Spin chains and string theory

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Abstract: Recently, an impressive agreement was found between anomalous dimensions of certain operators in $\mathcal{N} = 4$ SYM and rotating strings with two angular momenta in the bulk of $AdS_5 \times S^5$. A one-loop field theory computation, which involves solving a Heisenberg chain by means of the Bethe ansatz agrees with the large angular momentum limit of a rotating string. We point out that the Heisenberg chain can be equally well solved by using a sigma model approach. Moreover we also show that a certain limit, akin to the BMN limit, leads exactly to the same sigma model for a string rotating with large angular momentum. The agreement is then at the level of the action. As an upshot we propose that the rotating string should be identified with a coherent, semi-classical state built out of the eigenstates of the spin chain. The agreement is then complete. For example we show that the mean value of the spin $\langle \vec{S} \rangle$ gives, precisely, the position of the string in the bulk. This suggests a more precise formulation of the AdS/CFT correspondence in the large-N limit and also indicates a way to obtain string theory duals of other gauge theories.

Keywords: spin chains, string theory, QCD.
1. Introduction

Maldacena’s AdS/CFT correspondence [1] promised to reveal the deep relation between gauge theory and string theory that was conjectured to exist in ’t Hooft’s large-N limit [2]. In a recent paper Berenstein, Maldacena and Nastase [3] made the first steps in that direction by showing that certain operators in the boundary theory corresponded to string excitation in the bulk. After that, it was observed that such relation followed from a more general relation between that of semi-classical rotating strings in the bulk [4] and certain operators in the boundary. A lot of activity followed those papers. In particular many new rotating solutions were found\(^1\). In a parallel development, Minahan and Zarembo [5] observed that the one-loop anomalous dimension of operators composed of scalars in \(\mathcal{N} = 4\) SYM theory follows from solving and integrable spin chain\(^2\). Subsequently [9], much attention was devoted to a subset of operators given by

\[
O^{J_1,J_2} = \text{Tr}ZZXX\ldots ZX,
\]

\(^1\)There is a large body of literature in the subject. For the purpose of this paper the relevant reference is [6]. The reader is urged to consult the recent review [7] for a nice introduction to the subject and a complete set of references.

\(^2\)In QCD the relation between spin chains and anomalous dimensions had already been noted in [8].
where the right hand side contains an arbitrary permutation containing a number $J_1$ of $X$s and $J_2$ of $Z$s. Here we denote $X = \Phi^1 + i\Phi^2$, $Z = \Phi^3 + i\Phi^4$ and $\Phi^a, a = 1 \ldots 6$ are the adjoint scalars of $\mathcal{N} = 4$. There are as many such operators as different permutations of the $X$ and $Z$ one can make up to cyclic permutations. In the free theory all these operators have conformal dimension $\Delta_0 = J_1 + J_2 = J$. The one loop anomalous dimension can be obtained from the 1-loop dilatation operator which, acting on these operators, takes the form\(^3\) \cite{10}:

$$D_{1\text{-loop}} = \tilde{\lambda} \sum_i \frac{1}{4} (\tilde{S}_i \tilde{S}_{i+1}), \quad \text{with} \quad \tilde{\lambda} = \frac{\lambda}{4\pi^2} = \frac{g_{YM}^2 N}{4\pi^2}.$$ \hspace{1cm} (1.2)

To apply $D_{1\text{-loop}}$ to the $O^{J_1,J_2}$, one should consider $O^{J_1,J_2}$ as a spin 1/2 chain identifying e.g. $X$ with a spin down state and $Z$ with a spin up:

$$ZZXZXX \ldots ZX \quad \iff \quad | \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \rangle.$$ \hspace{1cm} (1.3)

After this identification the spin operators $\tilde{S}$ act in the usual manner. The only point is that the trace in (1.1) implies that we have to consider periodic chains ($\tilde{S}_{J_1+1} = \tilde{S}_1$) and zero momentum states, i.e. invariant under cyclic permutations. With all these in place, the computation of 1-loop anomalous dimensions reduces to the diagonalization of a spin $\frac{1}{2}$ ferromagnetic Heisenberg chain. The length of the chain\(^4\) is $J$ and the coupling constant is $\tilde{\lambda}$. The spin chain can be diagonalized by using the Bethe ansatz (for the case in hand see \cite{5,9}). This allowed the authors of \cite{9} to obtain an operator, linear combination of the $O^{J_1,J_2}$, with anomalous dimension given by

$$\gamma = \frac{2\lambda}{\pi^2 J} K(x)(E(x) - (1 - x)K(x)), \quad \frac{J_2}{J} = 1 - \frac{E(x)}{K(x)},$$ \hspace{1cm} (1.4)

where $x$ is an auxiliary parameter that should be eliminated using the second equation. The functions $E(x)$ and $K(x)$ are standard elliptic integrals. This gives the anomalous dimension $\gamma$ as a function of $J = J_1 + J_2$ and $J_2$. Although this is in itself a remarkable achievement, the most amazing thing is that, as shown in \cite{11}, it precisely agrees with a string computation done in the dual $AdS_5 \times S^5$ theory. The string calculation involves finding a rotating string with the same quantum numbers and identifying its energy with the conformal dimension $E = J + \gamma$ as in \cite{4}. The result is \cite{6}:

$$\left( \frac{J + \gamma}{K(x)} \right)^2 - \left( \frac{J_1}{E(x)} \right)^2 = \frac{4\lambda}{\pi^2} x, \quad \left( \frac{J_2}{K(x) - E(x)} \right)^2 - \left( \frac{J_1}{E(x)} \right)^2 = \frac{4\lambda}{\pi^2}. \hspace{1cm} (1.5)$$

\(^3\)All we say in this paper is valid (and is relevant) only in the large-$N$ limit.

