Impact of the finite volume effects on the chiral behavior of $f_K$ and $B_K$

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Abstract

We discuss the finite volume corrections to $f_K$ and $B_K$ by using the one-loop chiral perturbation theory in full, quenched and partially quenched QCD. We show that the finite volume corrections to these quantities dominate the physical (infinite volume) chiral logarithms.

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1 Introduction

Due to the limiting computing power, the current lattice computations of the hadronic matrix elements involving kaons are plagued by necessity for introducing three important approximations:

1. The (partially) quenched approximation;

2. The extrapolation in the light quark masses: because of the inability to simulate directly with the physical \( u/d \)-quarks, one works with masses not lighter than about a half of the physical strange quark and then extrapolate to the physical \( m_{u/d} \);

3. Degeneracy of valence quark masses in the kaon: matrix elements involving kaons are obtained with “kaons” consisting of degenerate valence quarks whose mass is tuned in such a way as to produce a pseudoscalar meson with its mass equal to the physical \( m_{K^0} = 0.498 \) GeV.

In view of the great importance of the \( K^0 - \overline{K^0} \) mixing amplitude in constraining the shape of the CKM unitarity triangle [1], a quantitative estimate of the systematic errors induced by the above listed approximations is mandatory. That is where chiral perturbation theory (ChPT) enters the stage and offers a systematic approach for quantifying (at least roughly) the size of these errors. In ChPT one computes the coefficients of the chiral logarithms for various hadronic quantities in order to: (a) examine whether or not the quenched approximation introduces potentially large systematic error; (b) guide the chiral extrapolations; (c) quantify the impact of the degeneracy in the quark masses on the evaluation of the hadronic matrix elements. The coefficients of the chiral logarithms are predicted by quenched and full ChPT. Although a convincing evidence for the presence of the chiral logarithms in any numerical lattice data is still missing, a slight discrepancy from the linear (or quadratic) dependence on the variation of the light quark mass is occasionally observed. Before identifying such a discrepancy as an indication for the presence of the chiral logarithmic behavior, one should make sure that the effects of finite volume are well under control. In particular, we would like to know how the finiteness of the lattice volume modifies the chiral logarithmic behavior of \( f_K \) and \( B_K \). In this paper we present the expressions obtained in three versions of ChPT that are relevant to the present and future lattice simulations, i.e. in quenched ChPT (QChPT), partially quenched ChPT (PQChPT) and in full (standard) ChPT. Concerning the PQChPT, we will consider the case of \( N_{\text{sea}} = 2 \) degenerate dynamical quarks, which is the current practice in the lattice community. Those expressions, obtained in both the finite and infinite volumes, are then used to: (i) show that chiral logarithmic behavior of \( f_K \) and \( B_K \) gets indeed modified by the finiteness of the volume, (ii) to assess the amount of systematic uncertainty induced by the finiteness of the lattice volume. As expected, finite volume effects increase as the mass of the valence light quark in the kaon, \( m_q \), becomes smaller (we keep the strange quark mass fixed to its physical value). For quark masses \( m_q \gtrsim m_s/3 \) and the volumes \( V \gtrsim (2 \text{ fm})^3 \), the finite volume effects are negligible. We will argue that even if one manages to push the quark masses closer to \( m_{u/d} \), finite volume effects will start overwhelming the effects of the physical (infinite volume) chiral logs (unless one uses very large volumes). This unfortunately complicates the efforts, currently made in the
lattice community, to observe the chiral log behavior directly from the lattice data. Our finite volume ChPT formulae for $f_K$ and $B_K$ may be used to disentangle the finite volume effect from the physical chiral logarithmic dependence. This obviously can be made only if the volume is sufficiently large and thus the finite volume corrections safely small to justify neglect of the unknown higher order corrections in the chiral expansion.

The remainder of this paper is organised as follows: in Sec. 2 we compute the chiral log corrections to $f_K$ and $B_K$ in all three versions of ChPT in infinite volume; in Sec. 3 we discuss the same chiral corrections but in the finite volume; these both sets of expressions are then combined in Sec. 4 to examine the impact of the finite volume artefacts on the chiral behavior of $f_K$ and $B_K$; in Sec. 5 we discuss the finite volume effects on $f_K$ and $B_K$ and we briefly summarise in Sec. ??.

