Eigenvector Expansion and Petermann Factor for Ohmically Damped Oscillators

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Thermal correlation functions \( C(t) \sim \langle \phi(t)\phi(0) \rangle \) in ohmically damped systems such as optical resonators can be expressed as a sum over modes \( j \). In contrast to the conservative case, each term is multiplied by the Petermann factor \( C_j \), which can exceed unity and lead to “excess noise”. A time-independent perturbation \( \sim \epsilon \) also leads to a response \( \sim \epsilon C_j \). Moreover, \( C_j^{-1} \) is proportional to a diagonal bilinear map central to eigenvector expansions for damped systems. “Giant excess noise” \( (C_j \to \infty) \) occurs in the limit of a critical point, where the divergent parts of \( J (>1) \) contributions to \( C(t) \) cancel and time-independent perturbations lead to non-analytic shifts \( \sim \epsilon^{1/J} \). A sum rule implies that (for the normal case of positive masses) the average value of \( |C_j| \) over all modes is \( \geq 1 \), showing that “excess noise” is not exceptional.

I. INTRODUCTION

In open optical cavities, spontaneous emission rates and thermal noise in a mode \( j \) are modified by a factor \( C_j \), which can paradoxically exceed unity and give rise to “excess noise”. This possibility was first pointed out by Petermann [1] for gain-guided semiconductor lasers, and \( C_j \) has come to be known as the Petermann factor (PF). This prediction was initially controversial [2], until Haus and Kawakami [3] pointed out the importance of “giant excess noise” — thus helping to demystify the PF. Section II develops the linear-space structure and expansion in eigenvectors \( f_j \) for ohmically damped systems, introducing the bilinear map \( (\psi, \phi) \) under which eigenvectors are orthogonal. The PF \( C_j \sim 1/(f_j, f_j) \) is then introduced for each mode \( j \). A sum rule implies that (for the normal case of positive masses) the average value of \( |C_j| \) is \( \geq 1 \). Section III derives the expansion for correlation functions \( C(t) \), showing that each term \( j \) involves \( C_j \). Section IV shows that the familiar Rayleigh-Schrödinger perturbation theory applies to the complex eigenvalues, provided eigenvectors are normalized by \( (f_j, f_j) \). Thus a time-independent perturbation \( \sim \epsilon \) leads to shifts \( \sim \epsilon C_j \), i.e., possibly “excess” response. Simple systems with finite (indeed few) degrees of freedom are studied, and in particular, examples of “giant excess noise” (\( C_j \) large, and divergent as a parameter is tuned) are easily constructed.

This invites the question: What happens if \( C_j \to \infty \), or equivalently \( (f_j, f_j) \to 0 \)? Section V shows that this happens if and only if \( J > 1 \) eigenvectors merge at a critical point, where the divergent parts of the \( J \) terms cancel, leaving a finite result. Section VI shows that at a critical point, the divergent PF is manifested as large and non-analytic frequency shifts \( \sim \epsilon^{1/J} \) in response to perturbations \( \sim \epsilon \). Concluding remarks and a sketch of generalizations are given in Section VII.

By placing the formalism in a general context, we make contact with two well-known results. First, critical damping (in the elementary sense of that word) of one single oscillator precisely provides an example of “giant excess noise” — thus helping to demystify the PF. Second, the factor \( (f_j, f_j) \) and thus also the PF, is shown to be related to the Zeldovich normalizing factor for quasinormal modes [5].

This paper emphasizes the role played by the PF. A
more systematic formulation of the linear-space properties will be given elsewhere.

II. LINEAR-SPACE STRUCTURE

A. Ohmic systems

Consider \( N \) coupled classical linear oscillators labelled as \( \alpha = 1, \ldots, N \), with coordinates \( \phi = (\phi(1), \ldots, \phi(N))^T \), described by

\[
(M d_t^2 + \Gamma d_t + K) \phi = 0 \quad ,
\]

where \( M(\alpha, \beta) = m_\alpha \delta(\alpha, \beta) \) is a diagonal mass matrix (but allowing for coordinate transformations, any symmetric positive-definite matrix), \( \Gamma \) is a symmetric non-negative damping matrix and \( K \) is a symmetric positive-definite matrix of force constants. Classical fields are included if \( \alpha \) is turned into a continuous position variable \( x \) (Appendix B). Two conditions can be relaxed: negative \( \Gamma \) describes a gain medium; negative \( M \) describes some dispersive media in a restricted frequency range.

Damping introduces an arrow of time and it is convenient to write the dynamics in first-order form, by introducing the momentum \( \hat{\phi} = (\hat{\phi}(1), \ldots, \hat{\phi}(N))^T \):

\[
\begin{align*}
&d_t \phi = M^{-1} \hat{\phi} \quad , \\
&d_t \hat{\phi} = -K \hat{\phi} - \Gamma M^{-1} \hat{\phi} \quad .
\end{align*}
\]

Denote the state in phase space as

\[
\phi = (\phi, \hat{\phi})^T = (\phi(1), \ldots, \phi(N); \hat{\phi}(1), \ldots, \hat{\phi}(N))^T = (\phi^1, \ldots, \phi^{2N})^T \quad .
\]

Underline denotes \( N \)-vectors, and boldface denotes \( 2N \)-vectors. Then the evolution takes the Schrödinger-like form

\[
\begin{align*}
&d_t \phi = -i \mathcal{H} \phi \quad ,
\end{align*}
\]

where \( \mathcal{H} = i \left( \begin{array}{cc} 0 & M^{-1} \\ -K & -\Gamma M^{-1} \end{array} \right) \quad .
\]

Incidentally, \( \phi_M \equiv M^{1/2} \phi \) satisfies (2.1) with \( M \mapsto I, \ \Gamma \mapsto \Gamma_M \equiv M^{-1/2} \Gamma M^{-1/2} \) and \( K \mapsto K_M \equiv M^{-1/2} K M^{-1/2} \), a standard representation that is sometimes convenient.

