Enhançon Solutions: Pushing Supergravity to its Limits

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Abstract

We extend the investigation of nonextremal enhançons, finding the most general solutions with the correct symmetry and charges. There are two families of solutions. One of these contains a solution with a regular horizon found previously; this previous example is shown to be the unique solution with a regular horizon. The other family generalises a previous nonextreme extension of the enhançon, producing solutions with shells which satisfy the weak energy condition. We argue that identifying a unique solution with a shell requires input beyond supergravity.


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1 Introduction

The enhançon mechanism [?] provides a very interesting novel example of singularity-resolution in string theory. Understanding the resolution of the singularity in the original supersymmetric solution of [?, ?] offers us an important insight into how string theory extends the notion of spacetime, and has been applied to obtain interesting physical results [?, ?, ?]. However, there are many questions concerning the nature of singularities in string theory which cannot be addressed in a supersymmetric context, so it is very important to try to extend any singularity resolution mechanism to address nonextremal, finite temperature geometries.

In [?, ?], it was found that there are nonextremal versions of the enhançon geometry, and it was noted that there are two different branches of solutions: the horizon branch, which always has a regular event horizon, and the shell branch, which always has an enhançon shell outside of the horizon (if any). The horizon branch approaches an uncharged black hole at large masses, so it is clearly physically relevant in this regime, but no solution on this branch exists for a finite range of masses above the BPS solution. Furthermore, the horizon branch solution does not exhibit the same physics as the extreme case, as it does not necessarily involve an enhançon shell. The shell branch, on the other hand, approaches the BPS solution as a parameter goes to zero, and always involves an enhançon shell. It thus represents a nonextremal generalisation of the singularity resolution in the BPS metric.

However, as shown in [?], this geometry is unphysical, as it violates the weak energy condition (WEC). Thus, to find a nonextremal generalisation of the enhançon, we must look for more general solutions. A further motivation for looking for more general solutions is the confusing two-branch structure in the existing solutions: near extremality, the only solution is the shell branch, which smoothly approaches the BPS solution of [?]. However, far from extremality, we would expect the horizon branch, which approaches an uncharged black hole solution for large masses, to be the correct solution. The transition between these two branches is an important unresolved problem (see [?, ?] for investigations of this issue).

In this paper, we will extend the investigation of nonextremal solutions in [?, ?], by finding the most general solution of the supergravity equations of motion with the correct symmetry and charges to correspond to a nonextremal enhançon solution. We will show that there are two families of asymptotically flat solutions, corresponding to extensions of the horizon branch and shell branch found previously. We demonstrate that the only solution with a regular event horizon is the horizon branch solution of [?]. Considering the shell branch, we show that the general family of solutions we have constructed satisfies the WEC for certain ranges of parameters. We then discuss the additional input that would be required to fix these parameters to obtain a physical solution describing a real nonextreme generalisation of the enhançon.

The general solution of the supergravity equations of motion is described in section 2. The physics of these solutions is then discussed in section 3. We conclude and discuss open issues in section 4.
## 2 Supergravity equations

Our aim is to extend previous studies of the extreme and nonextreme enhançon solutions, by finding the most general solutions of the supergravity equations consistent with the appropriate symmetries. In this section, we will write the metric in a convenient way, and reduce the supergravity equations of motion to a simple system of equations for the free functions in the metric.

We want to describe a system built up from excited D-branes wrapped on K3. As usual, we will focus mainly on the case of D6-branes, to simplify formulae. We describe the results of the analysis for wrapped D4- and D5-branes at the end of this section. For D6-branes, we should consider ten-dimensional metrics which are static and have two flat non-compact directions and a compact K3 factor along the branes. We assume that the metric is independent of the non-compact longitudinal directions, and that only the overall volume of the K3 varies over the transverse space. It is then natural to proceed by Kaluza-Klein reducing from ten to four dimensions.

