Maximal Bell’s Inequality Violation for Non Maximal Entanglement

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Abstract

Bell’s inequality violation (BIQV) for correlations of polarization is studied for a product state of two two-mode squeezed vacuum (TMSV) states. The violation allowed is shown to attain its maximal limit for all values of the squeezing parameter, \(\zeta\). We show via an explicit example that a state whose entanglement is not maximal allow maximal BIQV. The Wigner function of the state is non negative and the average value of either polarization is nil.

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Two mode squeezed vacuum (TMSV) states generated via nondegenerate parametric amplifiers are of interest, following the pioneering study of Grangier et al. [1] that exhibited Bell’s inequality violations (BIQV) [2], [3] via intensity correlation measurements [4] for such states. (The Bell inequality we consider in this paper is the so called CHSH inequality, [5].) These BIQV studies, aside from their intrinsic interest, are valuable for elucidation of the applicability of Bell’s inequalities for continuous variables problems in general, and to the
original Einstein, Podolsky and Rosen (EPR) [6] state in particular. The latter’s pertinency to BIQV was discussed by Bell [7]. The reason for this general interest is associated with the fact that the Wigner function [8] for the EPR state (and, in general, for the TMSV state) is non-negative and thus has some attributes of a probability density in phase space, i.e. the problem relates to the fundamental question as to whether the quantum behaviour of these states can be underpinned with a theory based on local hidden variables. Bell [7] considered the problem and discussed correlations among the particles’ positions. He suggested [7] that such correlations within the EPR state (whose Wigner function is non-negative) will not allow BIQV. Wodkiewicz and Banaszek [9] noted that, intrinsically, [10–12], it is the parity correlation that is expressed by the Wigner function. They [9] then showed that the TMVS allow BIQV via parity correlations measurements. This analysis was greatly advanced by explicit definitions of “rotation” in parity space in [13,14], where it is shown that BIQV can achieve its Cirelson’s limit [15] for maximal entanglement, i.e. \( \zeta \rightarrow \infty \), where \( \zeta \) is the (positive) squeezing parameter. The above underscores the importance of specifying the operators (i.e., the dynamical variables) involved in the definition of the Bell operator [16], whose correlation values are bounded by Bell’s inequality. In the present work we consider polarization as our dynamical variable (the representative operator is given below - above Eq. (17)) . We show that the state under study: \(|\zeta\rangle\), a product of two TMSV states, whose Wigner function is non-negative, exhibits the remarkable property of allowing maximal BIQV, i.e. attaining the Cirelson limit, for all values of the squeezing parameter \( \zeta \).

The state under study, \(|\zeta\rangle\), is given by ( [17])

\[
|\zeta\rangle = \exp(\zeta K_x)|0\rangle; \tag{1}
\]

\[
K_x = a_+ b_+ + a_- b_- - a_+ b_- - a_- b_+ . \tag{2}
\]

Here the +/- subscripts denote the polarization relative to some chosen axis, common to A and B, with A and B labeling the two different channels toward which the two beams
head - the operators $a$ and $b$ refer to the respective channels: e.g. $a^+_x$ designates the creation operator for horizontally, i.e., “$x$ polarized” [18] photon headed into the $A$ channel, etc. $|\zeta\rangle$ is a product of two two-mode squeezed vacua - one pertaining to $x$- polarized and one to $y$-polarized. The Wigner function $W(\alpha_A, \alpha_B)$ for the state is thus, likewise, a product of two functions each of the form [19] ($\alpha = q + ip$; we delete the polarization subscripts to reduce notational cluttering)

$$W(\alpha_A, \alpha_B) = \frac{4}{\pi^2} e^{2 \cosh(2\zeta) (|\alpha_A|^2 + |\alpha_B|^2) + 2 \sinh(2\zeta)(\alpha_A \alpha_B + c.c.)}.$$  (3)

The total Wigner function, being a product of two such gaussian functions, is clearly non-negative.

We further recall the symmetry operator [17],

$$K_0 = i[a^+_+a_- - a^+_+a_+ + b^+_+b_- - b^+_+b_+].$$  (4)

Since

$$[K_0, K_2]_n = 0,$$  (5)

and

$$K_0|0\rangle = 0,$$  (6)

we have

$$|\zeta'\rangle \equiv e^{i\delta K_0} |\zeta\rangle = |\zeta\rangle,$$  (7)

i.e. $e^{i\delta K_0}$ is a symmetry operator: it leaves the state invariant. Note that $K_0$ is made of , additively, two parts

$$K_0^A = i[a^+_+a_- - a^+_+a_+] \quad \text{and} \quad K_0^B = i[b^+_+b_- - b^+_+b_+].$$  (8)

Each of these parts acts on a distinct channel - which we take to be at a different locale, but

$$K_0 = K_0^A + K_0^B.$$  (9)
is a symmetry operator for the system as a whole. On the other hand, the operator 
\( \exp(i\delta_A K^A_0) \) is a symmetry breaking operator - it breaks the \( K_0 \) symmetry. This operator does not commute with \( K_x \).