B. Bilinear map

In phase space, the inner product \( \langle \psi | \phi \rangle \equiv \psi^\dagger \phi + \psi^\dagger \dot{\phi} \) satisfies \( \langle \phi | \phi \rangle \geq 0 \). Unfortunately, \( \langle \psi | \mathcal{H} \phi \rangle \neq \langle \phi | \mathcal{H} \psi \rangle^\dagger \), which renders it useless for projections or orthogonality. Instead, we introduce a bilinear map:

\[
\langle \psi, \phi \rangle = \langle \phi, \psi \rangle = i \left( \phi^\dagger \dot{\psi} + \psi^\dagger \dot{\phi} + \phi^\dagger \Gamma \phi \right) = \langle D \phi | \psi \rangle \quad ,
\]

where the duality map [cf. below (2.11)] is

\[
D \psi = -i \left( \begin{array}{cc} \Gamma & I \\ I & 0 \end{array} \right) \psi^* \quad , \quad D^{-1} \psi = i \left( \begin{array}{cc} 0 & I \\ I & -\Gamma \end{array} \right) \psi^* \quad .
\]

Note that \( M \) does not appear in the bilinear map, as is readily seen by transforming from the standard representation. This bilinear map is constructed to ensure the key property

\[
\langle \psi, \mathcal{H} \phi \rangle = \langle \phi, \mathcal{H} \psi \rangle \quad ,
\]

analogous to self-adjointness. In proving (2.8), the dissipative term in \( \mathcal{H} \) cancels the last term in (2.7). The diagonal entries \( \phi, \phi \) are not positive definite, not even necessarily real; they may also be small (or even zero) — a central concern of this paper.

All statements about the bilinear map can be expressed in terms of inner products with dual vectors, but the latter language obscures the important symmetry property. Duality maps and biorthogonal bases are general concepts; however, \( D \) and \( D^{-1} \) can here be written down explicitly without reference to the eigenvectors of \( \mathcal{H} \) — possible only because \( \hat{\phi} \) is considered together with \( \phi \) in a first-order formalism.

C. Eigenvectors

Eigenvalues are the roots of the characteristic function

\[
\chi(\omega) \equiv \det(\mathcal{H} - \omega) \quad ,
\]

and right eigenvectors are defined by

\[
\mathcal{H} f_j = \omega_j f_j \quad .
\]

The dual basis vectors \( f^j = D f_j \) are left eigenvectors satisfying \( \mathcal{H}^\dagger f^j = \omega^j f^j \). For an eigenvector \( f_j \), (2.10) implies the time-dependence \( e^{-i \omega_j t} \), so the coordinates and momenta are related: \( f_j = -i \omega_j M f_j \).

With dissipation, the eigenvalues are complex. For \( \Gamma \) non-negative and \( K \) positive, \( \text{Im} \ \omega_j \leq 0 \). By conjugating (2.10), it follows that \(-\omega_j^\dagger\) is also an eigenvalue,
with eigenvector $\propto f_j^\dagger$. Thus, except for $\Re \omega_j = 0$, eigenvalues are paired, ensuring that $\phi$ is real even though each eigenvector is complex.

The property (2.8) is the analog of self-adjointness and, as usual, leads immediately to $(f_j, f_k) = 0$ if $\omega_j \neq \omega_k$, henceforth to be called orthogonality. This condition is extended to level crossings (defined in Section 11E) by taking suitable linear combinations.

Since $\mathcal{H}$ is not self-adjoint, the eigenvectors may be incomplete, but only on a set of measure zero in parameter space, to be called critical points for reasons that will become apparent. Until Section VI, we assume that there are $2N$ eigenvectors forming a complete basis, so the dynamics given the initial condition $\phi(t=0) = \phi$ is readily solved:

$$\phi(t) = \sum_j \frac{(f_j, \phi)}{(f_j, f_j)} f_j e^{-i\omega_j t} . \quad (2.11)$$

Recognizing that (2.11) at $t = 0$ is valid for all $\phi$ leads to

$$\mathcal{I} = \sum_j \frac{(f_j, \cdot)}{(f_j, f_j)} f_j , \quad (2.12)$$

where $\mathcal{I}$ is the $2N \times 2N$ identity. In terms of $N \times N$ blocks,

$$0 = \sum_j f_j \otimes f_j / (f_j, f_j) ,$$

$$I = \sum_j \omega_j (M f_j) \otimes f_j / (f_j, f_j) ,$$

$$0 = \sum_j f_j \otimes (\Gamma f_j) / (f_j, f_j) , \quad (2.13)$$

$$0 = \sum_j [\omega_j^2 (M f_j) \otimes (M f_j) + i\omega_j (M f_j) \otimes (\Gamma f_j)] / (f_j, f_j) ,$$

where $a \otimes b$ stands for the matrix with elements $a(\alpha)b(\beta)$. The appearance of $(f_j, f_j)$ in the denominator and the possibility that it may be small is at the heart of this paper.

D. Normalization and Petermann factor

The diagonal bilinear map

$$(f_j, f_j) = \int (2\omega_j M + i\Gamma) f_j \quad (2.14)$$

is not a good measure of the length of an eigenvector. Instead we define the norm of an eigenvector to be

$$N_j = \int \| M f_j \| \quad (2.15)$$

which is different from $(f_j, f_j)$ even for $M = I$. In the limit of zero dissipation, $(f_j, f_j) = 2\omega_j N_j$, which motivates the definition of the PF

$$C_j = \frac{2\omega_j N_j}{(f_j, f_j)} . \quad (2.16)$$

Unlike the familiar situation, the length [cf. the norm (2.10)] and the projection [cf. the bilinear map (2.6)] do not relate to the same inner product; in a sense $C_j$ measures the difference between these two concepts. Since $N_j \propto f_j^* f_j$ whereas $(f_j, f_j) \propto f_j^2$, the phase of $C_j$ is a matter of convention. Interestingly, several physical quantities depend on $|C_j|$.