\(^4\)Note that this differs from the conventions used in the condensed matter literature where the strength of the interaction is usually denoted as $J$ and the length of the chain as $N$ or $L$. 

\hspace{1cm} – 2 –
This does not look the same as eq.(1.4). However we should remember that (1.4) is valid only at first order in $\lambda$. Interestingly, (1.5) also has an expansion in powers of $\lambda$ (more precisely $\lambda/J^2$) and the first term agrees exactly with (1.4). This is the main result of [11]. Subsequently, a more detailed agreement between the integrable structures\(^5\) of the classical string and the spin chain was found for these solutions in [14].

In this paper we go one step further and show that the spin chain system, in the limit of large number of sites, is described by a sigma model which precisely agrees with the sigma model obtained from the rotating string in the relevant limit. From the Heisenberg chain side the result is well-known [15], the low energy modes can be described through an effective action which, at the order in $1/J$ we need, has only one undetermined coefficient that can be fixed from the BMN limit. From the string theory side it amounts to taking a limit of the Polyakov action. The identification then goes beyond a particular solution. Moreover this gives a precise mapping of the states. For example we show that the mean value of the spin at a given site is the same as the position of the string in the bulk. That is, given a state of the spin chain we can map it into a string configuration making the identification complete. In the states we find, the spins are precessing with exactly the same motion as the string.

The results of [11, 14] are a manifestation of the agreement between these two ways of describing a spin chain, in terms of Bethe eigenstates or through a low energy effective action. The AdS/CFT correspondence, on the other hand, states that one of those descriptions also corresponds to a string rotating in $AdS_5 \times S^5$ (in a particular limit).

In the reminder of the paper, we analyze briefly the higher loop dilatation operator which leads us to suggest that a more precise formulation of the large-N Maldacena conjecture is that the classical action for string theory in global $AdS_5 \times S^5$ is equivalent to the long wave length limit of the dilatation operator acting on single trace operators. By long wave length limit we mean when taking traces of a large number of operators. Of course this is a well known idea since we are just saying that the dilatation operator is the hamiltonian in global coordinates. Perhaps the new ingredient, if any, is that we are suggesting a slightly more precise way to extract the string action from the field theory. Unfortunately, it is not as precise as we would like it to be since, as we shall see, the exact way of mapping the sigma models at higher loops is not completely clear to us. Even knowing the exact mapping, in practice it is not feasible to do an all loop computation in the field theory, but we believe it is interesting that there is a way, at least in principle to embed the string theory in the field theory. Here we do just that at 1-loop by using previously known results. Finally, it is clear that this type of calculation

\(^5\)See e.g. [12, 13] for an account of the relation between integrable systems and rotating strings.
can be extended to any gauge theory suggesting a way to obtain a string dual of a gauge theory. By constructing the corresponding spin chain and taking the long wave length limit, a sigma model will emerge that describes a string moving in some background. With some effort (i.e. computing more loops) or some guess-work one might be able to reproduce the whole background. The known alternative method is to construct the theory using D-branes and then finding the supergravity background.

2. Heisenberg chain and ferromagnetic sigma model

The ferromagnetic spin $s$ Heisenberg chain is exactly equivalent to a discrete sigma model known as the ferromagnetic sigma model. By discrete sigma model we mean that there are a discrete number of sites with continuum variables leaving on them. In the long wave length limit this reduces to a usual sigma model up to corrections $1/J$ where $J$ is the length of the chain.

Since this is well known, in this section we just sketch where this sigma model comes from and its main properties. We refer the reader to e.g. [15] for a detailed account and references to the original work. To obtain a sigma model one defines a set of coherent states labeled by two angles $\phi$ and $\theta$ and defined as

$$|\vec{n}\rangle = e^{iS_z\phi}e^{iS_y\theta}|ss\rangle,$$

where $\vec{n}$ is a unit vector with components

$$\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),$$

and $|ss\rangle$ is a state of total spin $s$ and maximal $z$-projection $S_z = s$. The identity can be written as

$$1 = \left(\frac{2s + 1}{4\pi}\right) \int \sin^2 \theta d\theta d\phi \langle \vec{n}|\langle \vec{n}|.$$ (2.3)

Inserting this in the standard derivation of the path integral from the Hamiltonian approach one finds that a transition amplitude can be computed in terms of a path integral

$$A = \int \mathcal{D}\vec{n}e^{iS(\vec{n})},$$

with action

$$S(\vec{n}) = s \sum_k \int dt \int_0^1 d\tau \vec{n}_k.(\partial_t \vec{n}_k \times \partial_\tau \vec{n}_k) - \frac{\tilde{\lambda}s^2}{2} \int dt \sum_k (\vec{n}_k - \vec{n}_{k+1})^2.$$ (2.5)

