Dynamics of quantum correlations and linear entropy in a multi-qubit-cavity system

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Abstract. We present a theoretical study of the relationship between entanglement and entropy in multi-qubit quantum optical systems. Specifically we investigate quantitative relations between the concurrence and linear entropy for a two-qubit mixed system, implemented as two two-level atoms interacting with a single-mode cavity field. The dynamical evolutions of the entanglement and entropy, are controlled via time-dependent cavity-atom couplings. Our theoretical findings lead us to propose an alternative measure of entanglement, which could be used to develop a much needed correlation measure for more general multi-partite quantum systems.

1. Introduction

The distinction between true quantum correlations (entanglement) and classical correlations, is a fundamental topic in physics. Entanglement plays a central role in quantum information (QI) science. However it is a brittle phenomenon: in any real experimental situation, the unavoidable interaction of the quantum system with its environment results in decoherence processes, and will eventually yield mixed quantum states. The environment of a single quantum system might be a complex collection of other quantum systems with an enormous number of degrees of freedom, or it might be as simple as a second single quantum system. The main properties of entangled/mixed states have recently been discussed in Refs. [1, 2, 3]. It is thought that a total increase in the ‘mixedness’ of a multi-partite system will generally produce a decrease in the entanglement. Nevertheless we believe that a full understanding of the connection between entanglement and mixedness, will require additional insight gained by considering specific physical realizations of such multi-partite systems.

Quantum optics, which considers systems of atoms interacting strongly with photons (quantum electrodynamics), provides an ideal setting for the study of open quantum systems [4]. Apart from being prime candidates for QI processing [5], such quantum optical systems are of fundamental theoretical interest. In particular, the experimental and theoretical study of quantum optical systems may yield insight into the connections between mixedness and entanglement. Mixedness, which is associated with the lack of purity of a quantum state, is usually measured by the linear entropy

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\( M = 1 - Tr(\rho^2) \) [6] where \( \rho \) is the open quantum system’s density operator. No general measure of entanglement in \( N \)-partite systems is known for \( N \geq 3 \). For bipartite systems, however, entanglement has been described by several measures. One of these is the entanglement of formation, which represents the asymptotic number of Bell pairs required to prepare the state using only local unitary transformations and classical communication. Intimately related to this entanglement measure is the notion of concurrence. An exact, closed expression for the concurrence in two-qubit systems, was obtained by Wooters [7].

In this paper we report on quantitative relations between the concurrence and linear entropy for a two-qubit mixed system, implemented as two two-level atoms interacting with a single-mode cavity field. Even when the whole system is evolving under a single unitary evolution, the atoms’ surrounding environment (i.e. the cavity field) can become entangled with the atomic subsystem causing mixedness of the latter. This analysis is carried out with generalized time-dependent cavity-atom couplings, yielding a generalized dynamical Dicke model. Finally we discuss how our results might extend to multi-particle systems.

2. Two-qubit-cavity system

We consider a pair of two-level atoms interacting on resonance with the quantized mode of an optical field during times shorter than both the photon and the atomic dipole lifetimes. We systematically compare the dynamics of the two-atom entanglement, atom-field entanglement, and the linear entropy of each of the three constituents of the system: atom \( A_1 \) (\( A_1 \)), atom \( A_2 \) (\( A_2 \)), and field (\( f \)). The atoms can interact with the cavity mode either one at a time or simultaneously. In both cases, different time-dependent couplings between each atom and the field are considered. Comparing these two situations serves to enhance our understanding of the relationships between entanglement and entropy during the generation, transfer or sharing of entanglement.

