Leptogenesis in seesaw models with a twofold-degenerate neutrino Dirac mass matrix

Walter Grimus*
Institut für Theoretische Physik, Universität Wien
Boltzmanngasse 5, A–1090 Wien, Austria

Luís Lavoura**
Universidade Técnica de Lisboa
Centro de Física das Interacções Fundamentais
Instituto Superior Técnico, P–1049-001 Lisboa, Portugal

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Abstract

We study leptogenesis in two seesaw models where maximal atmospheric neutrino mixing and $U_{e3} = 0$ result from symmetries. Salient features of those models are the existence of three Higgs doublets and a twofold degeneracy of the neutrino Dirac mass matrix. We find that in those models both leptogenesis and neutrinoless double beta decay depend on the same unique Majorana phase. Leptogenesis can produce a baryon asymmetry of the universe of the right size provided the mass of the heavy neutrino whose decays generate the lepton asymmetry is in the range $10^{11}–10^{12}$ GeV. Moreover, in these models, leptogenesis precludes an inverted neutrino mass spectrum since it requires the mass of the lightest neutrino to be in the range $10^{-3}–10^{-2}$ eV.

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*E-mail: walter.grimus@univie.ac.at
**E-mail: balio@cfif.ist.utl.pt
1 Introduction

Experimental cosmology has witnessed spectacular progress during the last few years. In particular, the WMAP experiment [1] has determined with fantastic precision the baryon asymmetry of the universe, which is given by the ratio of baryon number over the number of photons, experimentally measured to be

$$\eta_B \equiv \frac{n_B - n_{\bar{B}}}{n_{\gamma}} = 6.5^{+0.4}_{-0.3} \times 10^{-10},$$

(1)

where $n_B$ is the (present) baryon density of the universe, $n_{\bar{B}}$ is the anti-baryon density and $n_{\gamma}$ is the (present) density of photons. This value of $\eta_B$ is in excellent agreement with the one inferred from big bang nucleosynthesis [2].

Another field which has lately witnessed outstanding experimental progress is neutrino masses and lepton mixing. On the one hand, the first results of the KamLAND experiment [3] have conclusively proved that solar neutrinos oscillate; a global analysis of all solar neutrino results, including the recent SNO measurement [4] and also the KamLAND result, gave a mass-squared difference

$$\Delta m^2_{\odot} \equiv m^2_2 - m^2_1 = 7.1^{+1.2}_{-0.6} \times 10^{-5} \text{eV}^2$$

(2)

and a large but non-maximal mixing angle

$$\theta = 32.5^{+2.4}_{-2.3} \text{degrees},$$

(3)

where the errors reflect 1σ constraints in the two-dimensional $\theta-\Delta m^2_{\odot}$ region. Note that $\tan \theta$ is the ratio, in the decomposition of the electron neutrino $\nu_e$, of the amplitude for the heavier neutrino $\nu_2$ over the amplitude for the lighter neutrino $\nu_1$: $\tan \theta = |U_{e2}/U_{e1}|$. where $U$ is the lepton mixing matrix. On the other hand, the Super-Kamiokande experiment [5] has shown that atmospheric neutrinos oscillate with a mass-squared difference in the 90% CL range

$$1.3 \times 10^{-3} \text{eV}^2 < \Delta m^2_{\text{atm}} \equiv |m^2_3 - m^2_1| < 3.0 \times 10^{-3} \text{eV}^2,$$

(4)

with best-fit value $\Delta m^2_{\text{atm}} = 2.0 \times 10^{-3} \text{eV}^2$, and a (most likely) maximal mixing angle:

$$\sin^2 2\theta_{\text{atm}} \equiv 4 |U_{\mu 3}|^2 (1 - |U_{\mu 3}|^2) > 0.90$$

(5)

at 90% CL, the best-fit value being exactly 1. Finally, the CHOOZ experiment [6] and all other neutrino oscillation data yield the upper bound $|U_{e3}|^2 < 0.054$ at 3σ [7]. For recent reviews on neutrino oscillations see [8].

These experimental developments invite a renewed interest of theorists for leptogenesis [9, 10]. This is the possibility that the baryon asymmetry of the universe has been generated through the standard-model sphaleron transmutation of a previously existing lepton asymmetry, which in turn was generated at the decay of the heavy neutrinos involved in the seesaw mechanism [11]. In the standard version of that mechanism one introduces three gauge-singlet right-handed neutrinos $\nu_R$. Let us define

$$\nu'_L \equiv C \bar{\nu}_R^T,$$

(6)
where $C$ is the charge-conjugation matrix. Then, the mass terms for the neutrinos are given by \[12\]

$$
\mathcal{L}_{\nu_{\text{mass}}} = -\bar{\nu}_R M_D \nu_L - \bar{\nu}_L M_D^T \nu_R - \frac{1}{2} \bar{\nu}_R M_R C \bar{\nu}_R^T + \frac{1}{2} \nu_R^T C^{-1} M_R^* \nu_R
$$

(7)

where $M_R$ is a symmetric matrix. If the eigenvalues of $M_R M_R^*$ are all much larger than the eigenvalues of $M_D M_D^*$, then the approximate Majorana mass matrix for the light neutrinos is given by

$$
\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D
$$

(9)

while the Majorana mass matrix for the heavy neutrinos is approximately equal to $M_R$ \[13\]. In the weak basis where the mass matrix $M_\ell$ of the charged leptons is diagonal, $M_\ell = \text{diag}(m_e, m_\mu, m_\tau)$, one has

$$
U^T \mathcal{M}_\nu U = \text{diag}(m_1, m_2, m_3)
$$

(10)

where $m_{1,2,3}$ are real non-negative and $U$ is once again the lepton mixing matrix.