In ten dimensions, we have the Type IIA 10D supergravity action (in string frame)

$$S_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G_{10}} \left( e^{-2\Phi_{10}} [R_{10} + 4(\partial \Phi_{10})^2] - \frac{1}{4} |F_{(2)}|^2 - \frac{1}{8} |F_{(6)}|^2 \right).$$  \(1\)

In Kaluza-Klein reducing, we write the ten-dimensional metric in an ansatz

$$dS_{10}^2 = dS_4^2 + e^B dx_2^2 + e^{D/\sqrt{2}} ds_{K3}^2,$$  \(2\)

where \(dx_2^2 = dx_1^2 + dx_2^2\) is a flat metric on the non-compact longitudinal directions, we assume that \(F_{(6)} = f_2 \wedge \epsilon_{K3}\), where \(\epsilon_{K3}\) is the volume form determined by the unit K3 metric \(ds_{K3}^2\), and we assume that \(f_2\) and \(F_{(2)}\) are non-zero only in the four dimensions contained in \(dS_4^2\). Then, following the classic technique of Maharana & Schwarz \([7]\), we can obtain an action for the four-dimensional fields,

$$S_4 = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-G_4} \left( e^{-2\phi_4} \left[ R_4 + 4(\partial \phi_4)^2 - \frac{1}{2}(\partial B)^2 - \frac{1}{2}(\partial D)^2 \right] \right)$$

$$- \frac{1}{4} e^{B+\sqrt{2}D} |F_2|^2 - \frac{1}{4} e^{B-\sqrt{2}D} |f_2|^2, \quad (3)$$

where the four-dimensional dilaton \(\phi_4 = \Phi_{10} - B/2 - D/\sqrt{2}\). We can convert this 4D action to Einstein frame by writing

$$g_{\mu\nu} = e^{-2\phi_4} G_{\mu\nu}. \quad (5)$$

The result is

$$S_{4E} = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g_4} \left[ R_{4E} - \frac{1}{2}(\partial \Phi_4)^2 - \frac{1}{2}(\partial B)^2 - \frac{1}{2}(\partial D)^2 \right]$$

$$- \frac{1}{4} e^{B+\sqrt{2}D} |F_2|^2 - \frac{1}{4} e^{B-\sqrt{2}D} |f_2|^2, \quad (6)$$

where we have defined \(\Phi_4 = 2\phi_4\) to obtain canonically normalised kinetic terms. Henceforth, we will work in Einstein frame for the 4D metric.
This process of Kaluza-Klein reduction has already led to one striking simplification: the dilaton is completely decoupled,

$$\nabla^2 \Phi_4 = 0.$$  \hfill (8)

The other two scalars have slightly more complicated behaviour:

$$\nabla^2 B = 4e^B [F_2^2 e^{\sqrt{2}D} + |f_2|^2 e^{-\sqrt{2}D}],$$  \hfill (9)

$$\nabla^2 D = \frac{1}{\sqrt{2}} e^B [F_2^2 e^{\sqrt{2}D} - |f_2|^2 e^{-\sqrt{2}D}].$$  \hfill (10)

The equations of motion for the gauge fields take the usual form,

$$\nabla_\mu (e^B + \sqrt{2}D F^{\mu \nu}) = 0, \quad \nabla_\mu (e^B - \sqrt{2}D f^{\mu \nu}) = 0.$$  \hfill (11)

We now wish to specify our ansatz for the four-dimensional metric. We assume that the metric is spherically symmetric in the three-dimensional space transverse to the branes, so the metric and scalar fields will only depend on the radial coordinate $r$ in the transverse space. Thus, we take the metric ansatz

$$ds^2_4 = -e^{2A(r)} dt^2 + e^{2C(r)} \big( dr^2 + r^2 d\Omega_2^2 \big),$$  \hfill (12)

choosing an isotropic gauge for the radial coordinate. Since we wish to consider a system of D6-branes, which are magnetically charged under $F_{(2)}$, and carry an induced D2-brane charge, which is a magnetic charge under $F_{(6)}$, we take the ansatze for the field strengths to be

$$F_2 = Q_2 \epsilon_{S^2}, \quad f_2 = q_2 \epsilon_{S^2},$$  \hfill (13)

where $\epsilon_{S^2}$ is the volume form corresponding to the unit sphere metric $d\Omega_2^2$. As the D2-brane charge arises from a curvature coupling of the D6-branes wrapped on K3, it is related to the D6-brane charge through $|q_2| = (V_s/V)|Q_2|$ \cite{?}. These ansatze satisfy the gauge field equations of motion (11).