2 Results in the infinite volume

To simplify the presentation and for an easier comparison with results available in the literature, we first briefly explain the notation adopted in this work and then present the expressions for the chiral logarithmic corrections that we computed in all three versions of ChPT.

2.1 Chiral Lagrangians

For the full (unquenched) ChPT we use the standard Lagrangian [2, 3]:

$$L_{\text{ChPT}} = \frac{f^2}{8} \text{tr} \left[ \left( \partial_{\mu} \Sigma^\dagger \right) \left( \partial^\mu \Sigma \right) + \Sigma^\dagger \chi + \chi^\dagger \Sigma \right],$$

with $f$ being the chiral limit of the pion decay constant, $f_\pi = 132$ MeV. In addition,

$$\chi = 2 B_0 \mathcal{M} = \frac{-2(0\bar{u}u + 0\bar{d}d)}{f^2} \mathcal{M},$$

$$\mathcal{M} = \text{diag}(m_u, m_d, m_s),$$

$$\Sigma = \exp \left( \frac{2i\Phi}{f} \right),$$

$$\Phi = \begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{6}} & \frac{\pi^+}{\sqrt{2}} & \frac{K^+}{\sqrt{3}} \\ \frac{\pi^-}{\sqrt{2}} & -\pi^0 - \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}} \eta \end{pmatrix}. $$

For the calculation in QChPT we will use the Lagrangian introduced in refs. [4, 5]:

$$L_{\text{QChPT}} = \frac{f^2}{8} \text{str} \left[ \left( \partial_{\mu} \Sigma^\dagger \right) \left( \partial^\mu \Sigma \right) + \Sigma^\dagger \chi + \chi^\dagger \Sigma \right] - m_0^2 \Phi_0^2 + \alpha_0 (\partial_{\mu} \Phi_0)(\partial^\mu \Phi_0),$$

where $\Phi_0 \equiv \text{str}[\Phi]/\sqrt{6}$, is proportional to the graded extension of the $\eta'$, the trace over the chiral group indices has been replaced by the super-trace over the indices of the graded
We begin by collecting the ChPT expressions for the $SU(3)\times SU(3)$ group except that the indices now run over the graded group just defined above. Dimensional regularisation and the so called "MS + 1" renormalisation scheme of ref. [2].

2.2 1-loop chiral log corrections to $f_K$ and $B_K$

We begin by collecting the ChPT expressions for $f_K$ and $B_K$ in the infinite volume. We adopt the standard definition of the $B_K$ parameter, namely

$$B_K = \langle \bar{K}^0|\bar{s}\gamma_\mu(1-\gamma_5)d\bar{s}\gamma_\mu(1-\gamma_5)d|K^0\rangle \frac{8}{3}\langle \bar{K}^0|\bar{s}\gamma_\mu(1-\gamma_5)d|0\rangle \langle 0|\bar{s}\gamma_\mu(1-\gamma_5)d|K^0\rangle, \quad (5)$$

which is equal to 1 in the vacuum saturation approximation. The bosonised version of the relevant left–left ($\Delta S = 2$) operator reads

$$O_{27}^{\Delta S=2} = g_{27} f^4 16 \left( \Sigma \partial_\mu \Sigma^\dagger \right)_d \left( \Sigma \partial_\mu \Sigma^\dagger \right)_d . \quad (6)$$

To compute the chiral loop corrections to $f_K$, we use the standardly bosonised left handed current:

$$J^L_\mu = \bar{s}\gamma_\mu(1-\gamma_5)d \longrightarrow i f^2 4 \left( \Sigma \partial_\mu \Sigma^\dagger \right)_d . \quad (7)$$

In the following we will leave out the analytic terms (those accompanied by the low energy constants) and focus only onto the non-analytic ones. As we will see, the analytic terms are not relevant to the discussion of the finite volume effects.