Since leptogenesis needs CP violation, an intriguing question is whether there is a connection between the CP violation at the seesaw scale and the one at low energies. In the most general case, the answer to this question is negative \[14, 15\]. However, it is easy to find scenarios where such a connection exists—see, for instance, \[16, 17\]; in minimal scenarios only two heavy Majorana neutrinos are required \[18, 19, 20, 21\]. Inspired by grand unified theories, it is also quite common in studies of leptogenesis to assume hierarchies in the neutrino sector—see \[22, 23, 24, 25\] and references therein. In particular, one may assume that the Dirac neutrino mass matrix $M_D$ is strongly hierarchical, i.e. that the eigenvalues $|a|^2$, $|b|^2$, $|c|^2$ of $M_D M_D^*$ satisfy $|a| \ll |b| \ll |c|$, and subsequently reconstruct the masses of the heavy Majorana neutrinos from the low energy data \[22, 23\]. The assumption of a hierarchy in $M_D$ is justified by the relationship, existing in some grand unified theories, between $M_D$ and the up-type-quark mass matrix, and by the fact that the latter matrix is known to be strongly hierarchical.

In this paper we take a different departure and start from the fact that atmospheric neutrino mixing is maximal (or nearly maximal), which suggests an alternative possibility \[26, 27, 28\]. Since experimentally $|U_{\mu 3}| \simeq |U_{\tau 3}|$, there may exist in nature a $\mu-\tau$ interchange symmetry. We know that $m_\mu \neq m_\tau$, hence the $\mu-\tau$ interchange symmetry must be broken in the charged-lepton sector, but it may be kept intact in the neutrino sector. This can be achieved through the introduction of three Higgs doublets, one of them ($\phi_1$) with Yukawa couplings to the neutrino singlets and to the charged-lepton singlet $e_R$, and the other two ($\phi_2$ and $\phi_3$) with Yukawa couplings only to the charged-lepton singlets $\mu_R$ and $\tau_R$. Under the interchange $\mu \leftrightarrow \tau$ the doublet $\phi_2$ remains invariant while $\phi_3$ changes sign; this leads to $m_\mu \neq m_\tau$. On the other hand, the neutrino Dirac mass matrix is twofold degenerate because $\phi_1$ is invariant under $\mu \leftrightarrow \tau$:

$$
M_D = \text{diag}(a, b, b).
$$

(11)
A crucial feature of these models is the existence of some other symmetry—either the continuous lepton-number symmetries [26] or a discrete symmetry [27]—which forces $M_D$ and the charged-lepton mass matrix $M_\ell$ to be simultaneously diagonal. These other symmetries are allowed to be softly broken, hence the right-handed-neutrino Majorana mass matrix is non-diagonal and has the form

$$M_R = \begin{pmatrix}
m & n & n \\
n & p & q \\
n & q & p
\end{pmatrix}$$

(12)
due to the $\mu-\tau$ interchange symmetry. It is this matrix $M_R$ which produces lepton mixing.

The neutrino sectors of the $Z_2$ model of [26] and of the $D_4$ model of [27] are both characterized by equations (11) and (12); the $D_4$ model is more constrained than the $Z_2$ model since it has $q = 0$ in $M_R$. Note that the $CP$-violating phase analogous to the CKM phase is absent from the models under discussion, because $U_{e3} = 0$. This follows easily from equations (11) and (12), since

$$M_\nu = \begin{pmatrix}
x & y & y \\
y & z & w \\
y & w & z
\end{pmatrix}$$

(13)
in the basis in which $M_\ell$ is diagonal. Thus the only sources of $CP$ violation in the leptonic sector are the two physical Majorana phases in $U$.

The purpose of this paper consists in analyzing leptogenesis in the models of [26, 27]. In particular, we will show that they have the following properties:

1. Leptogenesis is a viable scenario.

2. Correctly reproducing $\eta_B$ constrains the spectra of both the light and the heavy neutrinos.

3. Only one of the two Majorana phases is responsible for leptogenesis, and that phase is also the only one which appears in the effective mass $|\langle m \rangle|$ for neutrinoless $\beta\beta$ decay.

In section 2 we review the computation of $\eta_B$ from the knowledge of $M_D$ and $M_R$, with emphasis on the three-Higgs-doublet structure of our models. We proceed in section 3 to derive the relevant analytic formulae for the diagonalization of $M_R$ and $M_\nu$, in order to calculate $\eta_B$. We apply those formulæ in section 4 to study the variation of $\eta_B$ with the parameters of the models. In section 5 we draw our conclusions. An appendix contains calculational details related to section 3.

## 2 Baryogenesis from leptogenesis

The “natural” basis for our models is given by diagonal matrices $M_D$ and $M_\ell$ while $M_R$ is non-diagonal—see equations (11) and (12). However, the basis in which the leptogenesis
formalism is established is the one where $M_\ell$ and the mass matrix of the right-handed neutrino singlets are diagonal; the latter matrix is then

$$\hat{M}_R \equiv \text{diag} (M_1, M_2, M_3),$$

(14)

with real non-negative diagonal elements. Defining a unitary matrix $V$ by

$$V^T M_R V = \hat{M}_R,$$

(15)

we find, using equation (7),

$$M'_D = V^T M_D$$

(16)

for the neutrino Dirac mass matrix in the leptogenesis basis. For the actual calculation of $\eta_B$ one needs

$$R \equiv M'_D M'_D^\dagger = V^T M_D M_D^\dagger V^*.$$  

(17)

We assume, for the masses of the heavy Majorana neutrinos $N_{1,2,3}$, that $M_1 \ll M_{2,3}$. Then, the $CP$ asymmetry $\epsilon_1$ produced in the decay of $N_1$ (the heavy neutrino with mass $M_1$) is [10, 29]

$$\epsilon_1 = \frac{1}{8\pi |v_1|^2 R_{11}} \sum_{j=2}^{3} f \left( \frac{M_j^2}{M_1^2} \right) \text{Im} \left[ (R_{1j})^2 \right],$$

(18)

where $v_1$ denotes the vacuum expectation value (VEV) of $\phi_1^0$. The function $f$ is given by

$$f(t) = \sqrt{\frac{2 - t}{1 - t}} + (1 + t) \ln \frac{t}{1 + t}.$$  

(19)

For $t \gg 1$, we have

$$f(t) = -\frac{3}{2} t^{-1/2} - \frac{5}{6} t^{-3/2} - \frac{13}{12} t^{-5/2} - \frac{19}{20} t^{-7/2} - \ldots.$$  

(20)

Thus, $f(t)$ is negative.