With this ansatz, the Einstein equations for the four-dimensional metric reduce to (where $'$ denotes $\partial_r$)

$$2C'' + (C')^2 + \frac{4}{r} C' = -\frac{1}{4} \left( (\Phi_4')^2 + (B')^2 + (D')^2 \right)$$  \hfill (14)

$$-\frac{1}{4} e^{B-2C} \left( e^{\sqrt{2}D} Q_2^2 e^{-\sqrt{2}D} q_2^2 \right),$$

$$2A'C' = \frac{1}{4} \left( (\Phi_4')^2 + (B')^2 + (D')^2 \right)$$  \hfill (15)

$$-\frac{1}{4} e^{B-2C} \left( e^{\sqrt{2}D} Q_2^2 e^{-\sqrt{2}D} q_2^2 \right),$$

$$A'' + C'' + (A')^2 + \frac{1}{r} (A' + C') = -\frac{1}{4} \left( (\Phi_4')^2 + (B')^2 + (D')^2 \right)$$  \hfill (16)

$$+\frac{1}{4} e^{B-2C} \left( e^{\sqrt{2}D} Q_2^2 e^{-\sqrt{2}D} q_2^2 \right),$$
and the scalar equations become

$$\Phi''_4 + \Phi'_4 \left(\frac{2}{r} + A' + C'\right) = 0, \quad (17)$$

$$B'' + B'(\frac{2}{r} + A' + C') = \frac{1}{2} \frac{e^{B-2C}}{r^4} (e^{\sqrt{2}D} Q^2 + e^{-\sqrt{2}D} q^2), \quad (18)$$

and

$$D'' + D'(\frac{2}{r} + A' + C') = \frac{1}{\sqrt{2}} \frac{e^{B-2C}}{r^4} (e^{\sqrt{2}D} Q^2 - e^{-\sqrt{2}D} q^2). \quad (19)$$

We have reduced the problem of finding the general solution subject to the assumed symmetries to solving this system of equations for the five unknown functions $A, B, C, D, \Phi_4$. This seems like a complicated coupled system of equations, but in fact it conceals some remarkable simplifications. If we introduce new functions

$$a(r) = A + C,$$\hspace{1cm} (15)

$$c(r) = C + \frac{B}{2},$$\hspace{1cm} (16)

then (14)+(15)+(18) gives

$$a'' + (a')^2 + \frac{3}{r} a' = 0, \quad (20)$$

a completely decoupled equation for $a$. Similarly, (14)+(15)+(18) gives

$$c'' + c' \left[\frac{2}{r} + a'\right] + \frac{1}{r} a' = 0, \quad (21)$$

which can be rearranged to write

$$[c' r^2 e^a]' = -re^a a'. \quad (22)$$

Similarly, (17) can be rewritten as

$$[\Phi'_4 r^2 e^a]' = 0. \quad (23)$$

These equations are solvable once we know $a$. Furthermore, if we define $x_6 = -B - D/\sqrt{2}$ and $x_2 = -B + D/\sqrt{2}$, then $-2(18) - \sqrt{2}(19)$ becomes

$$[x_6' r^2 e^a]' = -\frac{e^{a-2c}}{r^2} Q^2 e^{-2x_6}. \quad (24)$$

We choose to rewrite this as

$$r^2 e^a [x_6' r^2 e^a]' = -e^{2(a-c)} Q^2 e^{-2x_6}. \quad (25)$$

Similarly, $-2(18) + \sqrt{2}(19)$ can be rewritten as

$$r^2 e^a [x_2' r^2 e^a]' = -e^{2(a-c)} q^2 e^{-2x_2}. \quad (26)$$

We now have a much simplified system of equations in terms of the functions $a, c, x_2, x_6, \Phi_4$. Before proceeding to solve these equations, let us express our ansatz for the ten-dimensional fields in terms of these variables for future reference:

$$dS^{10}_{10} = -e^{\Phi_4 + 2(a-c)} e^{-\frac{x_6}{2} - \frac{x_2}{2}} dt^2 + e^{-\frac{x_6}{2} - \frac{x_2}{2}} dx_2^2 + e^{\Phi_4 + 2c} e^{\frac{x_6}{2} + \frac{x_2}{2}} (dr^2 + r^2 d\Omega_2^2) + e^{x_2 - \frac{x_6}{2}} ds_{K3}. \quad (27)$$
with ten-dimensional dilaton

\[ \Phi_{10} = \frac{\Phi_4}{2} + \frac{x_2}{4} - \frac{3x_6}{4} \]  

and gauge fields

\[ F_{(2)} = Q_2 \epsilon S^2, \quad F_{(6)} = q_2 \epsilon S^2 \wedge \epsilon K^3. \]  

Note the familiar way in which the functions \( x_2, x_6 \) appear in the metric and dilaton.