Rotating the polarizations, \[ \text{[18]}, \] i.e., e.g., \( a^+_\alpha \rightarrow a^+_{\delta\alpha} \), is accomplished via

\[
a^+_\alpha(\delta_A) = \exp^{i\delta_A K^A_0} a^+_\alpha \exp^{-i\delta_A K^A_0}, \quad \alpha = \pm,
\]

\[
b^\beta(\delta_B) = \exp^{i\delta_B K^B_0} b^\beta \exp^{-i\delta_B K^B_0}, \quad \beta = \pm,
\]

and their hermitian adjoints. We shall study the correlations between polarizations of the two channels. To this end our normalizer (i.e., reference correlation) is,

\[
C(A, B) = \langle \zeta | (a^+_+a_- - a^+_-a^+_-)(b^+_+b^-_ - b^+_-b^-_-) | \zeta \rangle. \tag{10}
\]

IIt is shown in the appendix of \[ \text{[17]} \] that the correlation function

\[
C^{\alpha\beta}(\delta_A, \delta_B) = \langle \zeta | \exp^{i\delta_A K^A_0+i\delta_B K^B_0} a^+_{\alpha}a_{\bar{\alpha}}b^\beta b_{\bar{\beta}} \exp^{-i\delta_A K^A_0-i\delta_B K^B_0} | \zeta \rangle, \quad \alpha, \beta = \pm,
\]

is a function of \( \delta = \delta_A - \delta_B \). Hence the operator, \( \exp(-i\delta K_0) \) implies

\[
C(A(\delta), B(\delta)) = \langle \zeta | (a^+_+a_+ - a^+_-a_+) (b^+_+b_+ - b^+_-b_-) | \zeta \rangle = C(A(0), B(0)) \equiv C(A, B), \tag{11}
\]

with \( \bar{\delta} = \delta \pm \pi/2 \) (cf. \[ \text{[18]} \]). The polarization along \( \bar{\delta} \) is orthogonal to the one along \( \delta \), i.e. the correlation is invariant under equal (common) rotation of the two channels. Direct calculation gives

\[
C(A(\delta), B(\delta')) = C^{++}(\delta, \delta') + C^{--}(\delta, \delta') - C^{-+}(\delta, \delta') - C^{--}(\delta, \delta'), \tag{12}
\]

thus,

\[
C(A, B) = 2 \cosh^2 \zeta \sinh^2 \zeta. \tag{13}
\]

Similarly, direct calculation yields

\[
C(A(\delta), B(\delta')) = C(A, B) \cos 2(\delta - \delta'). \tag{14}
\]

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Define the normalized expectation value of the polarization correlation by

\[ E(\delta, \delta') = \frac{C(A(\delta), B(\delta'))}{\sqrt{C(A(\delta), B(\delta)) C(A(\delta'), B(\delta'))}} = \cos 2(\delta - \delta'). \] (15)

Our expression for the Bell inequality [4] is, thus

\[ |E(\delta_A, \delta_B) + E(\delta_A, \delta'_B) + E(\delta'_A, \delta_B) - E(\delta'_A, \delta'_B)| \leq 2. \] (16)

Using Eq.(15), the left hand side can be written as

\[ |\cos 2(\delta_A - \delta_B) + \cos 2(\delta_A - \delta'_B) + \cos 2(\delta'_A - \delta_B) - \cos 2(\delta'_A - \delta'_B)| \leq 2. \] (17)

Clearly the choice \( \delta_A = 0, \delta'_A = \pi/4, \delta_B = \pi/8, \delta'_B = -\pi/8 \), gives for the LHS \( 2\sqrt{2} \), i.e. maximal violation of the inequality. This result is independent of \( \zeta \). We recall that for each polarization, \( \pm \), the entanglement of the state is maximized only for \( \zeta \to \infty \) [9], [13], [14], i.e., not only is the state under study a product of two states, but also both component states are not maximally entangled (for \( \zeta < \infty \)). However, BIQV discussed in these references are for parity as the relevant dynamical variable. In our study the relevant dynamical variable is the polarization in each channel, viz: \( (a^+_1 a^-_2 - a^+_2 a^-_1) \). For this variable the amount of BIQV can be estimated by evaluating its average value:

\[ \langle \zeta | (a^+_1 a^-_2 - a^+_2 a^-_1) | \zeta \rangle = 0 \],

which holds for all values of \( \zeta \) for the pure state \( |\zeta\rangle \) (Eq.(1)), and for both channels variables. This value, (nil) in Eq. (18), is analogous to the two spin state case, such as, e.g. (the numerical superscripts label the particles),

\[ |\psi\rangle = |\alpha\rangle \uparrow^{12} + |\beta\rangle \downarrow^{12}, \]

where \( \langle \psi | \sigma_z^{(1)} | \psi \rangle = \langle \psi | \sigma_z^{(2)} | \psi \rangle = 0 \) implies allowance of maximal BIQV and, in the spin case, \( |\alpha| = |\beta| \) i.e. maximal entanglement.

For comparison, we evaluate (just for one polarization, \( + \)) the expectation value of the parity operator [13], [14] as the relevant dynamical variable,
\[ (\zeta|\sum_{n=0}^{\infty} -1^n|n_+\rangle\langle n_+|)|\zeta\rangle = \frac{(1 - \tanh^2\zeta)}{(1 + \tanh^2\zeta)}. \]

Hence, with \textit{parity} as the relevant variable the state allows maximal BIQV only at \( \zeta \to \infty \) - only at this limit does average value (of the relevant dynamical variable for either channel) vanishes.

We have shown thus that correlations among polarizations as dynamical variables (with expectation values between \(-1\) and \(1\)) allow maximal Bell inequality violation for a product of two two-mode squeezed vacuum states. The Wigner function of the state is non-negative. This same state for \textit{parity} as the dynamical variable allows maximal violation of Bell’s inequality only in the limit of the squeezing parameter \( \zeta \to \infty \), i.e., only when the state’s expectation value for either parity is nil. Hence we have demonstrated that the violation incurred in Bell’s inequality is strongly related not only to the entanglement (of the state) and the particular dynamical variables - i.e. the variables involved in the correlations considered in the Bell operator but that maximal entanglement is not necessary for maximal Bell’s inequality violation.

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REFERENCES