The leptonic asymmetry produced through the decay of $N_1$ is written as [10, 30, 31]

$$Y_L \equiv \frac{n_L - \bar{n}_L}{s} = \frac{\epsilon_1 \kappa_1}{g_{*1}},$$

(21)

where $n_L$ is the lepton density, $\bar{n}_L$ is the anti-lepton density, $s$ is the entropy density, $\kappa_1$ is the dilution factor for the $CP$ asymmetry $\epsilon_1$ and $g_{*1}$ is the effective number of degrees of freedom at the temperature $T = M_1$. The effective number of degrees of freedom is given by (see for instance [32])

$$g_* = \sum_{j=\text{boson}} g_j + \frac{7}{8} \sum_{k=\text{fermion}} g_k.$$  

(22)

In the $SU(2) \times U(1)$ gauge theory with three Higgs doublets and supplemented by the seesaw mechanism one has

$$g_{*1} = \left[ 28 + \frac{7}{8} \times 90 \right]_{\text{SM}} + 8 + \frac{7}{8} \times 2 = 116.5,$$  

(23)
where the terms within the brackets are the Standard Model contributions and the last two terms in the sum take into account the two additional Higgs doublets and the lightest heavy neutrino $N_{1}$, respectively.

The baryon asymmetry $Y_{B}$ produced through the sphaleron transmutation of $Y_{L}$, while the quantum number $B - L$ remains conserved, is given by [33]

$$Y_{B} = \frac{\omega}{\omega - 1} Y_{L} \quad \text{with} \quad \omega = \frac{8N_{F} + 4N_{H}}{22N_{F} + 13N_{H}},$$

(24)

where $N_{F} = 3$ is the number of fermion families and $N_{H}$ is the number of Higgs doublets. This relation derives from the thermal equilibrium of sphalerons for $10^{2} \text{ GeV} \lesssim T \lesssim 10^{12} \text{ GeV}$ [10, 31]. Note that we must impose the condition $M_{1} \lesssim 10^{12} \text{ GeV}$, otherwise $Y_{L}$ could be erased before it transmutes into $Y_{B}$. From equation (24), $\omega = 12/35$ in three-Higgs-doublet models.

Next we discuss the relation between $Y_{B}$ and $\eta_{B}$, where the latter quantity is the ratio baryon-number density over photon density. Note that this is the present ratio, and $\eta_{B}$ has last changed at the time of $e^{+}e^{-}$ annihilation, at which time the photon density (temperature) has increased relative to the neutrino density (temperature). On the other hand, $Y_{B}$ did not change since its generation—the baryon number per comoving volume and the entropy per comoving volume remain constant. Thus we have

$$\eta_{B} = \frac{s}{n_{\gamma}} \bigg|_{0} Y_{B},$$

(25)

where $n_{\gamma}$ is the photon density. The index 0 denotes present time. We know that [32]

$$s = \frac{2\pi^{2}}{45} g_{*0} T^{3} \quad \text{and} \quad n_{\gamma} = \frac{2}{\pi^{2}} \zeta(3) T^{3},$$

(26)

where $T$ is the photon temperature and $\zeta(3) \approx 1.20206$, $\zeta$ being the zeta function. At present only photons and light neutrinos are relevant for $s$, since all other particles have annihilated apart from tiny remnants—actually, we are just discussing the baryon remnant. Thus

$$g_{*0} = 2 + \frac{7}{8 \times 6 \times \frac{4}{11}} = \frac{43}{11},$$

(27)

where we have taken into account that the neutrino temperature after $e^{+}e^{-}$ annihilation is a factor of $(4/11)^{1/3}$ lower than the photon temperature $T$. One thus obtains [20, 34]

$$\left. \frac{s}{n_{\gamma}} \right|_{0} = \frac{\pi^{4}}{120} \frac{43}{11} = 0.338 \quad \text{and} \quad \eta_{B} = 0.338 Y_{B} \simeq -3.15 \times 10^{-2} \kappa_{1} \epsilon_{1}.$$ 

(28)

We have used the values for three-Higgs-doublet models.

We now turn to the dilution factor, which is approximately given by [30, 35, 36]

$$\kappa_{1} \approx \frac{0.3}{K_{1} \ln K_{1}}^{3/5},$$

(29)

where

$$K_{1} = \frac{\Gamma_{1}}{H_{1}}.$$ 

(30)
In this equation, $\Gamma_1$ is the decay width of $N_1$, given at tree level by

$$\Gamma_1 = \frac{R_{11} M_1}{8\pi |v_1|^2},$$  \hfill (31)

and $H_1$ is the Hubble constant at temperature $T = M_1$,

$$H_1 = 1.66 \sqrt{g_{*1}} \frac{M_1^2}{M_{\text{Planck}}},$$  \hfill (32)

where $M_{\text{Planck}} = 1.221 \times 10^{19} \text{GeV}$. Thus,

$$K_1 = \frac{M_{\text{Planck}} R_{11}}{1.66 \sqrt{g_{*1}} 8\pi |v_1|^2 M_1}.$$  \hfill (33)

Equation (29) holds for $10 \lesssim K_1 \lesssim 10^6$, (30) 35 36. Numerically one obtains

$$K_1 \simeq \frac{895.6}{1 \text{eV}} \frac{R_{11}}{M_1} \left( \frac{174 \text{GeV}}{|v_1|} \right)^2.$$  \hfill (34)

For $g_{*1}$ we have used the value of equation (23).