\footnote{Some times the word sigma model implies a particular form of the action. Here we just mean a field theory with a target space given by a compact manifold, in this case the coset space $SU(2)/U(1) = S^2$.}
The first term is a Wess-Zumino term proportional to the area spanned between the trajectory and the North pole. Its definition requires the introduction of an additional coordinate $\tau$ with $\theta(\tau = 1) = 0$. The variation of the Wess-Zumino term is given just by a boundary term at $\tau = 0$ which is the only way in which it enters in the equations of motion. The important point to note is that this is exactly equivalent to the Heisenberg chain, no approximation was made. At first sight this does not seem possible since now we have, in appearance, an infinite number of states per site. However this is not so. The Wess Zumino term is equivalent to a magnetic charge $s$ at the center of the $S^2$ over which $\vec{n}$ is moving. Therefore, one should find the Landau levels. For charge $s$ one finds precisely $2s + 1$ levels. The gap to the other levels is infinite since no kinetic term $\partial_t \vec{n} \partial_t \vec{n}$ is present. Another consequence is that, since the action is linear in time derivatives, near the ground state the dispersion relation is quadratic ($\omega \sim k^2$) as is known to be for ferromagnetic magnons\(^7\). In our case this is the BMN limit of operators (1.1) where $\gamma \sim n^2$.

As it is, there is no advantage in using the sigma model. The interest arises when one takes the limit of very large chains, or large $J$ in operator language. In that limit it makes sense to consider the long wave length limit of (2.5) and consider a continuous coordinate $\sigma$ running from 0 to $J$. The action is

$$S(\vec{n}) = s \int dt d\sigma \int_0^1 d\tau \sin \theta (\partial_\tau \phi \partial_\theta - \partial_\phi \partial_\theta) - \frac{\tilde{\lambda} s^2}{2} \int d\sigma dt \left((\partial_\tau \theta)^2 + \sin^2 \theta (\partial_\sigma \phi)^2\right),$$

(2.6)

where we used the parameterization (2.2). From here we obtain the momenta, Hamiltonian and momentum density in direction $\sigma$:

$$P_\phi = S_z = -s \int d\sigma \int_0^1 d\tau \sin \theta \partial_\tau \theta = -s \int \cos \theta d\sigma,$$

(2.7)

$$P_\theta = s \int d\sigma \int_0^1 d\tau \sin \theta \partial_\tau \phi,$$

(2.8)

$$H = P_\phi \partial_t \phi + P_\theta \partial_t \theta - L = \frac{\tilde{\lambda} s^2}{2} \int d\sigma \left((\partial_\sigma \theta)^2 + \sin^2 \theta (\partial_\sigma \phi)^2\right),$$

(2.9)

$$P = \int T_{01} d\sigma = -s \int_0^J \cos \theta \partial_\sigma \phi = 0,$$

(2.10)

where the last equality is a condition on the solutions. Form the operator point of view this means that we consider operators invariant under cyclic permutations (see discussion below eq.(1.3)). The ground state of this system is with all spins aligned

\(^7\)This is not the case for antiferromagnets were $\omega \sim k$ and which are described by the sine-Gordon model.
parallel since it is ferromagnetic. On the other hand, we are interested here in highly excited states. Since quantum numbers are large we expect that they will be well described by classical solutions of the equations of motion. This classical solutions should emerge in quantum theory as superposition of a large number of eigenstates of the same approximate energy. Since this argument cannot be applied to individual spins because they have only two states, what we consider is that $\vec{n}$ describes the average spin over long domains that form since the ferromagnetic Heisenberg chain has an ordered ground state. Namely the spins tend to be parallel at short distances with fluctuations occurring only at long distances.

We should also comment that in taking the continuum limit the coupling constants can be renormalized. The number in front of the Wess-Zumino term has to be a half integer by a topological argument so it cannot be renormalized but the constant in front of the gradient in principle can. That such renormalization does not occur can be checked by computing the spectrum of magnons near the ground state and comparing with the known exact result\textsuperscript{8}. In our case one would say that one checks that there is no renormalization of the coupling constant by comparing with the BMN limit.

In principle this is all we wanted to do in this section. In the next one we show that exactly the same sigma model follows from taking the large angular momentum limit of a string rotating in $S^3$ (which is the relevant part of $AdS_5 \times S^5$). However the reader will probably feel much better if we use the sigma model to reproduce for example (1.4) showing that it actually gives the same result as the Bethe ansatz.

