma B} \sigma_{\beta} \xi_{\beta B} \].
\[ \delta[\tilde{O}^{\mu}]^{\dot{A}} = -\frac{1}{(j+2)(j+3)} \left\{ (6\tilde{O}^{\mu}_{j+1} - \frac{1}{8}(j+3)\mathcal{O}^{\mu}_{j+1}) + (6\tilde{\mathcal{O}}^{\mu}_{j+1} - \frac{1}{8}(j+3)\bar{\mathcal{O}}^{\mu}_{j+1}) \right\} \sigma^{-\dot{\alpha}\dot{\beta}} \xi_{\dot{A}}^A \]

\[-\frac{1}{2(j+2)(j+3)} \left\{ (j+1) \left[ (\bar{\mathcal{O}}^{\mu}_{j+1})^A + [\mathcal{O}^{\mu}_{j+1}]^A \right] \right\} \sigma^{-\dot{\alpha}\dot{\beta}} \xi_{\dot{A}}^A \]

\[+\frac{j+1}{2(j+2)} \left\{ (T^{\mu}_{j+1})^A + [T^{\mu}_{j+1}]^A \right\} \sigma^{-\dot{\alpha}\dot{\beta}} \xi_{\dot{A}}^A, \]

and for scalar-gaugino operators

\[ \delta[\mathcal{O}^{\mu}]_\alpha^A = \frac{1}{8(j+2)} \left\{ 3\tilde{\mathcal{O}}^{\mu}_{j+1} - 3\tilde{O}^{\mu}_{j+1} \right\} \sigma^{-\dot{\alpha}\dot{\beta}} \xi_{\dot{A}}^A \]

\[+\frac{1}{2(j+2)} \left\{ (\bar{\mathcal{O}}^{\mu}_{j+1})^A - (\tilde{\mathcal{O}}^{\mu}_{j+1})^A \right\} \sigma^{-\dot{\alpha}\dot{\beta}} \xi_{\dot{A}}^A \]

\[+\frac{1}{2(j+2)} \left\{ (T^{\mu}_{j+1})^A + (-1)^j(j+1)(T^{\mu}_{j+1})^A \right\} \sigma^{-\dot{\alpha}\dot{\beta}} \xi_{\dot{A}}^A, \]

\[ \delta[\mathcal{O}^{\mu}]_\dot{\alpha}^A = \frac{1}{8(j+2)} \left\{ 3\tilde{O}^{\mu}_{j+1} - 3\tilde{O}^{\mu}_{j+1} \right\} \sigma^{-\dot{\alpha}\dot{\beta}} \xi_{\dot{A}}^A \]

\[ -\frac{1}{2(j+2)} \left\{ (\bar{\mathcal{O}}^{\mu}_{j+1})^A - (\tilde{\mathcal{O}}^{\mu}_{j+1})^A \right\} \sigma^{-\dot{\alpha}\dot{\beta}} \xi_{\dot{A}}^A \]

\[ -\frac{1}{2(j+2)} \left\{ (T^{\mu}_{j+1})^A - (-1)^j(j+1)(T^{\mu}_{j+1})^A \right\} \sigma^{-\dot{\alpha}\dot{\beta}} \xi_{\dot{A}}^A, \]

where we have used Fierz identities \([B.10]\) and the properties \([B.15]\) and \([B.16]\).

One can easily recognize from these variations the result for \(N = 1\) supersymmetric Yang-Mills theory by replacing \(\frac{1}{8} \rightarrow \frac{1}{2}\) in coefficients on the right-hand side of the variations of \(\mathcal{O}^{\mu}_{j+1}\). From the latter one can easily read off the components of the supermultiplet of conformal operators in this theory \([35, 39, 10]\) which form the chiral superfield \([10]\).