With equations (18), (28), (29) and (33) one obtains, after a numerical evaluation,

$$\eta_B \simeq -1.39 \times 10^{-9} \frac{1}{(\ln |K_1|)^{3/5}} \sum_{j=2}^{3} f \left( \frac{M_j^2}{M_1^2} \right) \frac{\text{Im} \left[ (R_{1j})^2 \right]}{(R_{11})^2} \frac{M_1}{10^{11} \text{GeV}}.$$  \hfill (35)

This equation, which we shall use in conjunction with equation (34), clearly indicates the desired order of magnitude of $M_1$. Equation (35) shows that $\eta_B$ is approximately independent of $|v_1|$. Only the logarithm of $K_1$ in the denominator makes $\eta_B$ dependent on $|v_1|$. Also notice that the minus sign in equation (35) cancels with the negative sign of $f(t)$.

3 The baryon asymmetry in our models

It remains to calculate the quantities $\text{Im} \left[ (R_{1j})^2 \right]$ ($j = 2, 3$) and $R_{11}$ in our models. We will see that the $Z_2$ model allows a full analytical calculation of these quantities and, according to the discussion at the end of section 1 these quantities in the $D_4$ model are special cases of those in the $Z_2$ model.

We remind the reader that, in our models, the effective light-neutrino Majorana mass matrix $M_\nu$ is as in equation (13), while the right-handed-neutrino Majorana mass matrix $M_R$ has the same form as $M_\nu$ and is in equation (12). The matrix $M_\nu$ is diagonalized by $U$, see equation (10), while $M_R$ is diagonalized by $V$, see equations (14) and (15). The analogies between $M_\nu$ and $M_R$ and between $U$ and $V$ are used repeatedly throughout this paper.
It can be shown that, because \( \mathcal{M}_\nu \) is of the specific form in equation (13), i.e. with \( (\mathcal{M}_\nu)_{\mu\mu} = (\mathcal{M}_\nu)_{\tau\tau} \) and \( (\mathcal{M}_\nu)_{\mu\tau} = (\mathcal{M}_\nu)_{\tau\mu} \), the matrix \( U \) can be parametrized as

\[
U = \text{diag}\left(1, e^{i\alpha}, e^{i\beta}\right) \begin{pmatrix}
-c & s & 0 \\
\frac{c}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{c}{\sqrt{2}} & \frac{s}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix} \text{diag}\left(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}\right),
\]

with \( c \equiv \cos \theta \) and \( s \equiv \sin \theta \). We assume, without loss of generality, that \( \theta \) belongs to the first quadrant while \( m_2 > m_1 \). The phase \( \alpha \) is unphysical, the only physical phases are the differences

\[
\Delta \equiv 2(\beta_1 - \beta_2)
\]

and \( 2(\beta_1 - \beta_3) \). The matrix \( \mathcal{M}_\nu \) has six physical parameters: the moduli of \( x, y, z \) and \( w \) and the phases of \( zw^* \) and \( y^2z^*x^*z^* \). It is better to use as physical parameters the moduli of \( z + w \) and \( z - w \) instead of the moduli of \( z \) and \( w \), and the phases of \( (z - w)(z + w)^* \) and \( y^2z^*x^*z^* \). The six parameters \( |x|, |y|, |z + w|, |z - w|, \arg[y^2z^*x^*z^*] \) and \( \arg[(z - w)(z + w)^*] \) correspond to the six observables \( m_1, m_2, m_3, \theta, \Delta \) and \( 2(\beta_1 - \beta_3) \).

Since the matrices \( M_R \) and \( \mathcal{M}_\nu \) have the same form, the matrix \( V \) which diagonalizes \( M_R \) has the same form as the matrix \( U \) which diagonalizes \( \mathcal{M}_\nu \):

\[
V = \text{diag}\left(1, e^{i\gamma_1}, e^{i\gamma_3}\right) \begin{pmatrix}
-c' & s' & 0 \\
\frac{s'}{\sqrt{2}} & \frac{c'}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{s'}{\sqrt{2}} & \frac{c'}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix} \text{diag}\left(e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3}\right),
\]

with \( c' \equiv \cos \theta' \) and \( s' \equiv \sin \theta' \). We assume, without loss of generality, that \( \theta' \) belongs to the first quadrant while \( M_2 > M_1 \). Like in the previous paragraph, there are six observables originating in \( M_R \): \( M_1, M_2, M_3, \theta', 2(\gamma_1 - \gamma_2) \) and \( 2(\gamma_1 - \gamma_3) \).

We proceed to the calculation of \( R \)—see equation (17). Using equations (11) and (38), we obtain

\[
R = V^T M_D M_D^T V^* = \begin{pmatrix}
|a|^2 c'^2 + |b|^2 s^2 & c's' \left(||b|^2 - |a|^2\right) e^{i(\gamma_1 - \gamma_2)} & \left(|b|^2 - |a|^2\right) e^{i(\gamma_1 - \gamma_2)} & 0 \\
0 & |b|^2 c'^2 & |a|^2 s^2 + |b|^2 c'^2 & 0 \\
0 & 0 & 0 & |b|^2
\end{pmatrix}.
\]

The fact that

\[
R_{13} = R_{23} = 0
\]

implies that in this model the third heavy neutrino has no bearing on leptogenesis—even if its mass \( M_3 \) happens to be lower than \( M_1 \) and \( M_2 \), the masses of the other two heavy neutrinos.