The chiral logarithmic corrections to $f_K$ are

$$\left(f_K f^{\text{tree}}\right)^{\text{ChPT}} = 1 - \frac{3}{4(4\pi f)^2} \left[ m_\pi^2 \log \left( \frac{m_K^2}{\mu^2} \right) + 2m_K^2 \log \left( \frac{m_\eta^2}{\mu^2} \right) + m_\eta^2 \log \left( \frac{m_\eta^2}{\mu^2} \right) \right], \quad (8)$$

$$\left(f_K f^{\text{tree}}\right)^{\text{PQChPT}} = 1 - \frac{1}{2(4\pi f)^2} \left[ m_{SS}^2 - m_K^2 + (2m_K^2 - m_\pi^2 + m_{SS}^2) \log \left( \frac{m_{22}^2}{\mu^2} \right) \right]$$

$$+ \left( m_\pi^2 + m_{SS}^2 \right) \log \left( \frac{m_{13}^2}{\mu^2} \right) - \frac{m_K^2 m_{SS}^2 - m_\pi^2 (2m_K^2 - m_\pi^2)}{2 (m_K^2 - m_\pi^2)} \log \left( \frac{m_{22}^2}{m_\eta^2} \right), \quad (9)$$

$$\left(f_K f^{\text{tree}}\right)^{\text{QChPT}} = 1 - \frac{1}{3(4\pi f)^2} \left[ \left( m_0^2 - \alpha_0 m_K^2 \right) - \frac{m_0^2 m_K^2 - \alpha_0 m_\pi^2 m_{22}^2}{2 (m_K^2 - m_\pi^2)} \log \left( \frac{m_{22}^2}{m_\eta^2} \right) \right], \quad (10)$$
where ChPT, PQChPT and QChPT stand for the full, partially quenched \((N_{sea} = 2)\) and quenched chiral perturbation theory. In the above formulae,

\[
\begin{align*}
    m_{SS}^2 &= 2B_0 m_{sea}, \\
    m_{22}^2 &= 2B_0 m_s = 2m_K^2 - m_\pi^2, \\
    m_{23}^2 &= B_0 (m_s + m_{sea}), \\
    m_{13}^2 &= B_0 (m_q + m_{sea}).
\end{align*}
\]

We stress that we work in the exact isospin symmetry limit, i.e. \(m_q \equiv m_u = m_d\). The results listed above agree with the ones available in the literature: eq. (8) was first obtained in ref. [2], eq. (9) in ref. [6], and eq. (10) in refs. [4, 5].

For the \(B_{K}\) parameter, we obtain:

\[
\begin{align*}
    (B_{K} B_{K}^{\text{tree}})_{\text{ChPT}}^{\text{ChPT}} &= 1 - \frac{2}{(4\pi f)^2} \left[ m_K^2 + m_{\pi}^2 \log \left( \frac{m_K^2}{\mu^2} \right) + \frac{m_{\pi}^2 (m_K^2 + m_{\pi}^2)}{4m_K^2} \log \left( \frac{m_{\pi}^2}{\mu^2} \right) \right. \\
    &\left. + \frac{(7m_K^2 - m_\pi^2)m_{m_\eta}^2}{4m_K^2} \log \left( \frac{m_{m_\eta}^2}{\mu^2} \right) \right], \quad (12)
\end{align*}
\]

\[
\begin{align*}
    (B_{K} B_{K}^{\text{tree}})_{\text{PQChPT}}^{\text{PQChPT}} &= 1 - \frac{2}{(4\pi f)^2} \left\{ m_{SS}^2 + m_{\pi}^2 - \frac{m_\pi^4 + m_\pi^2}{2m_K^2} + m_K^2 \left[ \log \left( \frac{m_K^2}{\mu^2} \right) + 2 \log \left( \frac{m_{22}^2}{m_{\pi}^2} \right) \right] \right. \\
    &\left. - \frac{1}{2} \left( m_{SS}^2 - \frac{m_K^2 + m_{\pi}^2}{2m_K^2} - m_\pi^2 \frac{m_{SS}^2 - m_{\pi}^2}{m_K^2 - m_{\pi}^2} \right) \log \left( \frac{m_{22}^2}{m_{\pi}^2} \right) \right\}, \quad (13)
\end{align*}
\]

\[
\begin{align*}
    (B_{K} B_{K}^{\text{tree}})_{\text{QChPT}}^{\text{QChPT}} &= 1 - \frac{1}{3(4\pi f)^2} \left[ m_0^2 - \alpha_0 m_K^2 - \frac{m_0^2 m_K^2 - \alpha_0 m_\pi^2 m_{22}^2}{2(m_K^2 - m_\pi^2)} \log \left( \frac{m_{22}^2}{m_{\pi}^2} \right) \right]. \quad (14)
\end{align*}
\]

Also these results agree with the ones previously computed in full ChPT [7, 8], PQChPT [9], and QChPT [5, 10], where more details about the actual calculation can be found.