Before doing that however, we want to argue that since we are only going to look for classical solutions we can integrate by parts the Wess-Zumino term and convert it into a more conventional term. This amounts to write the volume form $\omega_2 = \sin \theta d\theta d\phi$ of the sphere as an exact form $\omega_2 = dA$, $A = -\cos \theta d\phi$. This is singular at the poles but for studying the classical solutions we can ignore that fact. The Lagrangian becomes

$$S(\vec{n}) = -s \int d\sigma dt \cos \theta \partial_\phi - \frac{\tilde{\lambda}s^2}{2} \int d\sigma dt \left((\partial_\sigma \theta)^2 + \sin^2 \theta (\partial_\sigma \phi)^2\right).$$

\textbf{2.1 Particular solutions}

We want to find now a particular classical solution with given angular momentum. Note that $J$ is already given by the length of the chain. On the other hand, we can fix $S_z$, the total component of the spin in direction $z$. For the operators $\mathcal{O}^{J_1,J_2}$ we have $S_z = (J_2 - J_1)/2$. The equations of motion that follow from the Lagrangian are

$$\sin \theta \partial_t \theta + \tilde{\lambda}s \partial_\sigma \left(\sin^2 \theta \partial_\sigma \phi\right) = 0,$$

$$\sin \theta \partial_t \phi + \tilde{\lambda}s \partial_\sigma^2 \theta - \tilde{\lambda}s \sin \theta \cos \theta (\partial_\sigma \phi)^2 = 0.$$\textsuperscript{8}

\textsuperscript{8}This comment refers to the Heisenberg chain, not the whole field theory.
The boundary conditions are
\[ \phi(\sigma = J, t) = \phi(\sigma = 0, t), \quad \theta(\sigma = J, t) = \theta(\sigma = 0, t), \quad (2.14) \]
and the zero momentum condition (2.10) is still
\[ P_\sigma = -\frac{1}{2} \int \cos \theta \partial_\sigma \phi = 0. \quad (2.15) \]
Now we make the ansatz \( \partial_\sigma \phi = 0 \) which immediately satisfies this last condition\(^9\). The first equation immediately implies that \( \partial_t \theta = 0 \). The second equation then implies that \( \partial^2_t \phi = 0 \). So we put \( \partial_t \phi = w \). Finally the only equation that we have to solve is
\[ \partial^2_\sigma \theta + \frac{w}{\lambda s} \sin \theta = 0. \quad (2.16) \]
This integrates to
\[ \partial_\sigma \theta = \pm \sqrt{a + b \cos \theta}, \quad \text{with} \quad b = \frac{2w}{\lambda s}, \quad (2.17) \]
and \( a \) is a constant of integration. Two possibilities arise. If \( a > |b| \) then the square root never vanishes. This means that \( \theta \) keeps increasing and then the periodic condition should read \( \theta(J) = \theta(0) + 2\pi n \), for some integer \( n \). The other possibility is that \( b > |a| \). Then, at \( \theta_0 = \arccos(-a/b) \) the square root becomes zero and we can change branches, namely the “particle” returns oscillating between \( -\theta_0 < \theta < \theta_0 \). In this paper we concentrate in this solution and leave the other case for the interested reader.

With this solution we compute the energy and angular momentum as:
\[ S_z = -4s \int_0^{\theta_0} \frac{\cos \theta}{\sqrt{a + b \cos \theta}} d\theta, \quad (2.18) \]
\[ J = \int_0^J d\sigma = 4 \int_0^{\theta_0} \frac{d\theta}{\sqrt{a + b \cos \theta}}, \quad (2.19) \]
\[ \gamma = E = 2\tilde{\lambda} s^2 \int_0^{\theta_0} \frac{a + b \cos \theta}{\sqrt{a + b \cos \theta}} d\theta = \frac{\tilde{\lambda} s^2}{2} \left( aJ - \frac{b}{s} S_z \right). \quad (2.20) \]
The integrals can be performed in terms of elliptic integrals [16]\(^{10}\) giving
\[ S_z = -4s \sqrt{\frac{2}{b}} \{2E(x) - K(x)\}, \quad (2.21) \]
\[ J = 4 \sqrt{\frac{2}{b}} K(x), \quad (2.22) \]
\[ x = \frac{a + b}{2b}. \quad (2.23) \]

\(^9\)It would be interesting to study solutions with \( \partial_\sigma \phi \neq 0 \) since (2.15) does not necessarily imply \( \partial_\sigma \phi = 0 \). This other solutions should also agree with supergravity.

\(^{10}\)For simplicity of comparison we follow the notation of [11] rather than that in [16] which differ in the argument of the elliptic integral being \( x \) or \( \sqrt{x} \).
Finally, using that \( S_z = \frac{J_2 - J_1}{2} \), simple algebra leads to

\[
\frac{J_2}{J} = (s + \frac{1}{2}) - 2s \frac{E(x)}{K(x)},
\]

\[
\gamma = E = \frac{\tilde{\lambda}}{J} 32 s^2 K(x) \left[ E(x) - (1 - x)K(x) \right].
\]

If we now replace \( s = \frac{1}{2} \) and, from (1.2), \( \tilde{\lambda} \to \lambda/4\pi^2 \) we get precisely (1.4).

What we have just obtained is a sigma model that reproduces the Bethe ansatz result. But, actually, that is already done by the string theory sigma model, albeit in a particular limit. It is obvious that both should be related. In the next section we derive this same sigma model directly from the string theory action.