### 2.1 General solutions of the field equations

We now proceed to solve the equations. The solution of (20) is

\[ a = \ln \left( 1 - \frac{r_h^2}{r^2} \right) + C_1. \]  

Then \( r^2 e^a = (r^2 - r_h^2) e^{C_1} \), and we can easily see that the solution of (23) is

\[ \Phi_4 = A_1 \ln \left( \frac{r + r_h}{r - r_h} \right) + C_2, \]  

and (22) is solved by

\[ c = 2 \ln \left( 1 + \frac{r_h}{r} \right) + A_2 \ln \left( \frac{r + r_h}{r - r_h} \right) + C_3. \]  

Then

\[ e^{2(a-c)} = \frac{r^4}{(r + r_h)^4} \left( \frac{r - r_h}{r + r_h} \right)^{2A_2} e^{-2C_3} \left( \frac{r^2 - r_h^2}{r^2} \right)^2 e^{2C_1} = \left( \frac{r - r_h}{r + r_h} \right)^{2(A_2+1)} e^{2(C_1-C_3)}. \]  

Plugging this into (25) gives

\[ (r^2 - r_h^2) \partial_{r}( (r^2 - r_h^2) \partial_{r} x_6 ) e^{2x_6} = -Q_2^2 e^{-2C_3} \left( \frac{r - r_h}{r + r_h} \right)^{2(A_2+1)} , \]  

and similarly

\[ (r^2 - r_h^2) \partial_{r}( (r^2 - r_h^2) \partial_{r} x_2 ) e^{2x_2} = -q_2^2 e^{-2C_3} \left( \frac{r - r_h}{r + r_h} \right)^{2(A_2+1)}. \]  

These are non-linear equations, but nonetheless they have a closed-form solution. To solve them, it is convenient to introduce a new independent variable,

\[ z = \ln \left( \frac{r - r_h}{r + r_h} \right) , \]  

so that these equations become

\[ \partial_z^2 x_6 e^{2x_6} = -\frac{Q_2^2 e^{-2C_3}}{4r_h^2} e^{2(A_2+1)z} , \]  

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\[ \partial^2_z x^2 e^{2x} = -\frac{q^2 e^{-2C_3}}{4 r_h^2} e^{2(2 z + a_1 z)}. \] (38)

The general solutions of these equations is

\[ x_6 = \ln \left( \alpha - \frac{Q^2 e^{-2C_3}}{16 r_h^2 (A_2 + \gamma + 1)^2} e^{2(A_2 + \gamma + 1) z} \right) - \gamma z, \] (39)

\[ x_2 = \ln \left( \beta - \frac{q^2 e^{-2C_3}}{16 r_h^2 (A_2 + \kappa + 1)^2} e^{2(A_2 + \kappa + 1) z} \right) - \kappa z. \] (40)

### 2.2 Other cases

We can carry out a similar analysis for the cases of D4-branes wrapped on K3 in IIA and D5-branes wrapped on K3 in IIB. We will just briefly state the results, pointing out a few minor differences relative to the D6-brane case discussed in detail above.

For the D4-branes, we write the ten-dimensional string frame metric in the form

\[ dS^2_{10} = -e^{2a - 6c} e^{-\frac{x_0}{2}} dt^2 + e^{2c} e^{-\frac{x_0}{2}} (dr^2 + r^2 d\Omega^2_4) + e^{\frac{x_0}{2} - \frac{x_4}{2}} ds^2_{K3}, \] (41)

and write the ten-dimensional dilaton as

\[ \Phi_{10} = -\frac{x_4}{4} + \frac{3x_0}{4}, \] (42)

and gauge fields as

\[ F_{(4)} = Q_4 \epsilon_{S^4}, \quad F_{(8)} = q_4 \epsilon_{S^4} \wedge \epsilon_{K3}. \] (43)

We then obtain simple equations for the functions \( a, c, x_4, x_0 \), as in the previous case. Note that the absence of any unwrapped directions along the brane implies that there is one less scalar field in the dimensional reduction here; it is the decoupled scalar that we lose.