Transformation of the maximal-helicity fermion operators

\[ \delta[\chi^{\mu}]_\alpha^A = \frac{1}{8(j+2)(j+3)} \left\{ (j+3) \left( \epsilon^{A B C D} (2(j+1)[T^{\mu}_{j+1}]^A_{C D} - [T^{\mu}_{j+1}]^A_{C D}) + 2[T^{\mu}_{j+1}]^A_{C D} \right) \right\} \]
\[ + (j + 1) \left( \varepsilon^{ABCD} \left( 2(j + 3) [T_{j+1}^{6}]^{\mu}_{AB} + [T_{j+1}^{8}]^{\mu}_{AB} \right) - 2 [T_{j+1}^{9}]^{\mu}_{AB} \right) \right] \sigma^{-\alpha}_D \xi_B^\beta \]

\[ - \frac{3}{2(j + 2)(j + 3)} \left\{ [T_{j+1}^{9}]^{\mu\nu} + [T_{j+1}^{9}]^{\mu\nu} \right\} \sigma^{-\alpha}_D \sigma^{-\beta}_C \xi_A^\gamma, \]  

(C.23)

and

\[ \delta [\Omega^{qq\mu\nu}] = \frac{-1}{8(j + 2)(j + 3)} \left\{ (j + 3) \left( \varepsilon_{ABCD} \left( 2(j + 1) [T_{j+1}^{6}]^{\mu\nu} + [T_{j+1}^{8}]^{\mu\nu} \right) - 2 [T_{j+1}^{9}]^{\mu\nu}_{AB} \right) \right. \]

\[ + (j + 1) \left( \varepsilon_{ABCD} \left( 2(j + 3) [T_{j+1}^{6}]^{\mu\nu} - [T_{j+1}^{8}]^{\mu\nu} \right) + 2 [T_{j+1}^{9}]^{\mu\nu}_{AB} \right) \right\} \sigma^{-\alpha}_D \xi_B^\beta \]

\[ - \frac{3}{2(j + 2)(j + 3)} \left\{ [T_{j+1}^{9}]^{\mu\nu} + [T_{j+1}^{9}]^{\mu\nu} \right\} \sigma^{-\alpha}_D \sigma^{-\beta}_C \xi_A^\gamma. \]  

(C.24)

The 20 transforms as

\[ \delta [\Omega^{qq}] = \frac{1}{j + 2} [P_{20}]^{C,DE} \left\{ \left( - [\Omega^{qq}]^{E}_{j+1} + (j + 2) [\Omega^{ss}]^{E}_{j+1} \right) \right. \]

\[ + \left( \left( [\Omega^{qq}]^{E}_{j+1} + (j + 2) [\Omega^{ss}]^{E}_{j+1} \right) \right. \]

\[ + \frac{1}{2} \left( \left( [\Omega^{qq}]^{E}_{j+1} + [\Omega^{ss}]^{E}_{j+1} \right) \right. \]

\[ + \frac{3}{2(j + 2)} \left( [T_{j+1}^{6}]^{\mu\nu} + [T_{j+1}^{8}]^{\mu\nu} \right) \sigma^{-\alpha}_D \xi_B^\beta \]

\[ + \frac{1}{4(j + 2)} \left( [T_{j+1}^{9}]^{\mu\nu} + [T_{j+1}^{9}]^{\mu\nu} \right) \sigma^{-\alpha}_D \xi_B^\beta \]

\[ - \frac{1}{2} \left( \left( [\Omega^{qq}]^{E}_{j+1} + [\Omega^{ss}]^{E}_{j+1} \right) \right. \]

\[ - \frac{3}{2(j + 2)} \left( [T_{j+1}^{6}]^{\mu\nu} - [T_{j+1}^{8}]^{\mu\nu} \right) \sigma^{-\alpha}_D \xi_B^\beta \]

\[ + \frac{1}{4(j + 2)} \left( [T_{j+1}^{9}]^{\mu\nu} + [T_{j+1}^{9}]^{\mu\nu} \right) \sigma^{-\alpha}_D \xi_B^\beta. \]  

(C.25)