Before doing so, let us analyze the solution we have just obtained a little further. At an instant in time the position of the spins is given by \( \theta(\sigma) \). We see that, as sigma is varied, the end point of \( \vec{n} \) goes from \( \theta = 0 \) to \( \theta = \theta_0 \) and back. Then to the other side. This looks like a folded string already. Furthermore, each point is precessing around the \( z \) axis with the same angular velocity \( w \). So the configuration looks exactly like a rotating string!. The only thing lacking from the Frolov-Tseytlin solution is the motion of the center of mass. We will see that to get this sigma model from the string picture one has to subtract the motion of the center of mass in a similar way as is done in the pp-wave limit \(^{11}\). A final caveat is that when we talk about time dependent solutions, time is the conjugate variable to the Hamiltonian which here is actually the dilatation operator. So, from the field theory point of view we are thinking of an operator with some weird properties under scaling. This makes sense however when we think of the state/operator correspondence which exists in conformal theories. Under that mapping, such an operator maps into a time dependent state which is precisely what we want since the string is a time dependent state. That the relation between the operator and the string includes a state/operator correspondence is well known from the rules of the AdS/CFT correspondence.

### 3. Rotating string

The rotating string solutions corresponding to the operators discussed in the previous section were found in [6]. The relevant part of the metric in the coordinates used in that paper is

\[
ds^2 = -dt^2 + d\Omega_2^2 = -dt^2 + d\psi^2 + \cos^2 \psi d\phi_1^2 + \sin^2 \psi d\phi_2^2.
\]

\(^{11}\)Actually, in our case we have to subtract also another “fast” rotation as it will become apparent in the next section. I am grateful to A. Tseytlin for pointing this out.
Changing coordinates to $\phi_1 = \varphi_1 + \varphi_2$, $\phi_2 = \varphi_1 - \varphi_2$ we get a metric

$$ds^2 = -dt^2 + d\psi^2 + d\varphi_1^2 + d\varphi_2^2 + 2\cos(2\psi)d\varphi_1d\varphi_2.$$  \hspace{1cm} (3.2)

In the Frolov-Tseytlin solution the center of mass moves with angular velocity $w$ in direction $\varphi_1$. In the limit of large angular momentum $w$ tends to one. It makes sense then to do a change of coordinates

$$\varphi_1 \to t + \varphi_1,$$  \hspace{1cm} (3.3)

and the metric becomes

$$ds^2 = 2dt^2 + d\psi^2 + d\varphi_1^2 + d\varphi_2^2 + 2\cos(2\psi)dt\varphi_1 + 2\cos(2\psi)d\varphi_1d\varphi_2.$$  \hspace{1cm} (3.4)

After making the gauge choice $t = \kappa \tau$, the Polyakov action describing a string in this background becomes:

$$S = \frac{R^2}{4\pi \alpha'} \int G_{\mu\nu}\partial_\tau X^\mu \partial_{\tau'} X^\nu - G_{\mu\nu}\partial_\sigma X^\mu \partial_{\sigma'} X^\nu$$  \hspace{1cm} (3.5)

$$= \frac{R^2}{4\pi \alpha'} \int 2\kappa \ddot{\varphi}_1 + \dot{\psi}^2 + \ddot{\varphi}_1 + \ddot{\varphi}_2 + 2\cos(2\psi)\kappa \dot{\varphi}_2 + 2\cos(2\psi)\dot{\varphi}_1 \dot{\varphi}_2$$  \hspace{1cm} (3.6)

$$- \psi'^2 - \varphi_1'^2 - \varphi_2'^2 - 2\cos(2\psi)\varphi_1' \varphi_2',$$  \hspace{1cm} (3.7)

where we denote derivatives with respect to $\tau$ with a dot and those with respect to $\sigma$ with a prime. The Virasoro constraints are:

$$G_{\mu\nu}\partial_\tau X^\mu \partial_{\sigma'} X^\nu = 2\kappa \ddot{\varphi}_1 + \dot{\psi}^2 + \ddot{\varphi}_1 + \ddot{\varphi}_2 + 2\cos(2\psi)\kappa \dot{\varphi}_2 + 2\cos(2\psi)\dot{\varphi}_1 \dot{\varphi}_2 +$$  \hspace{1cm} (3.8)

$$+ 2\cos(2\psi)\dot{\varphi}_2 \dot{\varphi}_1' = 0,$$

and

$$G_{\mu\nu}\partial_{\tau} X^\mu \partial_{\tau'} X^\nu + G_{\mu\nu}\partial_{\sigma} X^\mu \partial_{\sigma'} X^\nu = 2\kappa \ddot{\varphi}_1 + \dot{\psi}^2 + \ddot{\varphi}_1 + \ddot{\varphi}_2 + 2\cos(2\psi)\kappa \dot{\varphi}_2 + 2\cos(2\psi)\dot{\varphi}_1 \dot{\varphi}_2 +$$  \hspace{1cm} (3.9)

$$+ \psi'^2 + \varphi_1'^2 + \varphi_2'^2 + 2\cos(2\psi)\varphi_1' \varphi_2' = 0,$$

where we have also used the gauge choice $t = \kappa \tau$.