In the rotating-wave approximation (RWA) and assuming the atomic transitions are on resonance with the frequency of the single-mode cavity, the Hamiltonian in the interaction picture is given by (\( \hbar = 1 \))

\[
H_I = f_1(t_1, t, \tau_1)\left\{a^\dagger \sigma^+_1 + \sigma^+_1 a\right\} + f_2(t_2, t, \tau_2)\left\{a^\dagger \sigma^+_1 + a^\dagger \sigma^-_2 + \sigma^+_2 a\right\}
\]

where \( \sigma^+_i = |e_i\rangle\langle g_i|, \sigma^-_i = |g_i\rangle\langle e_i| \) with \( |e_i\rangle \) and \( |g_i\rangle \) \((i = 1, 2)\) being the excited and ground states of the \( i \)'th atom. Here \( a^\dagger \) and \( a \) are respectively the creation and annihilation operators for the cavity photons. The time-dependent coupling of the cavity field with the \( i \)'th atom, which is injected at \( t_i \) and interacts during a time \( \tau_i \), is given by a time-window function,

\[
f_i(t_i, t, \tau_i) = \left[\Theta(t - \tau_i) - \Theta(t - \tau_i - t_i)\right] \gamma_i(t)
\]

where \( \gamma_i(t) \) is the time-dependent atom-field coupling strength. For the situation in which the second atom flies through the cavity just after the first one leaves, i.e. \( t_2 = t_1 + \tau_1 \), the Hamiltonian corresponds to the Dynamical Jaynes-Cummings (DJC) interaction for each atom separately. Entanglement of two and three particles using this sequential interaction scheme has been experimentally achieved [8]. For the case in which \( t_1 = t_2 \) and \( \tau_1 = \tau_2 \), both atoms interact simultaneously with the cavity mode, although they can have different time-dependent couplings. The situation is described by the generalized two-atom Dynamical Dicke (DD) model. Elsewhere we discuss entanglement generation within this model [9].
For both the DJC model and the DD interaction, the number of excitations \( N = a^+a + \sum_{i=1,2} \sigma_i^+ \sigma_i^- \) is a conserved quantity within the RWA. This implies separable dynamics within subspaces having a prescribed eigenvalue \( N \) of \( N \). For \( N = 1 \) a basis is given by \( \{|e_1, g_2, N-1\}, |g_1, e_2, N-1\}, |g_1, g_2, N\} \) while for \( N \geq 2 \) a basis is given by \( \{|e_1, e_2, N-2\}, |e_1, g_2, N-1\}, |g_1, e_2, N-1\}, |g_1, g_2, N\} \). The third label indicates the number of photons. Although the number of excitations is conserved in both models (DJC and two-atom DD), the main differences in the time evolution of the system arise from the collective atomic dynamics governing the quantum dynamics in the DD model, which is of course not present in the DJC situation. For the sake of simplicity, we consider the initial state to be \( |\psi(0)\rangle = |e_1, g_2, 0\rangle \) in both situations, such that the unitary time evolution of the whole system’s state is given by

\[
|\psi(t)\rangle = a_1(t)|e_1, g_2, 0\rangle + a_2(t)|g_1, e_2, 0\rangle + a_3(t)|g_1, g_2, 1\rangle
\]  

(3)

This allows us to study the role of the cavity as as “quantum data bus’ to generate/transfer entanglement, as well as considering the cavity as a third qubit.

3. Entanglement and linear entropy

The two-qubit-cavity system can be thought of as a three-qubit system, hence the atom-atom and single-atom-field entanglements can be quantified using the concurrence \( [7] \). We have found simple expressions for these quantities:

\[
C(\rho_{a,a})(t) = 2|a_1(t)a_2(t)|
\]

\[
C(\rho_{a,f})(t) = 2|a_1(t)a_3(t)|
\]

\[
C(\rho_{a,f})(t) = 2|a_2(t)a_3(t)|
\]

(4)

The linear entropy of a particular sub-system \( (A) \) can be calculated using the relation \( M_A = 1 - Tr A \rho_A^2 \), where \( \rho_A \) is the partial trace of \( |\psi(t)\rangle \langle \psi(t)| \) over the BC sub-system. The linear entropies for the field \( (M_f) \), the atom \( (M_{a1}) \), and the atom \( (M_{a2}) \) are given by

\[
M_f = 1 - [(|a_1(t)|^2 + |a_2(t)|^2)^2 + |a_3(t)|^4]
\]

\[
M_{a1} = 1 - [(|a_2(t)|^2 + |a_3(t)|^2)^2 + |a_1(t)|^4]
\]

\[
M_{a2} = 1 - [(|a_1(t)|^2 + |a_3(t)|^2)^2 + |a_2(t)|^4]
\]