C.5 ... components of the supermultiplet

As we explained in the main text, in order to construct automatically components of the supermultiplet, one has to choose the right primary. To this end we use \( \Omega^{qq,20} \). Then, the first level descendants are, according to Eq. (C.10), the operators \( \Omega^{qq,20} \) and its complex conjugated
sibling. In turn their descendants are

\[
\delta[O_{[+]jl}^{sq,20}]^C_{\alpha AB} = \frac{1}{2(j+2)} \left\{ 2[P_{20}]^{C,DE}_{AB,F} \left( [S_{jl+1}^{1,15}]_{EF} + [\tilde{O}_{jl+1}^{sq,15}]_{EF} \right) - (j+2)[O_{jl+1}^{ss,20}]^{CD}_{AB} \right\} \bar{\sigma}^\alpha_{\beta D} \delta^\beta_\gamma
\]

\[
+ \frac{1}{4(j+2)} \left\{ 6\varepsilon_{ABEF} [T_{jl+1}^{eq,10}]^{\mu CE} \right\} - (j+3)[T_{[-jl+1]}^{3\mu}]_{\beta D} \left\{ \bar{\sigma}^\mu_{\alpha \beta} \delta^\gamma_\gamma \right\} \Omega_{jl+1}^{sq,15},
\]

\[
\delta[O_{[-jl]}^{eq,20}]^C_{\alpha AB} = \frac{1}{2(j+2)} \left\{ 2[P_{20}]^{C,DE}_{AB,F} \left( [S_{jl+1}^{1,15}]^{15}_{EF} - [\tilde{O}_{jl+1}^{eq,15}]^{15}_{EF} \right) - (j+2)[O_{jl+1}^{ss,20}]^{CD}_{AB} \right\} \bar{\sigma}^\alpha_{\beta D} \delta^\beta_\gamma
\]

\[
- \frac{1}{4(j+2)} \left\{ 6\varepsilon_{ABEF} [T_{jl+1}^{eq,10}]^{\mu CE} \right\} - (j+3)[T_{[-jl+1]}^{3\mu}]_{\beta D} \left\{ \bar{\sigma}^\mu_{\alpha \beta} \delta^\gamma_\gamma \right\} \Omega_{jl+1}^{sq,15},
\]

When one combines the operators with same quantum numbers and conformal spin they form the components of the supermultiplet, and, of course, the eigenfunctions of the anomalous dimension matrix as was demonstrated in section 4. Evaluating supersymmetric descendants of all arising bosonic components one finds the following fermionic components; namely,

\[
\delta[T_{[+]jl}^1]^{\mu}_{AB} = -\frac{2(j+1)}{3(2j+3)} \bar{\xi}_{\alpha[A} \sigma^\mu_{\beta \gamma]} \left\{ (j+3)\Omega_{[\gamma jl]}^{11}[\beta B] + 3(j+2)\Omega_{[\gamma jl+1]}^{2}[\beta B] \right\}
\]

\[
- \frac{j+1}{3} \bar{\xi}_{\alpha[C} \sigma^\mu_{\beta \gamma]} [\Omega_{[\gamma jl+1]}^{sq,20}]_{\beta AB} + 4(j+3)\varepsilon_{ABCD} \xi_{\alpha[C} [\tilde{T}_{[\gamma jl+1]}^{\mu D}],
\]

\[
\delta[T_{[+]jl}^2]^{\mu}_{AB} = \frac{2(j+2)}{3(2j+3)} \bar{\xi}_{\alpha[A} \sigma^\mu_{\beta \gamma]} \left\{ 3\Omega_{[\gamma jl]}^{11}[\beta B] + \Omega_{[\gamma jl+1]}^{11}[\beta B] \right\}
\]

\[
- \frac{j+2}{3} \bar{\xi}_{\alpha[C} \sigma^\mu_{\beta \gamma]} [\Omega_{[-jl+1]}^{sq,20}]_{\beta AB} + 4(j+2)\varepsilon_{ABCD} \xi_{\alpha[C} [\tilde{T}_{[-jl+1]}^{\mu D}],
\]

\[
\delta[T_{[-jl]}^3]^{\mu}_{AB} = -\frac{1}{3(2j+3)} \bar{\xi}_{\alpha[A} \sigma^\mu_{\beta \gamma]} \left\{ \Omega_{[-jl+1]}^{11}[\beta B] + (j+2)\Omega_{[-jl]}^{11}[\beta B] \right\}
\]

\[
+ \frac{1}{4(2j+3)} \bar{\xi}_{\alpha[C} \sigma^\mu_{\beta \gamma]} \left\{ (j+1)\Omega_{[-jl+1]}^{sq,20} \right\} + (j+2)\Omega_{[-jl+1]}^{sq,20} \right\}
\]