The matrices \( \mathcal{M}_\nu \) and \( M_R \) are related through equation (13), with \( M_D \) given by equation (14). That relation contains only two extra parameters: \( |a| \) and \( |b| \). This means that,

\begin{enumerate}
\item Equation (36) is clearly not the most general parametrization for a unitary matrix, rather it is a consequence of the specific form of \( \mathcal{M}_\nu \) in equation (13).
\item The physical phases in \( \mathcal{M}_\nu \) are the ones of its Jarlskog invariants. For instance, \( \arg(zw^*) \) is the phase of \( (\mathcal{M}_\nu)_{\mu\mu} (\mathcal{M}_\nu)_{\tau\tau} (\mathcal{M}_\nu)^*_{\mu\tau} (\mathcal{M}_\nu)^*_{\tau\mu} \), and \( \arg(y^2z^*x^*) \) is the phase of \( (\mathcal{M}_\nu)_{ee} (\mathcal{M}_\nu)^*_{ee} (\mathcal{M}_\nu)^*_{ee} \).
\end{enumerate}
out of the six observables originating in $M_R$, only two can be considered as independent of the six observables originating in $M_\nu$. We shall select $M_{1,2}$, together with the observables originating in $M_\nu$, as the basic observables of the theory, and express the four remaining ones in terms of these. It will turn out that $2(\beta_1 - \beta_3)$ and $m_3$ play no role in the computation of $\eta_B$; the basic observables that we need for that computation are

$$m_{1,2}, \; M_{1,2}, \; \theta \; \text{and} \; \Delta.$$ (41)

With the aim of using the experimental information on $\Delta m^2_{\odot}$ and $\Delta m^2_{\text{atm}}$, we express

$$m_2 = \sqrt{m_1^2 + \Delta m^2_{\odot}} \quad \text{and} \quad m_3 = \sqrt{m_1^2 + \sigma \Delta m^2_{\text{atm}}} \quad (\sigma = \pm 1)$$ (42)

as functions of $m_1$. For $m_3$ there is a twofold ambiguity stemming from the two possible neutrino spectra: normal spectrum $m_1 < m_2 < m_3$ ($\sigma = +1$) and inverted spectrum $m_3 < m_1 < m_2$ ($\sigma = -1$). The inverted spectrum is only allowed for $m_1 \geq \sqrt{\Delta m^2_{\text{atm}}}$; the experimental result for the atmospheric mass-squared difference—see equation (4)—then requires $m_1 \gtrsim 0.04$ eV, which is rather large.

As shown in the Appendix, the parameters $|x|$, $|y|$ and $|z + w|$ are given in terms of the observables in equation (41) by

$$|x| = \left| c^2 m_1 + s^2 m_2 e^{i \Delta} \right|,$$ (43)

$$|y| = \frac{c s}{\sqrt{2}} \left| m_1 - m_2 e^{i \Delta} \right|,$$ (44)

$$|z + w| = \left| s^2 m_1 + c^2 m_2 e^{i \Delta} \right|.$$ (45)

We may then compute

$$B = 4m_1 m_2 M_1 M_2 |y|^2 - m_1^2 m_2^2 \left( M_1^2 + M_2^2 \right),$$ (46)

$$C = m_1^2 m_2^2 M_1^2 M_2^2 |x|^2 |z + w|^2.$$ (47)

These are the coefficients of the equation

$$|x|^4 |b|^8 + B |x|^2 |b|^4 + C = 0,$$ (48)

which is derived in the Appendix and allows one to compute the parameter $|b|$ as a function of the observables in equation (41). Indeed,

$$|b|^4 = \frac{-B \pm \sqrt{B^2 - 4C}}{2 |x|^2}.$$ (49)

Moreover, the parameter $|a|$ is given by—see equation (A32)

$$|a|^4 = \frac{-B \mp \sqrt{B^2 - 4C}}{2 |z + w|^2},$$ (50)

and the mixing angle $\theta'$ is given, as a function of the observables in equation (41), by—see equation (A34)

$$c^2 - s^2 = \frac{\pm \sqrt{B^2 - 4C}}{m_1^2 m_2^2 (M_2^2 - M_1^2)}.$$ (51)
Thus, using $m_{1,2}$, $M_{1,2}$, $\theta$ and $\Delta$ as input, we are able to compute, with a twofold ambiguity, $|b|$, $|a|$ and $\theta'$. The twofold ambiguity corresponds to the interchanges $|m| \leftrightarrow |p + q|$ and $c^2 \leftrightarrow s^2$, as is made clear in the Appendix.

One must check that the condition $B^2 - 4C \geq 0$ is respected. This condition leads to a lower bound on $m_1$; in the limit $M_2 \gg M_1$ one finds approximately

$$\frac{m_1}{m_2} \gtrsim \frac{M_1}{M_2} \sin^2 2\theta \quad \text{or} \quad m_1 \gtrsim \sqrt{\Delta m^2_{\odot}} \frac{M_1}{M_2} \sin^2 2\theta. \quad (52)$$

On the other hand, it turns out that, in general, with $m_2$ given by the first equation (49), $|a|^2$ and $|b|^2$ increase with increasing $m_1$. In our models, $a$ and $b$ arise from the Yukawa Lagrangian

$$\mathcal{L}_Y = \frac{(-\phi^0, \phi^+)}{v_1} \left\{ a \bar{\nu}_e R \left( \frac{\nu_e L}{\mu L} \right) + b \left[ \bar{\nu}_\mu R \left( \frac{\nu_\mu L}{\mu L} \right) + \bar{\nu}_\tau R \left( \frac{\nu_\tau L}{\tau L} \right) \right] \right\} + \text{h.c.} - \cdots, \quad (53)$$

where $\cdots$ represents the Yukawa couplings of the right-handed charged-lepton fields. If we require that the Yukawa coupling constants $a/v_1$ and $b/v_1$ should at most be of order 1, this implies a loose upper bound on $m_1$ (when $m_2^2 - m_1^2$ is kept fixed), since $|a|$ and $|b|$ increase with $m_1$. For $M_2 \gg M_1$ and using the solution $|b|^2 > |a|^2$, i.e. the upper signs in equations (49) and (50), we find the approximate expressions

$$|a|^2 \simeq m_1 M_1 \left[ 1 - \sin^2 (\Delta/2) \sin^2 2\theta \right]^{1/2}, \quad (54)$$

$$|b|^2 \simeq m_1 M_2 \left[ 1 - \sin^2 (\Delta/2) \sin^2 2\theta \right]^{-1/2}, \quad (55)$$

where we have used $m_2 \simeq m_1$, valid for $m_1 \gg \sqrt{\Delta m^2_{\odot}}$. These equations may be used to compute the approximate upper bound on $m_1$. We see that $|b|^2 \sim m_1 M_2$. Therefore, requiring $|b|^2/|v_1|^2 \lesssim 1$ leads to

$$m_1 \lesssim \frac{|v_1|^2}{M_2}. \quad (56)$$

For instance, for $|v_1| = 10$ GeV and $M_2 = 10^{13}$ GeV, equation (56) yields $m_1 \lesssim 0.01$ eV. If we choose $|v_1| = 50$ GeV and $M_2 = 2.5 \times 10^{12}$ GeV, we have $m_1 \lesssim 1$ eV.