Incidentally, the trace of the second sum rule in (2.13) gives

$$2N = \sum_j \sigma_j C_j , \quad (2.17)$$

where

$$\sigma_j = \frac{f_j^T M f_j}{\int |f_j|^2} . \quad (2.18)$$

has a magnitude bounded by unity (for the normal case with $M$ positive definite). This implies that the average value of $|C_j|$ is $\geq 1$, showing that “excess noise” [cf. below (3.12)] is by no means exceptional.

E. Exceptions

The formalism so far relies on two assumptions: (a) the eigenvectors are complete, and (b) $(f_j, f_j) \neq 0$ for all $j$. These two conditions are related. To see that (a) implies (b), suppose that at a critical point an eigenvector is lost because of merging, say $f_k \to f_j$. Then, $(f_j, f_j) = \lim (f_j, f_k) = 0$. Conversely, suppose $(f_j, f_j) = 0$. Then if the eigenvectors are complete, $f_j$ would be orthogonal to every vector $\psi$. Taking $\psi = D^{-1} f_j$, one finds $(f_j, f_j) = 0$, contradicting $f_j \neq 0$.

If all eigenvalues are distinct, there will be $2N$ eigenvectors, which must be linearly independent. Thus, the eigenvectors can only be incomplete if $\chi(\omega)$ has a root of order $J > 1$. When $J$ roots merge (degeneracy) as a parameter is tuned, there are two possibilities: either there are still $J$ linearly independent eigenvectors, the usual and familiar case of level crossing, or the eigenvectors merge as well, which will be called criticality for reasons that will become apparent. (The non-generic case where some but not all of the $J$ eigenvectors merge will be ignored except for an example in Appendix A) Incompleteness occurs only at criticality.

We shall later show that criticality is more generic than level crossing — somewhat surprising since for conservative systems, criticality is not allowed whereas level
pair goes critical at $\gamma$ small family $k$

With the freedom to rotate coordinates, there is no loss linear maps vanish and we find eigenvectors are $N$ and the norm $\omega$ values merge, then so do their partners $\omega_{-j} = -\omega_{j}'$, $\omega_{-k} = -\omega_{k}'$.

**F. Examples**

The ideas can be illustrated with just one oscillator. Let $N = 1$, with $M = 1$, $K = k$ and $\Gamma = 2\gamma$. The eigenvalue equation

$$
\chi(\omega) = \det \begin{pmatrix} -\omega & i \\ -ik & -2i\gamma - \omega \end{pmatrix} = 0
$$

leads to $\omega = \omega_{\pm} = \pm \Omega - i\gamma$, where $\Omega = \sqrt{k^2 - \gamma^2}$. The eigenvectors are

$$
f_{\pm} = A_{\pm} \begin{pmatrix} 1 \\ -i\omega_{\pm} \end{pmatrix} ,
$$

with the bilinear maps

$$
(f_+, f_-) = 0 \quad (f_{\pm}, f_{\pm}) = \pm 2A_{\pm}^2 \Omega ,
$$

and the norm $N_{\pm} = |A_{\pm}|^2$. Choosing a convenient phase, we find

$$
C_{\pm} = \frac{\Omega \mp i\gamma}{\Omega} .
$$

We note that $C_{\pm} = 1$ in the limit of zero damping; in the underdamped regime $|C_{\pm}| \geq 1$, while in the overdamped regime, one of $|C_{\pm}|$ exceeds unity.

At the critical point $k = k_{\ast} = \gamma^2$, (a) the two eigenvalues merge: $\omega_{\pm} = -i\gamma$; (b) from (2.20) the two eigenvectors merge, leaving only one eigenvector in the 2-dimensional space; and (c) from (2.22) the diagonal bilinear maps vanish and $C_{\pm}$ diverge.

Consider another example, with $N = 2$, $M = I$ and $\Gamma = 2$.

$$
K = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix} , \quad \Gamma = 2 \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix} .
$$

With the freedom to rotate coordinates, there is no loss of generality in assuming that $\Gamma$ is diagonal, so here we take $\gamma_{12} = 0$. To be specific take the one-parameter family $k_{11} = k_{22} = 4$, $k_{12} = -2$, $\gamma_{11} = 2\gamma$, $\gamma_{22} = \gamma$. For small $\gamma$, there are two pairs of underdamped modes. One pair goes critical at $\gamma_{\ast 1} = 0.8599$, and the other pair at $\gamma_{\ast 2} = 2.1031$, beyond which all modes are overdamped and eigenvalues purely imaginary. Except at the two critical points, the eigenvectors are complete, and all bilinear maps etc. can be evaluated explicitly.

In these examples of critical damping (in the elementary sense of that word), a pair of conjugate eigenvalues $\omega_j$ and $\omega_{-j} = -\omega_j'$ merge on the imaginary axis. This occurs with codimension = 1 in parameter space, and is the most generic class of the merging of eigenvectors—which we therefore refer to as criticality in general. Less generic cases, occurring with codimension > 1 in parameter space, include (a) the merging of two eigenvalues off the imaginary axis, and (b) the merging of $J > 2$ eigenvalues. Some examples are given in Appendix A.

Incidentally, if there is level crossing, then several linearly independent eigenvectors $f_j$ satisfy $\{\{ -i\omega \}^2 M + (-i\omega) \Gamma + K \} f_j = 0$ for the same $\omega$, making the square bracket vanish on the subspace spanned by these eigenvectors. Separating the real and imaginary parts, one finds that both $\Gamma_M \propto I$ and $K_M \propto I$ on this subspace. This requires many conditions; for example, for a 2-dimensional subspace, the degeneracy of the eigenvalues of $\Gamma_M$ and $K_M$ requires two conditions each; see Appendix A. Thus, level crossing is more exceptional than criticality.

**III. CORRELATION FUNCTIONS**

**A. Thermal noise**

When the oscillators are placed in a bath at temperature $T$, (2.1) acquires on the RHS a noise term $\eta(\alpha, t)$, which satisfies the fluctuation–dissipation theorem

$$
\langle \eta(\alpha, \omega) \tilde{\eta}(\beta, \omega') \rangle = 4\pi T \delta(\omega + \omega') \Gamma(\alpha, \beta) ,
$$

in units where $k_B = 1$, where $\langle \cdots \rangle$ denotes thermal average and $\tilde{\cdots}$ denotes Fourier transform.