Up to now we did not do any approximation. Now we are going to assume that the motion of the string is mainly captured by the rotation we just did through the change of coordinates. Specifically, we assume that all time derivatives are small. If we drop time derivatives altogether then the string will be moving at the speed of light and the only solution is the BPS state corresponding to a massless string moving around the circle. Looking at the action and the constraints one sees that a non-trivial limit is obtained by taking

$$\hat{X}^\mu \to 0, \quad \kappa \to \infty, \quad \text{with} \quad \kappa \hat{X}^\mu \text{ fixed}.$$  \hspace{1cm} (3.10)
Here \( X^\mu \) denotes any coordinate except \( t \). Later on, we will see that taking this limit in the Frolov-Tseytlin solution precisely corresponds to the large angular momentum limit with \( J \sim \kappa \).

As a comment we want to indicate that this limit, although similar to the BMN limit\(^\text{[3]}\) is somewhat different. Here, we are not taking the limit directly in the metric but in the action. As a result we keep terms such as \( \varphi_2' \), which would not be there if we were zooming into a geodesic.

In the limit (3.10), the action reduces to

\[
S = \frac{R^2}{4\pi\alpha'} \int 2\kappa \dot{\varphi}_1 + 2 \cos(2\psi) \kappa \dot{\varphi}_2 - \psi'^2 - \varphi_1'^2 - \varphi_2'^2 - 2\cos(2\psi)\varphi_1'\varphi_2',
\]

and the constraints become

\[
\begin{align*}
2\kappa \varphi_1' + 2 \cos(2\psi) \kappa \varphi_2' &= 0, \\
2\kappa \dot{\varphi}_1 + 2 \cos(2\psi) \kappa \dot{\varphi}_2 + \psi'^2 + \varphi_1'^2 + \varphi_2'^2 + 2 \cos(2\psi)\varphi_1'\varphi_2' &= 0.
\end{align*}
\]

These constraints determine \( \varphi_1 \) as a function of \( \sigma \) and \( \tau \). Since from the action there is no equation of motion for \( \varphi_1 \) we can always satisfy them. The only caveat is that \( \varphi_1 \) is determined twice. However, one can check using the equations of motion that \( \dot{\varphi}_1' \) as determined by the first or second constraint agree. Also since the string is closed, \( \varphi_1 \) is a periodic function of sigma which implies

\[
0 = \int d\sigma \varphi_1' = - \int d\sigma \cos(2\psi)\varphi_2',
\]

which, as we will see below is the condition (2.10).

Replacing the first constraint in the action we get

\[
S = \frac{R^2}{4\pi\alpha'} \int 2\kappa \varphi_1 + 2 \cos(2\psi) \kappa \varphi_2 - \psi'^2 - \sin^2(2\psi)\varphi_2'^2,
\]

which already looks quite similar to the sigma model we had before. To make the agreement precise we first compute the angular momentum

\[
J = \mathcal{P}_{\varphi_1} = \frac{R^2}{4\pi\alpha'} 2\kappa \int_0^{2\pi} d\sigma = \int_0^J d\tilde{\sigma},
\]

where we defined

\[
\tilde{\sigma} = \frac{R^2}{4\pi\alpha'} 2\kappa \sigma.
\]

\(^\text{12}\)It also has some similarities with the so called wrapped or non-relativistic limit [17]. For the present case some related ideas appeared in [18] and [19].
so that the length of the chain is $J$ as before. To compare the energies we rescale $\tau$ into $t = \kappa \tau$. Also one can see that the angles are related by

$$\varphi_2 = -\frac{1}{2} \phi, \quad \psi = \frac{1}{2} \theta,$$  

(3.18)

which is related to the fact (at least for $\varphi$) that the chain has sites with spin 1/2. Finally we use the AdS/CFT relation $R^2/\alpha' = \sqrt{\lambda}$ to get the action

$$S = -\frac{1}{2} \int dtd\tilde{\sigma} \cos \theta \partial_t \phi - \frac{\lambda}{32\pi^2} \int dtd\tilde{\sigma} \left( (\partial_\sigma \theta)^2 + \sin^2 \theta (\partial_\sigma \phi)^2 \right),$$  

(3.19)

which precisely agrees with (2.11) after we put $s = 1/2$. Also the identification of the angles (3.18) implies that we can map directly a configuration $\vec{n}(\sigma)$ into a particular shape of the string in the bulk regardless if it is a solution or not. Since $\vec{n}$ is the average value of the spin at a site, $\langle \vec{n} | \vec{S} | \vec{n} \rangle = s \vec{n}$ one can identify the average value of the spin with the position of the corresponding portion of the string in the bulk. This is slightly more than we are entitled to argue since the calculations are valid in the long wave length limit only. However it is a natural conjecture to make. It is not trivial because it implies that if we look at the string carefully enough we will see the discrete nature of the spin chain. This can be a generic feature of string theory or just a peculiarity of the limit we are taking.