\[
+ \frac{2}{2j+3} \varepsilon_{ABCD} \xi_{\alpha[C} \left\{ (j+3)[\Omega_{[-jl]}^{sq,20}]^{\mu D} + (j+2)[\Omega_{[-jl+1]}^{sq,20}]^{\mu D} \right\},
\]

for the maximal-helicity operators and analogous relations holding for complex conjugated operators which can be read off from appendix C. Here and below we will not present these equations for their redundancy. Next

\[
\delta[\tilde{O}_{jl}^{sq,15}]_A^B = -\frac{2(j+2)}{3(2j+3)} [P_{15}]^{BD}_{AC} \left\{ \xi_{\alpha[C} \left( [\Omega_{[-jl]}^{11}]_{\alpha D} - (j+1)[\Omega_{[\gamma jl+1]}^{11}]_{\alpha D} \right) + \bar{\xi}_{\alpha D} \left( [\Omega_{[-jl]}^{11}]_{\beta C} - (j+1)[\Omega_{[-jl+1]}^{11}]_{\beta C} \right) \right\}
\]

\[
- \frac{(j+1)(j+2)}{3(2j+3)} \left\{ \xi_{\alpha[C} \left( \Omega_{[-jl+1]}^{sq,20} \right) - [\Omega_{[\gamma jl+1]}^{sq,20}]_{\alpha AC} \right\}
\]

\[
- \bar{\xi}_{\alpha[C} \left( \Omega_{[-jl+1]}^{sq,20} \right) - [\Omega_{[\gamma jl+1]}^{sq,20}]_{\alpha AC} \right\},
\]

37
\[
\delta[S_{jl}^{1,15}]_{AB} = -\frac{2(j+2)}{3(2j+3)}[P_{15}]^{BD}_{AC}\left\{(j+3)\xi^\alpha [\Omega^{1}_{|j|jl}]_\alpha D - 3(j+1)\xi^\alpha [\Omega^{2}_{[-j-\perp I]}]_\alpha D \\
-(j+3)\xi^\alpha D[\Omega^{2}_{[-j|jl]}]^{\alpha C} + (j+1)\xi^\alpha D[\Omega^{2}_{[-j-\perp I]}]^{\alpha C}\right\} \\
+\frac{j+2}{3}\left\{\xi^\alpha C[\Omega^{sq,20}_{[-j|jl]}]_{AC}^{\alpha B} + \xi^\alpha C[\Omega^{sq,20}_{[+j|jl]}]^{\alpha AC}\right\},
\]

\[
\delta[S_{jl}^{2,15}]_{AB} = \frac{2}{3(2j+3)}[P_{15}]^{BD}_{AC}\left\{\xi^\alpha C (3(j+2)[\Omega^{2}_{[+j|jl]}]_\alpha D - (j+1)[\Omega^{2}_{[-j-\perp I]}]_\alpha D) \\
-\xi^\alpha D (3(j+2)[\Omega^{2}_{[+j|jl]}]^{\alpha C} - (j+1)[\Omega^{2}_{[-j-\perp I]}]^{\alpha C}) \right\} \\
+\frac{j+1}{3}\left\{\xi^\alpha C[\Omega^{sq,20}_{[-j-\perp I]}]^{B}_{AC} + \xi^\alpha C[\Omega^{sq,20}_{[-j-\perp I]}]^{AC}\right\},
\]

for the rest. The subsequent step gives

\[
\delta[\Omega^{1}_{[+j|jl]}]_{\alpha A} = -\frac{1}{(j+2)(j+3)} \left\{3(S^{2}_{j+l+1} - P^{2}_{j+l+1}) \delta^B_A - (j+3)[S^{1,15}_{j+l+1}]_{AB} \\
+(2j+3)[\tilde{\Omega}^{qa,15}_{[+j|jl]}]_{AB}\right\} \sigma^{-}_{\alpha \beta} \tilde{\xi}^{\gamma}
\]