An expression for $\text{Im} \left( (R_{12})^2 \right)$ in terms of our basic observables is derived in the Appendix, see equation (A27). Using equation (53), the main result of this section is

$$\eta_B \simeq -1.39 \times 10^{-9} \frac{1}{(\ln K_1)^{3/5}} f \left( \frac{M_2^2}{M_1^2} \right) (|b|^2 - |a|^2)^2 \frac{M_1^2 M_2 (m_2^2 - m_1^2) c^2 s^2 \sin \Delta}{m_1 m_2 (M_2^2 - M_1^2) (R_{11})^2 (10^{11} \text{GeV})^2}. \quad (57)$$

Here, a convenient expression for $R_{11}$—see equation (59)—is

$$R_{11} = \frac{1}{2} \left[ |a|^2 + |b|^2 + (|a|^2 - |b|^2) (c^2 - s^2) \right], \quad (58)$$

where $c^2 - s^2$ is given by equation (51).
Since \( f (M_2^2 / M_1^2) \) is negative, we find from equation (57) that \( \sin \Delta > 0 \). Notice the important point that the violation of \( CP \) responsible for the generation of a non-zero \( \eta_B \) all comes from the Majorana phase \( \Delta \). This is the same Majorana phase entering the matrix element for neutrinoless double beta decay,

\[
|\langle m \rangle| = |(\mathcal{M}_\nu)_{ee}| = |x|, \tag{59}
\]

which is given by equation (43). Thus, in these models neutrinoless double beta decay and leptogenesis depend on the same Majorana phase \( \Delta \).

To conclude this section, we discuss the extra condition on \( M_R \) in the \( D_4 \) model of [27]. As mentioned before, in that model \( (M_R)_{\mu\tau} = q = 0 \). Since \( M_D \) is diagonal, this leads to

\[
2(M_{\nu1})_{\mu\tau} = 0, \tag{60}
\]

The Majorana phase \( 2(\beta_1 - \beta_3) \) is irrelevant for our purposes, therefore the important constraint is

\[
\frac{1}{m_3} = \left| \frac{s^2}{m_1} e^{i\Delta} + \frac{c^2}{m_2} \right|. \tag{61}
\]

This forces \( m_3 \) to be larger than both \( m_1 \) and \( m_2 \) (normal spectrum). The Majorana phase \( \Delta \) becomes a function of \( m_1 \) through

\[
\cos \Delta = \frac{(m_1 m_2 / m_3)^2 - c^4 m_1^2 - s^4 m_2^2}{2c^2 s^2 m_1 m_2}, \tag{62}
\]

and through equations (42) with \( \sigma = +1 \). Thus, the \( D_4 \) model of [27] has one degree of freedom less than the \( \mathbb{Z}_2 \) model of [26].

### 4 Numerical results

In this section we shall always use

\[
\theta = 33^\circ, \tag{63}
\]

\[
\Delta m_{\odot}^2 = 7.1 \times 10^{-5} \text{eV}^2, \tag{64}
\]

the best-fit values of [4]. Then the observables in our set (11) which are still free to be chosen are \( m_1, M_{1,2} \) and \( \Delta \); we furthermore have to choose \( |v_1| \)—see equations (54) and (57). The mass \( m_2 \) is fixed via equations (42) and (57).

From this input one obtains \( |x|, |y| \) and \( |z + w| \) using equations (43)–(45). One then computes \( |a|, |b| \) and \( \theta' \), with a twofold ambiguity, from equations (49)–(51). Thereafter, \( R_{11} \) is found in equation (58) and \( K_1 \) is given by equation (54).

As for the twofold ambiguity in the computation of \( |a| \) and \( |b| \), we use the upper signs in equations (49)–(51). Numerically, the choice of the lower signs yields a smaller \( \eta_B \).

The VEV \( |v_1| \) must be smaller than 174 GeV, in order that there is also room for the other two VEVs: \( \sum_{j=1}^3 |v_j|^2 = (174 \text{GeV})^2 \). Since in the models of [26] [27] both the
neutrino masses and the electron mass originate in Yukawa couplings to $\phi_1$, while the $\mu$ and $\tau$ masses originate in Yukawa couplings to $\phi_2$ and $\phi_3$, the smallness of the electron and neutrino masses suggests that $v_1$ should be relatively small.

In figure 1 we have plotted $\eta_B$ versus $m_1$ for three different values of $M_1$. We read off from that figure that $M_1$ must be larger than $10^{11}$ GeV in order to reproduce the experimental value of $\eta_B$ in equation (1). Since a successful leptogenesis requires $M_1 < 10^{12}$ GeV [10, 31], the order of magnitude of $M_1$ becomes quite constrained. We furthermore see that small values of $m_1$ are preferred; for large values of $m_1$, $\eta_B$ becomes too small. Actually from figure 1 we gather that $m_1$ cannot exceed $0.02$ eV. This rules out the inverted neutrino spectrum if we want to accommodate leptogenesis in our model. As a function of $m_1$, the maximum of the theoretical expression (57) for $\eta_B$ is roughly at $m_1 = 3 \times 10^{-3}$ eV. If we consider $\eta_B$ as a function of the CP-violating Majorana phase $\Delta$, a numerical study shows that the maximum of $\eta_B$ is attained for $\Delta$ close to 100°. As a function of $|v_1|$, $\eta_B$ increases by less than a factor of two when that VEV goes from 10 GeV to 100 GeV.