Write the equation of motion in two-component form:

$$
(\dot{d}_t + i\eta) \phi(t) = S(t) = \begin{pmatrix} 0 \\ \eta(t) \end{pmatrix} .
$$

Using the completeness of the eigenvectors, $\phi(t) = \sum_j a_j(t) f_j$, we find that the coefficients satisfy

$$
(\ddot{d}_t + i\omega_j) a_j(t) = \frac{(S(t), f_j)}{(f_j, f_j)} ,
$$

where from (3.2) and (2.20), and henceforth adopting the summation convention for Greek indices,

$$
(S(t), f_j) = i\eta(\alpha', t) f_j(\alpha') .
$$

Upon Fourier transform,
\[ \tilde{a}^2(\omega) = -\frac{1}{\omega - \omega_j} \tilde{\eta}(\omega', \omega) f_j(\omega') (f_j, f_j) \quad , \] (3.5)

from which \( \phi(t) \) is obtained. Again \( (f_j, f_j) \) appears in the denominator, and this will lead to possibly large response to noise.

**B. Evaluation of correlation function**

Consider the correlation function

\[ C(\alpha, \beta; t) = \langle \phi(\alpha, t) \phi(\beta, 0) \rangle \quad . \] (3.6)

Using (3.5) then gives

\[ \tilde{C}(\alpha, \beta; \omega) = -2T \sum_{jk} \frac{1}{(\omega - \omega_j)(\omega + \omega_k)} \frac{f_j(\alpha') f_j(\alpha') f_k(\beta') f_k(\beta')}{(f_j, f_j)(f_k, f_k)} \quad . \] (3.7)

From the definition of the bilinear map,

\[ f_j(\alpha') \Gamma(\alpha', \beta') f_k(\beta') = -i (f_k, f_k) \delta_{j,k} + i(\omega_j + \omega_k) f_j(\alpha') M(\alpha', \beta') f_k(\beta') \quad . \] (3.8)

When the second term of (3.8) is put into (3.7), there is a factor

\[ \frac{\omega_j + \omega_k}{(\omega - \omega_j)(\omega + \omega_k)} = \frac{1}{\omega - \omega_j} - \frac{1}{\omega + \omega_k} \quad , \] (3.9)

resulting in one term without \( \omega_k \) and another without \( \omega_j \). The former leads to a sum

\[ \sum_k f_k(\beta') f_k(\beta) = 0 \quad . \] (3.10)

by (3.13); likewise the latter vanishes.

The remaining first term in (3.5) then leads to the central result, which importantly involves a single sum over modes:

\[ \tilde{C}(\alpha, \beta; \omega) = 2iT \sum_{\beta} \frac{1}{\omega^2 - \omega_j^2} \frac{f_j(\alpha) f_j(\beta)}{(f_j, f_j)} \quad . \] (3.11)

Then, Fourier transforming (3.11) and evaluating the residues gives

\[ C(\alpha, \beta; t) = \frac{T}{2j} \sum_j \frac{1}{\omega_j^2} \frac{f_j(\alpha) f_j(\beta)}{N_j} C_j e^{-i\omega_j t} \quad . \] (3.12)

The square bracket in (3.11) and (3.12) is normalized in the sense that its trace with \( M(\alpha, \beta) \) is bounded by unity; if a mode has negligible dissipation (\( \omega_j \) real) or is overdamped (\( \omega_j \) imaginary), the eigenvector has a constant phase, and the trace has unit modulus. Thus \( C_j \) appropriately expresses the relative contribution of each mode. The familiar conservative case is recovered by setting all \( C_j = 1 \). Herein lies the paradox: the response per mode to thermal noise can be increased by dissipation (“excess noise”); near criticality, some \( C_j \)'s even diverge (“giant excess noise”). We have also shown in the last Section that the average value of \( |C_j| \) is \( \geq 1 \).

**C. Example**

The \( N = 1 \) example serves to demystify the PF and its possible divergence. In this case, \( f_\pm /N_\pm = 1 \) and \( C_\pm = \pm \omega_\pm /\Omega \). Some arithmetic leads to

\[ C(t) = \frac{T}{k} \frac{\omega e^{-i\omega t} - \omega e^{-i\omega t}}{\omega - \omega} \quad . \] (3.13)

Although each term has a large coefficient (“excess noise”) near criticality, the sum is not large. In particular, \( C(t) \) is manifestly finite at the critical point.

A much simpler derivation can be given for this trivial case. In general, for \( t \geq 0 \)

\[ \phi(t) = \phi(0) G_1(t) + \dot{\phi}(0) G_2(t) + \phi_\eta(t) \quad , \] (3.14)

where \( G_i \) are homogeneous solutions satisfying the initial conditions \( G_1(0) = 1 \), \( G_1(0) = 0 \), \( G_2(0) = 0 \), \( G_2(0) = 1 \), and \( \phi_\eta(t) \) is an inhomogeneous solution caused by and therefore proportional to \( \eta(t) > 0 \).

The last term has zero correlator with \( \phi(0) \) and \( \dot{\phi}(0) \), while \( \langle \phi(0)^2 \rangle = T/k \), \( \langle \phi(0) \dot{\phi}(0) \rangle = 0 \), which then leads immediately to \( C(t) = (T/k) G_1(t) \), in agreement with (3.13).