\[
\delta[\Omega^{1}_{[-j|jl]}]_{\alpha A} = -\frac{1}{(j+2)} \left\{3(S^{2}_{j+l+1} - P^{1}_{j+l+1}) \delta^B_A - (j+1)[S^{2,15}_{j+l+1}]_{AB} \\
-(2j+5)[\tilde{\Omega}^{qa,15}_{[-j|jl]}]_{AB}\right\} \sigma^{-}_{\alpha \beta} \tilde{\xi}^{\gamma}
\]

\[
\delta[\Omega^{2}_{[+j|jl]}]_{\alpha A} = -\frac{1}{(j+2)} \left\{3(S^{1}_{j+l+1} - P^{1}_{j+l+1}) \delta^B_A + (j+2)[S^{2,15}_{j+l+1}]_{AB} \\
-(j+1)[\tilde{\Omega}^{qa,15}_{[+j|jl]}]_{AB}\right\} \sigma^{-}_{\alpha \beta} \tilde{\xi}^{\gamma}
\]

\[
\delta[\Omega^{2}_{[-j|jl]}]_{\alpha A} = -\frac{1}{(j+2)} \left\{3(S^{1}_{j+l+1} - P^{2}_{j+l+1}) \delta^B_A + (j+2)[S^{1,15}_{j+l+1}]_{AB} \\
-(j+1)[\tilde{\Omega}^{qa,15}_{[-j|jl]}]_{AB}\right\} \sigma^{-}_{\alpha \beta} \tilde{\xi}^{\gamma}
\]

and for the maximal-helicity fermionic operators

\[
\delta[\bar{T}^{\mu^\alpha}_{[+j|jl]}]_{\alpha A} = -\frac{1}{8(j+2)(j+3)} \left\{2(j+1)[\bar{T}^{\mu^\alpha}_{[+j+1|jl]}]_{AB} \\
-(j+3)e^{ABC\mu} [\tilde{T}^{2}_{[+j|jl]}]_{CD} + 2(j+1)[\tilde{T}^{3}_{[-j-\perp I]}]_{CD}\right\} \sigma^{-}_{\alpha \beta} \tilde{\xi}^{\gamma}
\]

\[
\delta[\bar{T}^{\mu^\alpha}_{[-j|jl]}]_{\alpha A} = \frac{1}{8(j+2)(j+3)} \left\{2(j+3)[\bar{T}^{\mu^\alpha}_{[-j+1|jl]}]_{AB} \\
+(j+1)e^{ABC\mu} [\tilde{T}^{2}_{[-j|jl]}]_{CD} + 2(j+3)[\tilde{T}^{3}_{[-j-\perp I]}]_{CD}\right\} \sigma^{-}_{\alpha \beta} \tilde{\xi}^{\gamma}
\]
\[ \delta[-\Omega_{i+l}^{\mu\dot{\alpha}}] = -\frac{3}{2(j+2)(j+3)} \left\{ [T_{[-j+l+1]}^{gg}]^{\mu\nu} + [T_{[l+j+1]}^{gg}]^{\mu\nu} \right\} \sigma_{\nu A} \sigma_{-\beta} \xi_{\gamma A}, \]

\[ \delta[-\Omega_{i+l}^{\mu\dot{\alpha}}] = \frac{1}{8(j+2)(j+3)} \left\{ 2(j+1)[T_{[j+l+1]}^{\mu\nu}]_{AB} + (j+3)\varepsilon_{ABCD} \left( [T_{[-j+l+1]}^{\mu\nu}] + 2(j+1)[T_{[-j+l+1]}^{\mu\nu}]_{CD} \right) \right\} \sigma_{-\alpha} \xi_{\gamma B}, \]

\[ \delta[-\Omega_{i+l}^{\mu\dot{\alpha}}] = -\frac{3}{2(j+2)(j+3)} \left\{ [T_{[-j+l+1]}^{\mu\nu}] + [T_{[l+j+1]}^{\mu\nu}] \right\} \sigma_{\nu A} \sigma_{-\beta} \xi_{\gamma A}. \] (C.34)