(5)

Under this unitary evolution, the mixedness for the state of each element in the system only arises from entanglement with the other components: hence we make the conjecture that the individual linear entropy should be a good measure of how entangled each element is with the rest of the system. Therefore an appropriate addition (subtraction) of individual entropies should be capable of quantifying the degree of entanglement within each of the sub-systems, i.e. atom1-atom2, atom1-field, and atom2-field. We therefore define the intrinsic entanglement within the AB subsystem as

\[
E_{A,B} = M_A + M_B - M_{A,B}
\]

(6)

where \( M_{A,B} \) indicates how entangled (mixed) the AB sub-system is with the remaining constituents of the system. We compare the two-atom concurrence \( C(\rho_{a,a}) \) and the atom-field concurrences \( C(\rho_{a,f}) \) with the intrinsic entanglements \( E_{a,a} \) and \( E_{a,f} \). We find that these two measures for entanglement have identical functional forms, namely:

\[
E_{a,a} = M_{a1} + M_{a2} - M_f = |C(\rho_{a,a})|^2
\]

\[
E_{a,f} = M_{a1} + M_f - M_{a2} = |C(\rho_{a,f})|^2
\]

\[
E_{a2,f} = M_{a2} + M_f - M_{a1} = |C(\rho_{a2,f})|^2
\]

(7)
Notice that $M_{A,B} = M_C$ since the whole system is in a pure state. Some important aspects that follow from these relations are:

- The definition in Eq.(6) and the relations given in Eq.(7) agree with general inequalities that quantum entropies should satisfy [10]: In particular, the subadditivity inequality ($M_{A,B} \leq M_A + M_B$) if the system is considered as a mixed two-qubit state, and the strong subadditivity inequality ($M_{A,B,C} + M_B \leq M_{A,B} + M_{B,C}$) if the system is assumed as a composite system of three elements $A_1$, $A_2$, and field.

- The definition in Eq.(6) satisfies all the relevant criteria for an entanglement measure [11]: (a) it is semipositive, i.e. $E_{A,B} \geq 0$ where the equality sign holds for a separable state, (b) $E_{A,B}(t)$ is a continuous function of time, as shown in the next sections, and (c) $E_{A,B}$ is invariant and nonincreasing under local unitary operations in a similar way to the quantum entropies defining it.

Now we discuss the above results in the two interaction models, namely, DJC and two-atom DD interaction.

3.1. Transfer of entanglement in the DJC situation

For this sequential interaction, the dynamics can be divided into two steps. Before the first atom leaves the cavity, i.e. $t \leq \tau_1$, the unitary evolution is given by

\[
\begin{align*}
  a_1(t) &= \cos[\theta_1(t)] \\
  a_2(t) &= 0 \\
  a_3(t) &= -i \sin[\theta_1(t)]
\end{align*}
\]

Once $A_1$ has left the cavity, i.e. $t > \tau_1$, it is easy to show that

\[
\begin{align*}
  a_1(t) &= \cos[\theta_1(\tau_1)] \\
  a_2(t) &= -\sin[\theta_1(\tau_1)] \sin[\theta_2(t)] \\
  a_3(t) &= -i \sin[\theta_1(\tau_1)] \cos[\theta_2(t)]
\end{align*}
\]

where $\theta_i(t) = \int_{t_i}^t \gamma_i(t')dt'$ for $t_i \leq t \leq \tau_i$. Using this sequential interaction scheme, it is possible to produce a maximally entangled two-atom state as well as a hybrid entangled atom-atom-cavity $W$-state. By adjusting the interaction times such that $\theta_1(\tau_1^*) = \pi/4$ and $\theta_2(\tau_2^*) = \pi/2$, a maximally entangled atomic state is obtained, which is separable from the field state, i.e. $|\Psi(\tau_n)\rangle = (1/\sqrt{2})\{|e_1, g_2\} - |g_1, e_2\} \otimes |0\rangle$.