**IV. PERTURBATION THEORY**

The familiar Rayleigh-Schrödinger perturbation theory can be transcribed, everywhere replacing the usual inner product with the bilinear map (2.6). However, the analogy implicitly assumes that the bilinear map is unperturbed, so we only consider changes in \( K \) (2), namely \( \Delta H = H_0 + \epsilon \Delta H \), with

\[ \Delta H = i \begin{pmatrix} 0 & \epsilon \Delta K \\ -\Delta K & 0 \end{pmatrix} \quad . \] (4.1)
For example, the first-order frequency shift is

$$\Delta \omega_j = \epsilon \frac{\langle f_j, \Delta H f_j \rangle}{\langle f_j, f_j \rangle}. \quad (4.2)$$

Higher-order terms in analogy to the conservative case will not be displayed. Consider the example of two oscillators in Section III with $\gamma = 0.5$ for $H_0$, and subject it to a perturbation $k_{11} \rightarrow k_{11} + \epsilon$. Figure 1 shows $\Delta \omega/\epsilon$ versus $\epsilon$ for the mode with smaller damping, originally at $\omega = 2.267 - 0.699\pi$. The points are obtained by directly solving $\chi(\omega) = 0$, while the straight line is second-order perturbation. The intercept (slope) verifies first-order perturbation. The remaining discrepancy is consistent with an $\epsilon^2$ term in $\Delta \omega$. Interestingly, the same formula now correctly gives the imaginary part of $\omega$, which is the obvious way to define the normalization.

The normalizing factor, originally involving a regulator, will not be displayed. Consider the example of two oscillators $\{f_j\}_{j=1,\ldots,J}$ which merge as a parameter $\lambda$ approaches a critical point $\lambda_c = 0$ without loss of generality. Let the eigenvalues be $\omega_j = \omega_* + \lambda \zeta_j \rightarrow \omega_*$. (Note that $\lambda$ measures the changes in eigenvalues, and is not proportional to the size $\epsilon$ of any perturbation in $H$ causing these differences.) The coefficients $\zeta_j$ label the directions and relative rates at which the frequencies approach the limit; from below, these are the roots of unity: $\zeta_j = e^{2\pi j/J}$. A consequence that will be needed below is that for $|m| \leq J-1$,

$$S_m = \frac{1}{J} \sum_j C_j^m = \delta_{m,0}. \quad (5.1)$$

Now introduce a minimal polynomial

$$f(\omega) = \sum_{n=0}^{J-1} g_n (\omega - \omega_*)^n \quad (5.2)$$

to interpolate the $J$ eigenvectors:

$$f_j = f(\omega_j) = f(\omega_* + \lambda \zeta_j). \quad (5.3)$$

However, one can normalize the eigenvectors $f_j$ before interpolating. [Unit normalization is inappropriate, since $(f_j, f_j) \to 0$.] In terms of the interpolating function, this amounts to $f(\omega) \rightarrow N(\omega) f(\omega)$, under which

$$g_n = \frac{1}{n!} \frac{\partial^n}{\partial \omega^n} f(\omega)$$

$$\rightarrow \frac{1}{n!} \frac{\partial^n}{\partial \omega^n} [N(\omega) f(\omega)] = \sum_{k=0}^{n} c_k g_{n-k}. \quad (5.4)$$

where $c_k = \frac{\partial^k N(\omega)}{k!}$ and all expressions are understood to be evaluated at $\omega = \omega_*$. Using this freedom, we can choose

$$(g_n, g_{n'}) = \delta_{n+n',J-1}. \quad (5.5)$$

Then the diagonal bilinear map is
\( (f_j, f_j) = \sum_{n,n'} (g_n, g_{n'}) (\lambda \zeta_j)^{n+n'} = J(\lambda \zeta_j)^{J-1} \). (5.6)

The above definition of \( g_n \) applies at each small \( \lambda \), and the implicit assumption is that each \( g_n \) approaches a finite limit as \( \lambda \to 0 \). Thus \( \{g_n\} \) is more convenient than \( \{f_j\} \) near criticality, and goes over to the canonical Jordan normal basis at criticality \( \zeta \).

We again illustrate with the \( N=1 \) example. Take \( k = \gamma^2 - \lambda^2 \), so that the eigenvalues are \( \omega_{\pm} = -i(\gamma \mp \lambda) \) and the eigenvectors are given by \( (2.20) \). The critical point occurs at \( \omega_* = -i \gamma \), and \( \zeta_{\pm} = \pm i \). The minimal polynomial interpolation is

\[
 f(\omega) = \frac{f_+ + f_-}{2} + \frac{\omega - \omega_*}{\omega_+ - \omega_-} \frac{f_+ - f_-}{2} = \left( \frac{1}{-\gamma} \right) + (\omega - \omega_*) \begin{pmatrix} 0 \\ -i \end{pmatrix}. \quad (5.7)
\]

The coefficients \( g_n \) are independent of \( \lambda \) as expected and also satisfy \( (5.5) \).

**B. Small denominators near criticality**

Consider \( (6.1) \), use \( (6.0) \) for \( (f_j, f_j) \), put \( \omega_j = \omega_* + \lambda \zeta_j \), and expand in powers of \( \lambda \), giving terms proportional to \( \sum_j (\lambda \zeta_j)^{-J+1+\ell} \), with \( \ell \geq 0 \). But from \( (6.1) \), all the negative powers sum to zero. This resolves the small denominator problem, namely the paradox that as \( \lambda \to 0 \), the contribution of each mode diverges (“giant excess noise”). We do not spell out the remaining finite terms, which are in any event more appropriately expressed in terms of the Jordan normal basis \( \{g_n\} \).

Although the divergent parts cancel in the sum for \( C(\alpha, \beta; t) \), one could choose linear combinations that single out only one mode. What happens to the cancellation in that case? For simplicity suppose two modes \( j = 1, 2 \) are close to criticality, and consider

\[
 \langle (d_1 + i \omega_1) \chi(\alpha, t) \chi(\beta, 0) \rangle. \quad (5.8)
\]

The operator \( (d_1 + i \omega_1) \) eliminates the \( j=1 \) contribution, through the appearance of an extra factor \( -i(\omega_j - \omega_1) \) in \( (6.1) \) and \( (6.2) \). However, the remaining \( j=2 \) term will then carry a factor \( -i(\omega_2 - \omega_1) \), which also vanishes at criticality. Thus projecting out one near-critical mode also suppresses other such modes, so that again the physical result is not large. Contrast the conservative case, where the projection operator annihilating one mode does not at the same time suppress the others.