In order to understand the dependence of $\eta_B$ of equation (57) on $M_{1,2}$, it is useful to perform a scale transformation $M_{1,2} \rightarrow \lambda M_{1,2}$ where $\lambda$ is an arbitrary positive number.

Figure 1: $\eta_B$ as function of $m_1$ for three different values of $M_1$. In producing this figure we have chosen $M_2/M_1 = 10$, $|v_1| = 50$ GeV and $\Delta = 90°$. The lowest allowed $m_1$ for this value $M_2/M_1 = 10$ is $m_1 = 0.71 \times 10^{-3}$ eV. The horizontal lines indicate the experimental result for $\eta_B$, as given in equation (1).
Figure 2: $\eta_B$ as a function of $m_1$ for $M_1 = 2.5 \times 10^{11}$ GeV and three different values of $M_2$. We have used $|v_1| = 50$ GeV and $\Delta = 90^\circ$.

From equations (49) and (50) we see that $|a|^2$ and $|b|^2$ scale with one power of $\lambda$; the same holds for $R_{12}$ and $R_{11}$. It follows that $\eta_B$ also scales with one power of $\lambda$; in other words, $\eta_B$ is a homogeneous function of order one in $M_{1,2}$.

In figure 2 we have plotted $\eta_B$ versus $m_1$ for $M_1 = 2.5 \times 10^{11}$ GeV and different values of $M_2$. This figure shows that fixing $M_1$ and varying $M_2$ with $M_2 \gg M_1$ does not drastically alter $\eta_B$, except for very small values of $m_1$. The lower bound on $m_1$ as a function of the ratio $M_2/M_1$ is also illustrated in figure 2.

5 Conclusions

In this paper we have computed the baryon asymmetry of the universe in the $Z_2$ model of [26] and in the $D_4$ model of [27]. These models are characterized by a neutrino Dirac mass matrix $M_D$ with two degenerate eigenvalues, and by an interchange ($Z_2$) symmetry between the $\mu$ and $\tau$ families. Both models predict maximal atmospheric neutrino mixing and $U_{e3} = 0$ as a consequence of their symmetries.

We have shown that these models can easily accommodate baryogenesis via leptoge-
Figure 3: The effective Majorana mass in neutrinoless $\beta\beta$ decay as a function of $m_1$. Starting at the upper curve and descending to the lowest of the four curves, the values used for $\Delta$ are $0^\circ$, $90^\circ$, $135^\circ$ and $180^\circ$, respectively.

We see that, if our models are to have successful leptogenesis, then $|\langle m \rangle|$ is at most $0.01$ eV; if the evidence for neutrinoless $\beta\beta$ decay of [37], with $|\langle m \rangle| > 0.1$ eV is confirmed, then leptogenesis is not enough to generate an $\eta_B$ of the observed size.

We stress that the neutrino mass matrices of our models allow an analytical calculation of the $CP$ asymmetry $\epsilon_1$ of equation (18). The results of this paper for the $\mathbb{Z}_2$ and $D_4$ models may be valid in a wider context of general models with a twofold degenerate Dirac mass matrix $M_D$; the reason is that the mass matrix of the light neutrinos must have the form in equation (13) if one assumes maximal atmospheric neutrino mixing and $U_{e3} = 0$.

---

3On the other hand, as shown in [23], a mass hierarchy in $M_D$ requires finetuning of the masses in the heavy neutrino sector in order to reproduce $\eta_B$. 
assumptions that, as experiment shows, cannot be far from true.

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A Algebraic details

The matrices $M_R$ and $M_\nu$ Those matrices are given in equations (12) and (13), and their relationship to each other is expressed by equations (9) and (11). We define

$$d \equiv 2n^2 - m(p + q),$$  \hspace{1cm} (A1)

$$f \equiv 2y^2 - x(z + w),$$  \hspace{1cm} (A2)

so that

$$\det M_R = (q - p) d \quad \text{and} \quad \det M_\nu = (w - z) f.$$  \hspace{1cm} (A3)

Then, by explicitly inverting $M_R$ and by using equation (9), we find

$$x = a^2 \frac{p + q}{d},$$  \hspace{1cm} (A4)

$$y = ab \frac{-n}{d},$$  \hspace{1cm} (A5)

$$z + w = b^2 \frac{m}{d}$$  \hspace{1cm} (A6)

and

$$(z - w)(q - p) = b^2.$$  \hspace{1cm} (A7)

It is also useful to invert equation (9):

$$M_R = -M_D M_\nu^{-1} M_D^T.$$  \hspace{1cm} (A8)

From this relation we compute

$$m = a^2 \frac{z + w}{f},$$  \hspace{1cm} (A9)

$$n = ab \frac{-y}{f},$$  \hspace{1cm} (A10)

$$p + q = b^2 \frac{x}{f}.$$  \hspace{1cm} (A11)

In addition, we obtain equation (A7) again.