### 3 Results in the finite volume

The calculation of the chiral logarithmic corrections in the finite box of volume \(V = L^3\), with the periodic boundary conditions, is completely analogous to the one in the infinite volume, except for the fact that loop integrals now become sums over discretised 3-momenta. Like on the lattice, at the end of the calculation, the times of the kaon fields in the correlation function are sent to infinity. To abbreviate the expressions, we first introduce

\[
\begin{align*}
    \omega_\pi^2 &= \vec{q}^2 + m_\pi^2, \\
    \omega_K^2 &= \vec{q}^2 + m_K^2, \\
    \omega_{22}^2 &= \vec{q}^2 + m_{22}^2.
\end{align*}
\]

and analogously for \(\omega_{13}, \omega_{23}\), with the corresponding masses already defined in eq. (11). As in the infinite volume, \(m_0\) and \(\alpha_0\) are the \(\eta\)-parameters of the quenched theory.
For the decay constant \( f_K \) in all three versions of the ChPT, we obtain,

\[
(f_K f_{\text{tree}})^{\text{ChPT}} = 1 - \frac{3}{8 f^2 L^3} \sum_q (1 \omega_\pi + 2 \omega_K + 1 \omega_\eta),
\]

(16)

\[
(f_K f_{\text{tree}})^{\text{PQChPT}} = 1 + \frac{1}{8 f^2 L^3} \sum_q \left[ \frac{m_{SS}^2 - m_{\pi}^2}{2 \omega_\pi^3} + \frac{m_{SS}^2 - m_{22}^2}{2 \omega_{22}^3} - 4 (1 \omega_{13} + 1 \omega_{23}) 
\right. 
+m_{SS}^2 - m_{22}^2 m_{22}^2 (1 \omega_{22} - 1 \omega_\pi) 
\]

(17)

\[
(f_K f_{\text{tree}})^{\text{QChPT}} = 1 - \frac{1}{24 f^2 L^3} \sum_q \left\{ m_0^2 \left[ \frac{2}{m_K^2 - m_\pi^2} \left( \frac{1}{\omega_\pi} - \frac{1}{\omega_{22}} \right) - \frac{1}{\omega_{22}} - \frac{1}{\omega_\pi} \right] 
\right. 
- \alpha_0 \left[ \frac{2m_K^2}{m_K^2 - m_\pi^2} \left( \frac{1}{\omega_\pi} - \frac{1}{\omega_{22}} \right) - \frac{m_{22}^2}{\omega_{22}^3} - \frac{m_{22}^2}{\omega_\pi^3} \right] \right\},
\]

(18)

while for the \( B_K \) parameter we have

3.1 Evaluation of the chiral loop sums

The sums that appear in the calculation of the tadpole diagrams are of the form:

\[
\xi_s(L, M) = \frac{1}{L^3} \sum_q \frac{1}{(q^2 + M^2)^s} - \frac{\sqrt{4\pi}}{\Gamma(s + \frac{1}{2})} \int \frac{d^3q}{(2\pi)^4} \frac{1}{(q^2 + M^2)^{s + \frac{1}{2}}}
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-\tau M^2} \frac{1}{L^3} \sum_q e^{-q^2 \tau} - \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-\tau M^2} \int d^3q (2\pi)^3 e^{-q^2}
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-\tau M^2} \left\{ \left[ \frac{1}{L} \vartheta \left( \frac{4\pi^2 \tau}{L^2} \right) \right]^3 - \frac{1}{8(\pi \tau)^{3/2}} \right\}
\]

(19)

\[
= \frac{L^{2s-3}}{(2\pi)^{2s} \Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-\tau (M_L^2 / 4\pi^2)} \left\{ [\vartheta(\tau)]^3 - \left( \frac{\pi}{\tau} \right)^{3/2} \right\},
\]

where the elliptic theta-function, \( \vartheta(\tau) \), is defined as

\[
\vartheta(\tau) = \sum_{n=-\infty}^\infty e^{-\tau n^2},
\]