**VI. Perturbation around critical points**

From Section \( \[ \] \) a perturbation \( \sim \epsilon \) produces a frequency shift on \( \sim \epsilon C_j \), and \( C_j \to \infty \) at a critical point. So what happens if a system originally at a critical point is perturbed? In this Section, we show that the shifts are non-analytic, and generically \( \sim \epsilon^{1/J} \gg \epsilon \).

Consider a perturbation \( H = H_0 + \epsilon \Delta H \), in which \( H_0 \) describes a system at a critical point \( \omega_* \), where a block of \( J \) eigenvectors have merged. In powers of \( \epsilon \), the characteristic polynomial \( \chi(\omega) \) is (considering this block only)

\[
 (-1)^J \chi(\omega) = \chi_0(\omega) + \epsilon \chi_1(\omega) + \epsilon^2 \chi_2(\omega) + \cdots \\
 = (\Delta \omega)^J + \epsilon [\chi_1(\omega_*) + \Delta \omega \chi'_1(\omega_*) + \cdots] + \epsilon^2 \chi_2(\omega_*) + \cdots, \quad (6.1)
\]

where \( \chi_0(\omega) \) has a \( J \)-th order root \( \omega_* \) and \( \Delta \omega = \omega - \omega_* \). Setting \( \chi(\omega) = 0 \) gives, to leading order,

\[
 \Delta \omega = [\chi_1(\omega_*)]^{1/J} \epsilon^{1/J} \zeta_j + \cdots \quad (6.2)
\]

for \( j = 1, \ldots, J \), and

\[
 \zeta_j = e^{2\pi i j/J}. \quad (6.3)
\]

Thus (a) the shifts go as \( \epsilon^{1/J} \) (non-analytic in \( \epsilon \) and \( \gg \epsilon \)); (b) the eigenvalues split into \( J \) different ones, shifting in equiangular directions in the complex frequency plane, all at the same rate; (c) the directions of splitting for \( \epsilon < 0 \) bisect those for \( \epsilon > 0 \). These features are already contained in the trivial \( N=1 \) example: for \( k = k_* + \epsilon \), the eigenvalues are \( -i \gamma \pm \sqrt{\epsilon} \), approaching the critical point along the real (imaginary) direction for \( \epsilon > 0 \) (\( \epsilon < 0 \)).

The myriad non-generic possibilities will not be exhausted. For example, if \( \chi_1(\omega_*) = 0 \) but \( \chi_1(\omega_*) \neq 0 \), then to leading order,

\[
 0 = \Delta \omega [(\Delta \omega)^{J-1} + \epsilon \chi_1(\omega_*) + \ldots] \quad (6.4)
\]

Thus one state is unshifted to lowest order, and the other \( J-1 \) states split like a generic block of order \( J-1 \), viz., with \( |\Delta \omega| \propto \epsilon^{1/(J-1)} \) and splitting in \( J-1 \) equiangular directions. [For \( J = 2 \), the \( \epsilon^2 \chi_2(\omega_*) \) term is of the same order and must be retained as well.]

A more systematic analysis involving the basis vectors as well will be given elsewhere [\[ \]].

**VII. Conclusion**

In ohmically damped linear systems, including a broad class of optical resonators, the correlation function can be expressed as a sum over eigenvectors \( j \), differing from the
conservative case only through the appearance of the PF $C_J$ multiplying each term. In this paper we have demonstrated this in a broad context, allowing many interesting properties to be established in a systematic way, without reference to the details of the modes in optical cavities. For example, (complex) frequency shifts in response to time-independent perturbations are also proportional to $C_J$. We have also shown that $C_J \to \infty$ at critical points, but (a) in the correlation function the divergent parts cancel, while (b) in time-independent perturbation theory the shifts $\sim e^{C_J}$ go over to $\sim \epsilon^{1/J}$.

In conservative systems one is used to dealing with Hilbert spaces, in which vectors have magnitudes (associated with diagonal inner products $\langle \phi | \phi \rangle$) and directions (non-diagonal cosine associated with off-diagonal inner products $\langle \psi | \phi \rangle$). Both concepts refer to the same inner product. In dissipative systems, lengths of eigenvectors are given by $N_j$ whereas projections are given by the bilinear map $(\psi, \phi)$. The PF arises because these are different mathematical constructs. That is, the linear-space structure for dissipative systems is significantly different from that for conservative systems, and will be given in more detail elsewhere \[27\]. The inner product and bilinear map are related by the duality $D$, whose representation \[27\] is crucial here. This requires a first-order formalism, involving both coordinates and momenta (in the optics case, both the magnetic and electric fields), an ingredient previously missing in the literature.

The present discussion for a finite number of linear classical oscillators is readily generalized. (a) Many models of dispersion can be accommodated by enlarging the linear space, or equivalently postulating hidden ohmic oscillators \[21\]. (b) Turning $\alpha$ into a continuous variable $x$ gives continuum models, in which nearest-neighbor couplings (i.e., only between $\alpha$ and $\alpha \pm 1$) turn into a second-order spatial derivative $\partial_x^2$; see Appendix B. Electromagnetic waves in optical resonators are then included. (c) The variables $\phi(\alpha)$ and $\dot{\phi}(\alpha)$ can be promoted to operators satisfying the equal-time commutation relation

\[
[\phi(\alpha, t), \dot{\phi}(\beta, t)] = i\hbar\delta(\alpha, \beta).
\]

Eigenvector expansions remain formally unchanged, while equations of motion are modified only by the presence of quantum noise (which however needs to be handled with care). The coefficients $a^\dagger$ and $(a^\dagger)^* \dot{a}^k$ then become annihilation and creation operators. (d) Certain two-point correlations are just the Feynman propagators, which can be used to construct a perturbative expansion for interacting (i.e., nonlinear) fields, much in the usual way. Interestingly, $\Delta^R(t) \sim \langle a^j(t)^* a^k(0) \rangle$ is not diagonal. In fact, the usual logical chain from plane electromagnetic waves to electromagnetic propagators to free photons to interacting photons can be simply repeated — with the difference that each mode in the expansion is now a quasinormal mode, and each term carries the PF $C_J$. These further developments, which will be given elsewhere, open the way to a description of cavity quantum electrodynamics in terms of the damped quasinormal modes, convenient because of their discrete nature.