The parameters $x$, $y$, $z$ and $w$ From equations (10), (13) and (36) one may write

$$\begin{pmatrix} x & ye^{ia} & ye^{ia} \\ ye^{ia} & ze^{2ia} & we^{2ia} \\ ye^{ia} & we^{2ia} & ze^{2ia} \end{pmatrix} = \begin{pmatrix} -c & s & 0 \\ s/\sqrt{2} & c/\sqrt{2} & 1/\sqrt{2} \\ s/\sqrt{2} & c/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \hat{\mu} \begin{pmatrix} -c & s/\sqrt{2} & s/\sqrt{2} \\ s & c/\sqrt{2} & c/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix},$$  \hspace{1cm} (A12)

where

$$\hat{\mu} = \text{diag}(m_1 e^{-2i\beta_1}, m_2 e^{-2i\beta_2}, m_3 e^{-2i\beta_3}).$$  \hspace{1cm} (A13)
Using the analogy between the matrices $M$ and $R$, especially equations (A3), (A15) and (A17) lead to

$$y^2 x^*(z + w)^* = \frac{c^2 s^2}{2} \left( m_1 - m_2 e^{i\Delta} \right)^2 \left( c^2 m_1 + s^2 m_2 e^{-i\Delta} \right) \left( s^2 m_1 + c^2 m_2 e^{-i\Delta} \right), \quad (A14)$$

$$|z - w| = m_3, \quad (A15)$$

$$(z - w) (z + w)^* = m_3 \left[ s^2 m_1 e^{2i(\beta_1 - \beta_3)} + c^2 m_2 e^{2i(\beta_2 - \beta_3)} \right]. \quad (A16)$$

In analogy to equation (A15) we also have, for $M_R$ instead of $M_{\nu}$,

$$|p - q| = M_3. \quad (A17)$$

**The parameters $a$ and $b$** We now express $|a|$ and $|b|$ as functions of the neutrino masses. Using equations (A7), (A15) and (A17) we obtain

$$|b|^2 = m_3 M_3. \quad (A18)$$

Comparing $\det M_{\nu}$ with $\det M_R$ we readily find that $|a|^2 |b|^4 = m_1 m_2 m_3 M_1 M_2 M_3$. Therefore, with equation (A18) we conclude that

$$|a|^2 = \frac{m_1 m_2 M_1 M_2}{m_3 M_3}. \quad (A19)$$

Moreover, equations (A3), (A15) and (A17) lead to

$$|d| = M_1 M_2, \quad (A20)$$

$$|f| = m_1 m_2. \quad (A21)$$

**The imaginary part of $(R_{12})^2$** From equation (A14) one derives

$$2 \text{Im} \left[ y^2 x^*(z + w)^* \right] = c^2 s^2 m_1 m_2 \left( m_2^2 - m_1^2 \right) \sin \Delta. \quad (A22)$$

Using the analogy between the matrices $M_{\nu}$ and $M_R$, and between $U$ and $V$, one sees that

$$2 \text{Im} \left[ n^2 m^*(p + q)^* \right] = c^2 s^2 M_1 M_2 \left( M_2^2 - M_1^2 \right) \sin (2\gamma_1 - 2\gamma_2). \quad (A23)$$

The matrix $R$ is given in equation (36). Using equation (A23) one sees that

$$\text{Im} \left[ (R_{12})^2 \right] = \left( |b|^2 - |a|^2 \right)^2 \frac{2 \text{Im} \left[ n^2 m^*(p + q)^* \right]}{M_1 M_2 (M_2^2 - M_1^2)}. \quad (A24)$$

Now, from equations (A1) to (A6) one finds the relation

$$y^2 x^*(z + w)^* = |a|^4 |b|^4 \frac{n^2 m^*(p + q)^*}{|d|^4}. \quad (A25)$$

Therefore, equations (A22) and (A24) give

$$\text{Im} \left[ (R_{12})^2 \right] = \left( |b|^2 - |a|^2 \right)^2 \frac{m_1 m_2 (m_2^2 - m_1^2)}{M_1 M_2 (M_2^2 - M_1^2)} \frac{|d|^4}{|a|^4 |b|^4} c^2 s^2 \sin \Delta \quad (A26)$$

$$= \left( |b|^2 - |a|^2 \right)^2 \frac{M_1 M_2 (m_2^2 - m_1^2)}{m_1 m_2 (M_2^2 - M_1^2)} c^2 s^2 \sin \Delta, \quad (A27)$$

where we have used equations (A18)–(A20) in order to go from equation (A26) to equation (A27).
The quadratic equation giving \(|a|\) and \(|b|\) Equations (A18) and (A19) contain \(m_3\) and \(M_3\). We shall now derive expressions for \(|a|\) and \(|b|\) which are functions solely of the observables in equation (41). With equations (43)–(45) it is easy to check that
\[
m_1^2 + m_2^2 = |x|^2 + 4|y|^2 + |z + w|^2. \tag{A28}
\]
For \(M_R\) instead of \(M_\nu\), the analogous relation is
\[
M_1^2 + M_2^2 = |m|^2 + 4|n|^2 + |p + q|^2. \tag{A29}
\]
Using equations (A9)–(A11) and (A21), we find
\[
m_1^2m_2^2(M_1^2 + M_2^2) = |a|^4|z + w|^2 + 4|a|^2|b|^2|y|^2 + |b|^4|x|^2. \tag{A30}
\]
Multiplying this equation by \(|x|^2|b|^4\) and using \(|a|^2|b|^2 = m_1m_2M_1M_2\) we finally obtain equations (46)–(48). The solutions to equation (48) are
\[
m_1^2m_2^2|p + q|^2 = |x|^2|b|^4 = \frac{-B \pm \sqrt{B^2 - 4C^2}}{2}, \tag{A31}
\]
where we have used equations (A11) and (A21). Since \(C = \left(|b|^4|x|^2\right)\left(|a|^4|z + w|^2\right)\), cf. equation (47), we also see that, together with equation (A31),
\[
m_1^2m_2^2|m|^2 = |z + w|^2|a|^4 = \frac{-B \mp \sqrt{B^2 - 4C^2}}{2}. \tag{A32}
\]
In order to determine \(\theta'\) one notes, from equations (43) and (45), that
\[
|z + w|^2 - |x|^2 = \left(c^2 - s^2\right)\left(m_2^2 - m_1^2\right). \tag{A33}
\]
The analogous relation for \(M_R\) is
\[
c^2 - s^2 = \frac{|p + q|^2 - |m|^2}{M_2^2 - M_1^2} = \pm \frac{\sqrt{B^2 - 4C}}{m_1^2m_2^2(M_2^2 - M_1^2)}. \tag{A34}
\]
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