Finally, one observes that the supersymmetric algebra closes by varying the flavor-singlet bosonic components of the supermultiplet,

\[ \delta S^1 = (j+1) \left\{ \xi^{\alpha A} [\Omega^2_{[-j-l]}]_{\alpha A} + \bar{\xi}_{\dot{\alpha} A} [\bar{\Omega}^2_{[-j-l]}]^{\dot{\alpha} A} \right\}, \]

\[ \delta S^2 = \frac{j+2}{6(2j+3)} \left\{ 2(j+1) \left( \xi^{\alpha A} [\Omega^1_{[-j-l]}]_{\alpha A} + \bar{\xi}_{\dot{\alpha} A} [\bar{\Omega}^1_{[-j-l]}]^{\dot{\alpha} A} \right) \right\}, \]

\[ \delta S^3 = (j+2)(j+3) \left\{ \xi^{\alpha A} [\Omega^2_{[-j-l]}]_{\alpha A} + \bar{\xi}_{\dot{\alpha} A} [\bar{\Omega}^2_{[-j-l]}]^{\dot{\alpha} A} \right\}, \]

\[ \delta P^1 = -\frac{j+2}{2(2j+3)} \left\{ j \left( \xi^{\alpha A} [\Omega^2_{[-j-l]}]_{\alpha A} - \bar{\xi}_{\dot{\alpha} A} [\bar{\Omega}^2_{[-j-l]}]^{\dot{\alpha} A} \right) \right\}, \]

\[ \delta P^2 = \frac{-j+2(j+2)(j+3)}{2(2j+3)} \left\{ (\xi^{\alpha A} [\Omega^1_{[-j-l]}]_{\alpha A} - \bar{\xi}_{\dot{\alpha} A} [\bar{\Omega}^1_{[-j-l]}]^{\dot{\alpha} A}) \right\}. \] (C.35)

The remaining transformation laws which are no displayed above, i.e., for \([T_{[j+l]}^{gg}]^{\mu\nu}, [\tilde{T}_{[j+l]}^{gg\mu\nu}]_{AB}\) and \([\Omega_{[j+l]}^{ss,20}]_{CD}\) are given in Eqs. (C.13), (C.14) and (C.10), respectively, where, for the uniformity of notations used in the present section, one has to add the corresponding subscript \([\pm]\) depending on the \(j\)-parity.

### D One-loop renormalization in light-cone gauge

Let us give a few details of the derivation of anomalous dimensions which fix the scaling weight of the supermultiplet. We perform the calculation in the light-cone gauge \(A^+ = 0\) which preserves supersymmetry of the Lagrangian. The residual gauge freedom is fixed by imposing the antisymmet-
ric boundary condition on the gauge potential at the light-cone infinity, i.e., $A^\mu_+(\infty) + A^\mu_-(\infty) = 0$, which results into the principal value prescription on the spurious pole in the gluon density matrix.

### D.1 Propagators

Let us present first the propagators in $\mathcal{N} = 4$ Yang-Mills theory fixed with the light-cone gauge with a principal value prescription.

- **Scalar propagator:**

$$
\langle 0 | T \bar{\phi}^a_{AB}(z_1) \phi^{bCD}(z_2) | 0 \rangle = i\delta^{ab} \delta^A_B \delta^D_C \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (z_1 - z_2)} \frac{Z(k)}{k^2 + i0}.
$$

- **Gluon propagator:**

$$
\langle 0 | T A^\mu_a(z_1) A^\mu_b(z_2) | 0 \rangle = -i\delta^{ab} \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (z_1 - z_2)} U_{\mu\rho}(k) d^{\rho\sigma}(k) U_{\sigma\nu}(k) \frac{Z(k)}{k^2 + i0}.
$$

The spurious pole in the gluon density matrix

$$
d_{\mu\nu}(k) = g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{[k^+]},
$$

is regularized via the principal value $1/[k^+] = \frac{1}{2} \left[ 1/(k^+ + i0) + 1/(k^+ - i0) \right]$. The presence of the divergent tensor $U_{\mu\nu}$,

$$
U_{\mu\nu}(k) = g_{\mu\nu} - \tilde{Z}(k) \frac{k_\mu n_\nu + k_\nu n_\mu}{k^+},
$$

is an artifact of the Lorentz symmetry breaking effects by the gauge fixing vector $n_\mu$ and the use of the principal value prescription which leads to different renormalization constants for components of “good” and “bad” components of the tensor fields.