The $W$-state is obtained when $\theta_1(\tau_1^w) = \arccos(1/\sqrt{3})$ and $\theta_2(\tau_2^w) = \pi/4$, i.e. $|\Psi(\tau_w)\rangle = (1/\sqrt{3})\{|e_1, g_2, 0\} - |g_1, e_2, 0\} + i|g_1, g_2, 1\}$.

Figures 1 and 2 show the time evolution of individual entropies while forming a singlet and a $W$-state, respectively. As expected with this DJC interaction, their combined state is pure while atom$_1$ is interacting with the cavity – hence the atom and field entropies increase identically, i.e $M_{a1}(t) = M_f(t)$ for $t \leq \tau_1$. During this period $M_{a2}$ equals zero since $A_2$ is not interacting with the field or with $A_1$. After the first atom leaves the cavity, its entropy $M_{a2}(\tau_1)$ remains constant ($A_1$ does not interact with the field anymore) and only $M_{a2}$ and $M_f$ change. In each of these periods, the maximum individual linear entropy equals 1/2 as it should for a maximally mixed state.

In the case of singlet formation, $M_f$ goes to zero while $M_{a2}$ approaches the constant value $M_{a2}(\tau_1^*) = 1/2$. The fact that $M_f(\tau_n) = 0$ indicates that the field is in a pure state (i.e. it is not entangled with either of the atoms). Hence atom-field
concurrences vanish as well as atom-field intrinsic entanglements (as defined in Eq. (7)). This effect can be seen in figure 3. On the other hand, since both $M_{a1}(\tau_s)$ and $M_{a2}(\tau_s)$ approach the same finite maximum value (1/2), the atom-atom concurrence and atom-atom entanglement $E_{a,a}$ equal one, as expected for a maximally entangled two-qubit state. When forming the W-state, the field and $A_1$ become maximally entangled before the latter leaves the cavity. Hence $M_f(\tau_w^1) < 1/2$. After $A_1$ leaves, $M_{a2}$ increases and $M_f$ varies non-monotonically until they both equal $M_{a1}(\tau_w^1) < 1/2$ (see Fig. 2), indicating that at $t = \tau_w$ the field and atoms are identically mixed (entangled) with each other. All subsystem concurrences approach the same value $2/3$ (as should happen for a $W$-state). A similar behaviour is observed for subsystem entanglements: at $t = \tau_w$ all of them equal $4/9$ (see Fig. 4). In both singlet and hybrid $W$-state formations, one can interpret these dynamics as processes of converting (full or partial) initial $A_1$–field entanglement into $A_1 – A_2$ entanglement.

3.2. Sharing of entanglement in the two-atom DD interaction

We have previously demonstrated that the use of different time-dependent and asymmetric on-resonance atom-field couplings, allows the generation of two-qubit states with any degree of entanglement and a prescribed symmetry [9]. This is achieved by exploiting a trapping vacuum state condition, which does not arise for identical couplings [9]. Here we focus on the situation in which the couplings satisfy the relation $\gamma_1(t) = r\gamma_2(t)$. With this condition, we have found that the unitary evolution for the two-atom DD model with initial state $|\Psi(0)\rangle = |e_1, g_2, 0\rangle$ is given by [9]:

$$
\begin{align*}
    a_1(t) &= 1 + r a_2(t) \\
    a_2(t) &= -2\alpha\sin^2[\theta(t)/2] \\
    a_3(t) &= -i\sqrt{r}\alpha\sin[\theta(t)]
\end{align*}
$$

$\Box$

Figure 1. One-by-one DJC interaction during formation of a singlet state. (a) Coupling strengths and (b) individual linear entropies as functions of time. Time in units of the interaction time corresponding to $A_1$, $\tau_1$. 

$\Box$
Figure 2. One-by-one DJC interaction during formation of a hybrid atom-atom-cavity $W$-state. (a) Coupling strengths and (b) individual linear entropies as functions of time. Time is measured in units of $\tau_1$.