### 3.1 Particular solutions

It is clear that, since the actions agree, we get the same solutions. However we still need to check that at least in the case of the Frolov-Tseytlin solution the limit we have taken agrees with the one taken in [11]. The solution for the rotating string in [6] follows from the ansatz:

$$t = \kappa \tau, \quad \phi_1 = w_1 \tau, \quad \phi_2 = w_2 \tau, \quad \psi = \psi(\sigma).$$  

(3.20)

The relations between the energy, angular momentum and angular velocities can be cast in the form [11]:

$$x \equiv \frac{\kappa^2 - w_1^2}{w_2^2 - w_1^2},$$  

(3.21)

$$E = \kappa,$$  

(3.22)

$$1 = \frac{1}{w_1} J_1 + \frac{1}{w_2} J_2,$$  

(3.23)

$$J_1 = \frac{2w_1}{\pi \sqrt{w_2^2 - w_1^2}} E(x),$$  

(3.24)

$$\frac{2}{\pi} K(x) = \sqrt{w_2^2 - w_1^2},$$  

(3.25)
where we remind the reader that $E$, $J_1$ and $J_2$ are rescaled by $\sqrt{\lambda}$ in these equations. The limit we described before corresponds to taking

$$w_1 = \kappa - \epsilon_1, \quad w_2 = \kappa + \epsilon_2, \quad \epsilon_{1,2} \to 0, \quad \kappa \to \infty, \quad \text{with } \kappa \epsilon_{1,2} \text{ fixed}. \quad (3.26)$$

For the moment however we just assume that $\epsilon_{1,2} \to 0$. In this limit we get

$$J_1 = \frac{1}{\pi} \sqrt{\frac{2\kappa}{\epsilon}} E(x_0), \quad (3.27)$$

$$J_2 = \kappa - \frac{1}{\pi} \sqrt{\frac{2\kappa}{\epsilon}} E(x_0), \quad (3.28)$$

where $\epsilon = \epsilon_1 + \epsilon_2$ and $x_0 = 1/(1 - \epsilon_2/\epsilon_1)$ is the limiting value of $x$. This implies

$$\frac{J_2}{J_1} = -1 + \frac{\pi}{\sqrt{2}} \frac{\sqrt{\kappa \epsilon}}{E(x_0)}. \quad (3.29)$$

We see that in this limit both angular momenta diverge. However, if we want the ratio $J_2/J_1$ to be fixed at an arbitrary value, we need to consider $\kappa \to \infty$ with $\kappa \epsilon$ fixed. It is also easy to show that at lowest order we have

$$E = J = \kappa, \quad (3.30)$$

so $\kappa \to \infty$ precisely corresponds to $J \to \infty$.

### 4. Higher loops

In this section we begin the study of the sigma model for higher loops in the gauge theory side. It was shown in [10] that the two-loops dilatation operator has the form$^{13}$

$$D_{2\text{-loops}} = \frac{\lambda^2}{128 \pi^4} \left[ -\frac{3}{2} J + 8 \sum_k \tilde{S}_k \tilde{S}_{k+1} - 2 \sum_k \tilde{S}_k \tilde{S}_{k+2} \right]. \quad (4.1)$$

This is a spin chain with first and second neighbors interactions. One might think that this is trivial since the sigma model for a second neighbor interaction is the same as for first neighbors. However if that would be the case the correction would be $\lambda^2/J^3$ instead of the $\lambda^2/J^3$ as expected. We can see what happens by actually doing the calculation using:

$$\tilde{n}_{k+p} = n_{k+p} + \frac{p}{2} \partial_\sigma n_{k+p} + \frac{p^2}{24} \partial^2_\sigma n_{k+p} + \frac{p^3}{8} \partial^3_\sigma n_{k+p} + \frac{p^4}{16} \partial^4_\sigma n \ldots, \quad (4.2)$$

$$\tilde{n}_k \tilde{n}_{k+p} = 1 + \frac{p^2}{4} \tilde{n} \partial^2_\sigma \tilde{n} - \frac{p^2}{4} \partial_\sigma \tilde{n} \partial_\sigma \tilde{n} = \frac{p^4}{48} \partial^4_\sigma \tilde{n} \partial^2_\sigma \tilde{n} + \frac{p^4}{64} \partial^2_\sigma \tilde{n} \partial^2_\sigma \tilde{n} + \frac{p^4}{192} \partial^4_\sigma \tilde{n} + \ldots \quad (4.3)$$

$$\sum_k \tilde{n}_k \tilde{n}_{k+p} = \int d\sigma \left( 1 - \frac{p^2}{2} \partial_\sigma \tilde{n} \partial_\sigma \tilde{n} \ldots \right). \quad (4.4)$$

$^{13}$In [10] it was actually written in terms of permutation operators. The form given here is equivalent.
By replacing in (4.1) we see that all terms up to (and including) cubic order in derivatives cancel. That is, the sigma model is higher order in derivatives, in this case four. Let us see how this affects the results. Since we want states with lowest energy for a given spin we expect $\vec{n}$ to vary slowly. If the variations are of order one and the length of the chain is $J$ then we estimate that

$$\partial_\sigma n \sim \frac{1}{J}. \quad (4.5)$$

For the 1-loop calculation this gives

$$\gamma_1 \sim \lambda \int (\partial_\sigma \vec{n})^2 \sim \lambda J \frac{1}{J^2} \sim \frac{\lambda}{J}; \quad (4.6)$$

as we obtained before. At two loops we have

$$\gamma_2 \sim \lambda^2 \int \partial_\sigma^2 \vec{n} \partial_\sigma^2 \vec{n} \sim \frac{\lambda^2}{J^3}, \quad (4.7)$$

as we expect if the BMN limit is well defined. Furthermore, in the $n$-loop sigma model, all the terms having $2n - 1$ derivatives or less should vanish. Those higher than $2n$ in derivatives are irrelevant in this limit. Therefore, for the full dilatation operator, including the Wess-Zumino term $S_{WZ}$, we should have the schematic expansion

$$D = J + S_{WZ} + \lambda \int (\partial_\sigma \vec{n})^2 + \lambda^2 \int (\partial_\sigma \vec{n})^4 + \ldots + \lambda^n \int (\partial_\sigma \vec{n})^{2n} \ldots \quad (4.8)$$