- **Fermion propagator:**

$$
\langle 0 | T \lambda^a_\alpha(z_1) \bar{\lambda}^b_{\dot{\beta}B}(z_2) | 0 \rangle = i\delta^{ab} \delta^A_B \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (z_1 - z_2)} U^{\alpha\beta}(k) k_\mu \bar{\sigma}_\mu \delta^\beta_{\dot{\gamma}} U^\gamma_{\dot{\beta}}(k) \frac{Z(k)}{k^2 + i0}.
$$
Here the rotation matrix reads
\[ U_{\alpha}^{\beta}(k) = \delta_{\alpha}^{\beta} - \frac{\tilde{Z}(k)}{2k^+} \bar{\sigma}^{+ \alpha} k^\mu \sigma_{\mu}^{\gamma} \beta, \] (D.6)
and its conjugate is
\[ U_{\dot{\alpha}}^{\dot{\beta}} = (U_{\alpha}^{\beta})^*. \] (D.7)

Since the light-cone gauge fixing does not break the supersymmetry at the Lagrangian level, the renormalization constants of all elementary fields are equal. At one loop they are given by the expressions
\[ Z(k) = 1 + \frac{g^2 N_c S_\varepsilon}{4\pi^2} \varepsilon \int dq^+ \frac{k^+}{q^+ - k^+} \vartheta_0^{11}(q^+, q^+ - k^+), \] (D.8)
and
\[ \tilde{Z}(k) = \frac{g^2 N_c S_\varepsilon}{8\pi^2} \varepsilon \left\{ 1 + \int dq^+ \frac{k^+}{q^+ - k^+} \vartheta_0^{11}(q^+, q^+ - k^+) \right\}, \] (D.9)
where \( \varepsilon \equiv (4 - d)/2 \) is the parameter of the dimensional regularization. Note that the “good” components of the fields are renormalized by \( Z \) and are not affected by \( \tilde{Z} \). Here and below we introduced the functions
\[ \vartheta_k^{a_1 \ldots a_j}(x_1, \ldots, x_j) \equiv \int \frac{d\beta}{2\pi i} \beta^k \prod_{\ell=1}^{j} (x_\ell \beta - 1 + i0)^{-a_\ell}. \] (D.10)
The first few of them, which appear in lowest-order calculations, read
\[ \vartheta_0^{11}(x_1, x_2) = \frac{\theta(x_1) - \theta(x_2)}{x_1 - x_2}, \] (D.11)
\[ \vartheta_0^{111}(x_1, x_2, x_3) = \frac{x_2}{x_1 - x_2} \vartheta_0^{11}(x_2, x_3) - \frac{x_1}{x_1 - x_2} \vartheta_0^{11}(x_1, x_3). \] (D.12)
They are expressed in terms of the step function \( \theta(x) = \{1, x \geq 0; 0, x < 0\}. \)

Note that the field renormalization constants (D.8) and (D.9) are infrared sensitive due to the non-causal choice of the regularization of the spurious poles in the light-cone propagator. Had we chosen the Mandelstam-Leibbrandt prescription, the field renormalization constants are free from infrared singularities and moreover they are identically zero, demonstrating the ultraviolet finiteness of the \( \mathcal{N} = 4 \) super-Yang-Mills theory [15].

### D.2 Kernels

Here we give the result for the one-loop dilatation operator for the maximal-helicity bosonic operators in \( \bar{6} \) of \( SU(4) \). The calculation of the renormalization group kernels of these light-cone operators is most easily performed in the Fourier transformed space, i.e.,
\[ \mathcal{O}(x_1, x_2) \equiv \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} e^{-ix_1 \xi_1 - ix_2 \xi_2} \mathcal{O}(\xi_1, \xi_2), \] (D.13)
where \( O = \mathcal{T}_{\text{sg}} \ldots \). The computation of the one-loop diagrams in Figs. 2, 3 and 4 along the line of Refs. 53, 54, results into the following matrix equation

\[
[\mathcal{T}(x_1, x_2)]_{\text{1-loop}} = \frac{-g^2 N_c S_c}{8\pi^2} \varepsilon \int dy_1 dy_2 \delta(x_1 + x_2 - y_1 - y_2) K(x_1, x_2|y_1, y_2)[\mathcal{T}(y_1, y_2)]_{\text{0-loop}},
\]

with the two-vector

\[
\tilde{T}(x_1, x_2) = \left( \begin{array}{c} T^{\alpha\beta} \\ T_{\text{sg}} \end{array} \right)(x_1, x_2),
\]

evolving with the matrix kernel

\[
K = \begin{pmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{pmatrix}.
\]