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**APPENDIX A: NON-GENERIC CRITICALITY**

Section \[11\] showed examples of generic criticality with codimension one in parameter space, in which a pair of eigenvalues coalesce on the imaginary axis, corresponding to the classic case of critical damping. This Appendix describes some non-generic examples. It suffices to consider a system of $N=2$ oscillators with $M=I$, and $K$, $\Gamma$ given by \[22\], but allowing $\gamma_{12} \neq 0$, so there are 6 parameters.

1. Fourth-order block

To obtain a block where $J=4$ eigenvalues merge, we require $\chi(\omega)$ to have a 4th-order zero on the imaginary axis: $\det(\mathcal{H} - \omega) = (\omega + i)^4$ by a choice of scale. This identity gives four equations, leading to a 2-parameter family of solutions, of which a simple 1-parameter subset is: $k_{11} = k_{22} = \cosh(x)$, $k_{12} = \sinh(x)$, $\gamma_{11} = 1 + s_3$, $\gamma_{22} = 1 - s_3$, $\gamma_{12} = s_2$, where $s_n = \sinh^n(x/2)/\sinh(x)$. Positivity of $\Gamma$ requires $\cosh(x) \leq 3$.

The eigenvector(s) are found from $(K - \Gamma + I) \phi = 0$, and generically there is only one solution, hence a $J=4$ block. However, exceptionally there can be more than one eigenvector, leading to the crossing of different blocks. For a $2 \times 2$ system, this requires $K - \Gamma + I = 0$, and happens only for $x = 0$ — the trivial case of two independent but identical oscillators, each generating a $J=2$ block at the critical point. Interestingly, one cannot produce crossing between $J=3$ and $J=1$ blocks with only two oscillators.

Perturbations around a $J=4$ block can be studied, e.g. by letting $k_{11} \to k_{11} + \epsilon$ in the above. The eigenvalues solved numerically from $\chi(\omega) = 0$ shift by an amount $\propto \epsilon^{1/4}$ in 4 equiangular directions.
2. Two second-order blocks

To illustrate critical points off the imaginary axis, consider two second-order blocks at $\omega = -i \pm b$: $\det(\mathcal{H} - \omega) = (\omega + i b)^2 (\omega - i b)^2$. One family of solutions is given by $k_{11} = k_{22} = (1 + b^2) \cosh(x)$, $k_{12} = (1 + b^2) \sinh(x)$, $\gamma_{11} = 1 + 2 s_3 \xi$, $\gamma_{22} = 1 - 2 s_3 \xi$, $\gamma_{12} = 2 s_2$, $\xi = [1 + b^2 \coth^2(x/2)]^{1/2}$. The choice $b = 2$, $\sinh(x/2) = 1$ results in the simple special case

$$K = 5 \left( \frac{3}{2\sqrt{2}} \quad \frac{2\sqrt{2}}{3} \right) , \quad \Gamma = \left( \frac{2 + 3\sqrt{2}}{\sqrt{2}} \quad \frac{\sqrt{2}}{2 - 3\sqrt{2}} \right). \quad (A1)$$

This system at the critical point can be perturbed by $k_{11} \mapsto k_{11} + \epsilon$: the eigenvalues solved from $\chi(\omega) = 0$ for $\epsilon = n^2 c_0$ are shown in Figure 2, for $c_0 = \pm 1 \times 10^{-4}$ and $n = 0, 1, 2, \ldots$. The equal spacing verifies that the shifts go as $\epsilon^{1/2}$, and the shifts for opposite signs of $\epsilon$ bisect each other — for $J = 2$, the shifts in orthogonal directions resemble critical damping in the elementary sense.

These non-generic examples are much easier to construct and analyze than for continuum models (such as optical cavities), and readily verify the properties of the PF when $J$ eigenvectors are close to each other. Further details will not be presented.

APPENDIX B: CONTINUUM MODEL

Consider a scalar model of electromagnetism:

$$\mu \varepsilon(r) \frac{\partial^2 \phi(r, t)}{\partial t^2} + \mu \sigma(r) \frac{\partial \phi(r, t)}{\partial t} - \nabla^2 \phi(r, t) = 0 , \quad (B1)$$

where $\mu$ is the permeability of free space, $\varepsilon$ is the dielectric constant and $\sigma$ is the conductivity. Let the model be defined on $|x| \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, with $\phi = 0$ on the boundary. Further assume that the system is uniform in $y$ and $z$: $\varepsilon = \varepsilon(x)$, $\sigma = \sigma(x)$; then the $y$ and $z$ dependence can be expressed as $\phi \propto \sin(n_1 \pi y/b) \sin(n_2 \pi z/c)$, and the wave equation reduces to the 1-dimensional model

$$\rho(x) \frac{\partial^2 \phi(x, t)}{\partial t^2} + \Gamma(x) \frac{\partial \phi(x, t)}{\partial t} + (K \phi)(x, t) = 0 , \quad (B2)$$

where $\rho = \mu \varepsilon$, $\Gamma = \mu \sigma$ and $K = q^2 - \partial_x^2$, with $q^2 = (n_1 \pi/b)^2 + (n_2 \pi/c)^2$; the boundary condition is $\phi(0, t) = 0$. Our formulation studies the $t$-dependence, with eigenvectors being solutions that evolve as $e^{-\omega t}$; many works in the literature study instead the $x$-dependence (i.e., how a mode grows as it propagates along the optic axis), with eigenvectors being solutions that behave as $e^{i k x}$. It is trivial to adapt the formulation to the latter case.