This is just schematic since, at each order, one can have several terms with the same number of derivatives. This is a peculiar low energy expansion where, each order in perturbation theory contributes at a certain order in momentum (or higher). It would be interesting to argue for this using supersymmetry perhaps as in [20]. It is also tempting to speculate that the absence of higher loop corrections to the $(\partial_\sigma \vec{n})^2$ term is related to the non-renormalization of the coupling constant in $\mathcal{N} = 4$ SYM.

An important question is what is exactly the relation between the all-loop sigma model and the classical string action in $AdS_5 \times S^5$ (or in $\mathbb{R}^1 \times S^3$ for the case in hand). The Maldacena correspondence would indicate that they are the same. However the sigma model we just wrote is first order in time derivatives and infinite order in spatial derivatives. The Polyakov action is quadratic in both. It seems that the precise relation involves integrating out the quadratic time derivatives and trading that for a non-local action in $\sigma$. In this first approach we leave a more detailed analysis for future work.

5. Conclusions

In this paper we have shown that, as far as the one-loop anomalous dimension is concerned, the computation can be done using a sigma model in both, the field theory
side and the string theory side. Fortunately, the sigma models agree. This shows that, at least in this particular sector of operators, agreement is guaranteed for any solution that can be found. It seems an interesting problem to generalize this to the full bosonic sigma model and perhaps to include the fermions. In that case, from [21, 22] we expect to reproduce a limit of the Metsaev-Tseytlin action [23]. This might also shed light in the comparison of integrable structures [24, 25, 26].

The most striking feature of the calculation is that the string appears in the field theory side. The time dependent solution that we found is precisely the same as the rotating string. The expectation values of the spin is the position in space time of the corresponding point of the string (after we discount the center of mass motion). It seems plausible that this can allow us to derive a string dual for field theories for which they are not known. Of course there is no guarantee that the resulting string action corresponds to a critical superstring.

We also discussed briefly the extension to higher loop order. For the solutions we analyzed, it does not seem too difficult to compute the two or higher loop corrections using the sigma model once the Hamiltonian for the spin chain is known. This is not so if one uses the Bethe ansatz. Of course now one would be more ambitious and try to match the sigma models without referring to particular solutions. It would be interesting to see what picture emerges from there.

In particular it suggests that there is a precise formulation of the Maldacena correspondence which says that the sigma model obtained from the all-loop spin chain Hamiltoninan is equivalent to the classical action of the string in $AdS_5 \times S^5$. At 1-loop the mapping is simply that they are both the same. At higher loops the picture seems more complicated but we believe a careful analysis should indicate how to make the formulation of the correspondence completely precise. By that I mean exactly what calculation one has to do in the field theory to recover the action of the string. We should perhaps clarify that this would be a formulation of the weak version of the conjecture valid for large $N$. The strong version valid for any $N$ is simply that the Hamiltonians and space of states completely agree. To go in that direction one has to include multitrace operators and relate the “splitting” of a single trace operator into a double trace one with a string interaction [3, 27].

Many interesting field theories do not have scalar fields in the adjoint. Similar operators can be constructed however as

$$\mathcal{O} = \text{Tr} \nabla^{n_1} F \nabla^{n_2} F \nabla^{n_3} F \cdots \nabla^{n_N} F,$$

where $F$ is the field strength and we suppressed spatial indices. We see now that in each site an infinite number of derivatives (or states can appear). This is probably
related to the emergence of a fifth dimension in the string dual, the radial dimension in \textit{AdS} or the “thickness” in the QCD string.

Let us end with two speculations. What we have argued here is that, at least in a very restricted sector, the string action emerges as the infrared limit of the dynamics of a discrete lattice. It is tempting then to argue that the usual, conformally invariant action should be only the infrared fixed point of a more general, non-conformal model. There is going to be a regime where the discrete model is applicable and another where the conformal theory is. However there has to be another, intermediate regime, where a continuum but non-conformal model is appropriate. Since the discrete description corresponds to the field theory whose description is valid for very large curvature and the conformal theory for small curvature, it is natural to conjecture that a string in a background of curvature of order one (in string units) is described by a non-conformal theory. How this can be true is not clear to us.

The other speculation concerns the behavior of strings at finite temperature. Since we argued that the expectation value of the spin gives the position of the string in space time, a high temperature disordered state of the spin chain looks in space time as a string whose shape resembles a random walk rather that a continuum curve. We speculate that this can be a good description for strings above the Hagedorn temperature.

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NOTE ADDED

After the first version of this paper appeared, in the works [28] some related issues were clarified.

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