Figure 3. Dynamics of two-qubit entropies and concurrences during generation of a singlet state in the DJC model. (a) Atom-atom intrinsic entanglement $E_{a,a}$ and concurrence $C(\rho_{a,a})$, (b) $A_1$-field intrinsic entanglement $E_{a_1,f}$ and concurrence $C(\rho_{a_1,f})$, and (c) $A_2$-field intrinsic entanglement $E_{a_2,f}$ and concurrence $C(\rho_{a_2,f})$, as functions of time. Time is measured in units of $\tau_1$. 
This manifests itself in the fact that both the maximally entangled triplet state $|\Phi^+\rangle = |e_1, g_2\rangle + |g_1, e_2\rangle$ while forming the more complex dynamics. To demonstrate this, we now discuss the dynamics of entropy dynamics for both atoms. By contrast, asymmetric couplings lead to a simultaneous interaction scheme, identical atom-field couplings lead to a symmetric separability in the atom-atom states as well. This is due to the fact that in this when the couplings are identical the separability of atomic and field states implies the two-atom state becomes maximally entangled, i.e. $|\Phi^+\rangle$. Hence its entropy $M_{a1}$ increases faster than $M_{a2}$.

This manifests itself in the fact that both the $A_1$-field concurrence $C(\rho_{a1,f})$ and entanglement $E_{a1,f}$ grow faster than $C(\rho_{a2,f})$ and $E_{a2,f}$, respectively. They follow non-monotonic evolutions in such a way that, at the end of the interaction period ($t = \tau^*$), both $C(\rho_{a1,f})$ and $C(\rho_{a2,f})$ vanish while the atom-atom concurrence and intrinsic entanglement go to the maximum value of unity.

We will finish by discussing how the present study might be used to develop correlation measures for multi-partite systems. Our results have shown that the
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Figure 5. Dynamics of individual linear entropies during the formation of a triplet state $|\Phi^+, 0\rangle$. (a) Coupling strengths as functions of time.

Figure 6. Dynamics of two-qubit entanglement $E$ and concurrences while forming a triplet state $|\Phi^+, 0\rangle$ in the two-atom DD model. (a) Atom-atom entropy $E_{a,a}$ and concurrence $C(\rho_{a,a})$, (b) $A_1-$field entropy $E_{a_1,f}$ and concurrence $C(\rho_{a_1,f})$, and (c) $A_2-$field entropy $E_{a_2,f}$ and concurrence $C(\rho_{a_2,f})$, as a function of time.
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The additivity of the linear entropy seems to have a clear physical meaning in the particular cases here considered. We have shown that by properly adding linear entropies we can define an alternative measure for two-qubit entanglement which has the same functional form as the concurrence, and satisfies the relevant criteria for such a measure. In a similar way, one might be able to extend this analysis to multi-particle systems by studying the dynamics of quantum correlations in \(N\)-particle mixed states under unitary evolutions, i.e. in a \(N + 1\)-particle entangled state. In particular it has recently been shown for \(N = 3\), based on the strong subadditivity inequality for von Neumann entropies, that one can define a possible measure for four-particle entanglement[12]. By studying such inequalities for linear entropies, one might gain deeper insights into the open problem concerning entanglement measures for many-particle states. An extension of the present study to the case of \(N \geq 3\) qubits interacting with a cavity mode, is underway and will be reported elsewhere.

4. Summary and Future Directions

We have found quantitative relations between entanglement and linear entropy for a two-qubit-cavity system governed by unitary evolutions. By considering the behaviour of individual sub-system’s entropies, we have been able to propose a measure of bipartite entanglement. Furthermore, we have discussed how these relations might be extended to multi-partite systems. In parallel with future extensions to many-particle systems, it will be interesting to consider the full effects of open systems with regards to decoherence. In particular, it will be of fundamental interest to understand the role of the environment with regards to mixedness, entanglement and entropy. The atom-cavity quantum optics systems of the type studied here, should provide ideal theoretical and experimental systems for pursuing such studies.

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