The obvious discretization, with $x_n = -a + \alpha_n \Delta$, $\phi(x_n) \to \phi(\alpha)$, $\alpha = 1, \ldots, N = 2a/\Delta$, then leads to the problem

$$M(\alpha, \beta) = \rho(x_\alpha) \delta(\alpha, \beta) , \quad \Gamma(\alpha, \beta) = \Gamma(x_\alpha) \delta(\alpha, \beta) , \quad K(\alpha, \beta) = q^2 \delta(\alpha, \beta) - \frac{1}{\Delta^2} \left[ \delta(\alpha-1, \beta) - 2 \delta(\alpha, \beta) + \delta(\alpha+1, \beta) \right] . \quad (B3)$$

By reversing this mapping, it is straightforward to derive the bilinear map in the continuum model, namely (up to an irrelevant overall factor of $\Delta$)

$$(\psi, \phi) = i \int_{-a}^{a} dx \left[ \hat{\psi}(x) \phi(x) + \psi(x) \hat{\phi}(x) + \psi(x) \Gamma(x) \phi(x) \right] . \quad (B4)$$

Examples with a $J = 2$ critical point on the imaginary axis are trivial to construct. Take $\Gamma(x) = 2 \gamma \rho(x)$, with $\gamma$ to be tuned. The system with $\gamma = 0$ has a complete set of eigenfunctions $f_j(x)$:

$$K f_j = \Omega_j^2 \rho f_j , \quad (B5)$$

where $\Omega_1^2 < \Omega_2^2 < \ldots$ are real. It then follows that $f_j$ are also eigenfunctions of $[B3]$, but with complex eigenvalues

$$\omega_j = -i \gamma \pm \sqrt{\Omega_j^2 - \gamma^2} . \quad (B6)$$

So as $\gamma$ is increased from zero, the modes $j = 1, 2, \ldots$ go through criticality in turn.

Examples with a critical point off the imaginary axis require that two parameters are tuned. We take

$$\rho(x) = \begin{cases} \rho_1 & \text{for } 0 < |x| < a_1 \\ \rho_2 & \text{for } a_1 < |x| < a \end{cases} ,$$

$$\gamma(x) = \begin{cases} 2 \rho_1 \gamma_1 & \text{for } 0 < |x| < a_1 \\ 2 \rho_2 \gamma_2 & \text{for } a_1 < |x| < a \end{cases} , \quad (B7)$$

with $a = 2$, $a_1 = 1$. We fix $q^2 = 1$, $\rho_2 = 1$, $\gamma_2 = 1$ and tune $\rho_1$, $\gamma_1$. A critical point is found at $\rho_{1*} = 5.891$, $\gamma_{1*} = 1.994$. To display the nature of the splitting near the critical point, we let $\rho_1 \mapsto \rho_{1*} + \epsilon$, with $\epsilon = n^2 c_0$. The positions of the eigenvalues for $c_0 = \pm 1 \times 10^{-4}$ and $n = 0, 1, 2, \ldots$ are shown in Figure 3, demonstrating the same features as the discrete example in Figure 2.

The PF is also evaluated, with the bilinear map given by $[B4]$, and the norm $N_j$ given by

$$N_j = \int_{-a}^{a} dx \rho(x) |f_j(x)|^2 . \quad (B8)$$

Figure 4 shows $|C_j|$ for one of the near-critical modes versus $\epsilon$, verifying that $|C_j| \propto \epsilon^{-1/2}$. In particular, models with very large values of $C_j$ are readily constructed.
[10] The equation of motion can be given a quantum pedigree if one starts with a bath which is then eliminated from the equations of motion or path integral.
[11] A systematic notation would distinguish upper and lower indices, and the elements of $\mathcal{H}$ are denoted as $\mathcal{H}^\alpha_\beta$. The duality map lowers indices, and its inverse raises indices; see Ref. [1] for details.
[12] This is readily demonstrated as follows. Multiply the defining equation by $f_\beta(\alpha)^*$ and sum over $\alpha$. Define $m = f_\beta^\dagger MF_\beta > 0$, $2\gamma = f_\beta^\dagger \Gamma f_\beta > 0$ and $k = f_\beta^\dagger K f_\beta > 0$, giving $-\omega^2 f_\beta m - 2\omega_\beta \gamma + k = 0$ familiar from the case of a single oscillator.
[13] For an eigenvector, the momentum $f_\beta$ is completely determined by the coordinate $f_\beta$, so it is reasonable to define the norm only in terms of the latter.
[14] The definition is arbitrary up to any reasonable factor that reduces to unity in the conservative limit. For example, $2\omega_\beta$ can be replaced by its absolute value.
[15] Tuning one parameter in general causes the eigenvalues of a self-adjoint matrix to undergo level repulsion without crossing.
[16] More generally, changes in $M$ are allowed as well.
FIG. 1: The real (a) and imaginary (b) parts of the $\Delta \omega / \epsilon$, where $\Delta \omega$ is the shift in the eigenvalue for the system described in Section II, due to a perturbation $k_{11} \rightarrow k_{11} + \epsilon$. Straight lines are second-order perturbation theory; points are exact roots of the characteristic polynomial.
FIG. 2: Position of eigenvalues in the complex plane when a $J = 2$ critical point in a discrete model (see Appendix A) is split upon $k_{11} \mapsto k_{11} + \epsilon$, with $\epsilon = n^2 \epsilon_0$, $n = 0, 1, 2, \ldots$; $\epsilon_0 = 1 \times 10^{-4}$ (plus signs) $\epsilon_0 = -1 \times 10^{-4}$ (crosses). The nearly equal spacing shows that the shifts are $\propto \epsilon^{1/2}$.
FIG. 3: Position of eigenvalues in the complex plane when a $J = 2$ critical point in a continuum model (see Appendix B) is split upon $k_{11} \rightarrow k_{11} + \epsilon$, with $\epsilon = n^2 \epsilon_0$, $n = 0, 1, 2, \ldots$; $\epsilon_0 = 1 \times 10^{-4}$ (plus signs) $\epsilon_0 = -1 \times 10^{-4}$ (crosses). The nearly equal spacing shows that the shifts are $\propto \epsilon^{1/2}$. 
FIG. 4: $|C_j|$ versus $\epsilon$ (log-log plot) for one of the near-critical modes shown in Figure 3, showing $|C_j| \propto \epsilon^{-1/2}$. 