The elements of the evolution kernels are

\[
K^{11}(x_1, x_2|y_1, y_2) = \frac{x_1}{y_1} \left[ \frac{y_1}{x_1 - y_1} \varrho^{01}_{11}(x_1, x_1 - y_1) \right] + \frac{x_2}{y_2} \left[ \frac{y_2}{x_2 - y_2} \varrho^{01}_{11}(x_2, x_2 - y_2) \right] + \rho^{01}_{11}(x_1, x_1 - y_1),
\]

\[
K^{12}(x_1, x_2|y_1, y_2) = -\frac{4}{y_2} \left[ x_1 \varrho^{01}_{11}(x_1, x_1 - y_1) + x_2 \varrho^{01}_{11}(x_2, x_2 - y_2) \right],
\]

\[
K^{21}(x_1, x_2|y_1, y_2) = -\frac{1}{4} \left[ \varrho^{01}_{11}(x_1, x_1 - y_1) + \varrho^{01}_{11}(x_1, x_1 - y_2) \right],
\]

\[
K^{22}(x_1, x_2|y_1, y_2) = \left[ \frac{y_1}{x_1 - y_1} \varrho^{01}_{11}(x_1, x_1 - y_1) \right] + \left( \frac{x_2}{y_2} \right)^2 \left[ \frac{y_2}{x_2 - y_2} \varrho^{01}_{11}(x_2, x_2 - y_2) \right] + \frac{y_1}{y_2} \varrho^{01}_{11}(x_1, x_1 - y_2) - \frac{x_2}{y_2} \varrho^{01}_{11}(x_1, x_1 - y_2).
\]

Here the regularization of the end-point behavior \( y_i \to x_i \) arises from the self-energy diagrams and results into the conventional plus-prescription

\[
\left[ \frac{y_1}{x_1 - y_1} \varrho^{01}_{11}(x_1, x_1 - y_1) \right] + \frac{y_1}{x_1 - y_1} \varrho^{01}_{11}(x_1, x_1 - y_1) - \delta(x_1 - y_1) \int dx_i' \frac{y_1}{x_1' - y_1} \varrho^{01}_{11}(x_1', x_1' - y_1).
\]

The kernels are diagonal in the basis of Jacobi polynomials, i.e.,

\[
K^{ab}(x_1, x_2|y_1, y_2) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{x_1^{\alpha} x_2^{\beta}}{n_j(\alpha, \beta)} P_j^{(\alpha, \beta)} \left( \frac{x_1 - x_2}{x_1 + x_2} \right) \gamma_j^{ab} P_j^{(\alpha, \beta)} \left( \frac{y_1 - y_2}{y_1 + y_2} \right),
\]

where the normalization factor is

\[
n_j(\alpha, \beta) = \frac{\Gamma(j + \alpha + 1) \Gamma(j + \beta + 1)}{(2j + \alpha + \beta + 1) \Gamma(j + 1) \Gamma(j + \alpha + \beta + 1)}.
\]

The transformation of these kernels to the one acting on the light-cone coordinates is achieved via the Fourier transformation and reads

\[
K(x_1, x_2|y_1, y_2) \equiv -\int_0^1 dy \int_0^1 dz \theta(1 - y - z) \delta(x_1 - y y_1 - z y_2) \mathbb{K}(y, z),
\]

under the condition of the conservation of momentum in the \( t \)-channel: \( x_1 + x_2 = y_1 + y_2 \). Here \( \bar{y} \equiv 1 - y \). The explicit form of coordinate-space kernels as well as the corresponding local anomalous dimensions is given in the main text in section 4.1.
References


[29] B. Vallilo, Flat currents in the classical AdS$_5 \times S^5$ pure spinor superstring, hep-th/0307018.


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