(20)

\[1\] The function \( \vartheta(\tau) \) is obtained from the commonly used function \( \vartheta_3(u, q) = \sum_{n=-\infty}^\infty q^n e^{2\pi i u} \), after replacing, \( u = 0 \) and \( q = e^{-\tau} \). For the numerical analysis, we use the function predefined in “Mathematica”, namely \texttt{EllipticTheta[3, 0, e^{-\tau}]} \texttt{. For more details on the elliptic functions see ref. [11].}
and satisfies the Poisson summation formula \[12\]

\[
\vartheta(\tau) = \sqrt{\pi \tau} \vartheta(\pi^2 \tau).
\] (21)

Applying the formula (21) to eq. (19), we get

\[
\xi_s(L, M) = \frac{1}{(4\pi)^{3/2} \Gamma(s)} \int_0^\infty d\tau \; \tau^{s-5/2} e^{-\tau M^2} \left[ \vartheta^3 \left( \frac{L^2}{4\tau} \right) - 1 \right].
\] (22)

In the asymptotic limit \(L \to \infty\), the theta-function behaves as \(\vartheta(L^2/4\tau) \sim 1 + 2 e^{-L^2/4\tau}\), so that in the same limit we can write

\[
\xi_s(L, M) \to \frac{3}{(2\pi)^{3/2} (ML)^{2-s}} (2M^2)^{3/2-s}.
\] (23)

4 Impact of the finite volume effects on the chiral behavior of \(f_K\) and \(B_K\)

In recent years a considerable effort has been invested to control the chiral extrapolations of the hadronic matrix elements computed on the lattice. To guide the extrapolation from the directly accessible quark masses, \(r \approx 0.5\), down to the physical \(r \to r_{u/d} = 0.04\), one can rely on the expressions obtained in ChPT (quenched, partially quenched or full). Those expressions, however, contain chiral logarithmic terms which so far have not been observed in the numerical studies. An important task in front of the lattice community is to lower the quark mass and get closer to the region in which the chiral logarithms become clearly visible. However, by decreasing the quark mass, the sensitivity to the finiteness of the lattice box of the side \(L\) becomes more pronounced. Moreover, the finite volume effects modify the linear light quark dependence in the same way, i.e. they enhance the chiral logs. The problem is that the non-linearity induced by the finite volume is larger than the one due to the presence of physical chiral logarithms. To illustrate that statement, in fig. 1 we plot the chiral log contributions in the finite and infinite volumes, by using the expressions presented in the previous section for both \(f_K\) and \(B_K\) in all three versions of ChPT. From that plot we see that it is difficult (if not impossible) to distinguish between physical chiral logarithms (thick curves) and finite volume effect. For smaller masses, at which the chiral logarithms are expected to set in, the finite volume effects completely overwhelm the physical non-linearity.

A possible way out would be to fit the lattice data to the finite volume forms (see Sec. 3) and not to the ones of the infinite volume, given in Sec. 2. That, of course, is legitimate if one assumes the validity of the NLO ChPT formulae. Finally, the curves corresponding to \(L = 1\) fm should be taken cautiously because it may be too small for ChPT to set in, as recently discussed in ref. [13].

5 Finite volume corrections

In this section we combine the formulae derived in Secs. 2 and 3 to discuss the shift of \(f_K\) and \(B_K\) induced by the finite volume effects. Before embarking on this issue, we first briefly
Figure 1: From up to down, we plot the chiral logarithmic corrections as predicted in full, partially quenched ($r_{\text{sea}} = m_{\text{sea}}/m_{\text{phys}} = 0.5$) and quenched ChPT, respectively, as a function of the light valence quark mass $r = m_q/m_s$, where the strange quark mass is fixed to its physical value. In each plot the thick line corresponds to the physical (infinite volume) chiral logarithm, whereas the other four curves correspond to the logarithmic contributions computed in the finite volume $V = L^3$, where for $L$ we choose the values shown in the legend. The renormalisation scale is chosen to be $\mu = 1$ GeV.