The general solution is

\[ a(r) = \ln \left( 1 - \frac{r_h^6}{r^6} \right) + C_1, \] (44)

\[ c(r) = \frac{1}{3} \left[ 2 \ln \left( 1 + \frac{r_h^3}{r^3} \right) + A_1 \ln \left( \frac{r^3 + r_h^3}{r^3 - r_h^3} \right) + C_2 \right], \] (45)

\[ x_4(r) = \ln \left( \alpha - \frac{Q^2 e^{-2C_2}}{144 r_h^6 (A_1 + \gamma + 1)^2} e^{2(A_1 + \gamma + 1) z} \right) - \gamma z, \] (46)

\[ x_0(r) = \ln \left( \beta - \frac{q^2 e^{-2C_2}}{144 r_h^6 (A_1 + \kappa + 1)^2} e^{2(A_1 + \kappa + 1) z} \right) - \kappa z, \] (47)

where

\[ z = \ln \left( \frac{r^3 - r_h^3}{r^3 + r_h^3} \right). \] (48)
For the case of D5-branes in type IIB, we write the ten-dimensional string frame metric in the form
\[
\begin{align*}
\frac{dS_{10}^2}{ds^2} &= -e^{2\varphi + 2a - 4c}e^{-\frac{x_5^2}{2}}dt^2 + e^{-\frac{x_1^2}{2}}dx^2 + e^{2\varphi + 2c}e^{\frac{x_5^2 + x_1^2}{2}}(dr^2 + r^2 d\Omega^2_3) \\
&+ e^{\frac{x_5^2 + x_1^2}{2}}ds_{K3}^2,
\end{align*}
\]
where \(x\) is the single unwrapped brane direction, and write the ten-dimensional dilaton as
\[
\Phi_{10} = \frac{3}{2} \varphi - \frac{x_5^2}{2} + \frac{x_1^2}{2}
\]
and gauge fields as
\[
F_{(3)} = Q_3 \epsilon_{S^3}, \quad F_{(7)} = q_3 \epsilon_{S^3} \wedge \epsilon_{K3}.
\]
We then obtain simple equations for the functions \(a, c, \varphi, x_5, x_1\). In this case, the combination \(\varphi\) which decouples is not the same as the five-dimensional dilaton.

The general solution is
\[
\begin{align*}
a(r) &= \ln \left(1 - \frac{r_h^4}{r^4}\right) + C_1, \\
\varphi(r) &= A_1 \ln \left(\frac{r^2 + r_h^2}{r^2 - r_h^2}\right) + C_2, \\
c(r) &= \frac{1}{2} \left[2 \ln \left(1 + \frac{r_h^2}{r^2}\right) + A_2 \ln \left(\frac{r^2 + r_h^2}{r^2 - r_h^2}\right) + C_3\right], \\
x_5(z) &= \ln \left(\alpha - \frac{Q_3^2 e^{-2C_3-C_2}}{64 r_h^4 (A_2 + \frac{1}{2} A_1 + \gamma + 1)^2 \alpha} e^{2(A_2 + \frac{1}{2} A_1 + \gamma + 1) z}\right) - \gamma z, \\
x_1(z) &= \ln \left(\beta - \frac{q_3^2 e^{-2C_3-C_2}}{64 r_h^4 (A_2 + \frac{1}{2} A_1 + \kappa + 1)^2 \beta} e^{2(A_2 + \frac{1}{2} A_1 + \kappa + 1) z}\right) - \kappa z,
\end{align*}
\]
where
\[
z = \ln \left(\frac{r^2 - r_h^2}{r^2 + r_h^2}\right).
\]
We see that the solutions obtained in both these cases are very similar in form to the case of D6-branes.

### 3 New enhançons?

In the last section, we found the general solution of the supergravity equations of motion subject to the symmetries associated with an enhançon-like solution. The solution has a simple closed form. It generalises the known solutions, introducing a number of constants of integration. We would now like to see if this leads to any new physical enhançon solutions.\(^1\) We will just discuss the D6-brane case; the other cases will clearly be very similar.

\(^1\)Note that we have not introduced any enhançon shells, so at this stage we are really looking for more general analogues of the repulson solution—that is, what we are discussing is the solution exterior to any enhançon shell.
We first need to impose the condition of asymptotic flatness, which will fix some of the constants. To impose asymptotic flatness, we require that all the functions fall off as $1/r$ at large $r$. In the case of $\Phi_4$, this corresponds to a choice of gauge, defining the ten-dimensional dilaton so that $\Phi_{10}(\infty) = 0$. Examining (30,31,32), we see that this fixes $C_1 = C_2 = C_3 = 0$. From (39,40), we obtain non-trivial equations for $\alpha$ and $\beta$,

$$\alpha - \frac{Q_2^2}{16r_h^2(A_2 + \gamma + 1)^2\alpha} = 1,$$

$$\beta - \frac{q_2^2}{16r_h^2(A_2 + \kappa + 1)^2\beta} = 1,$$

with solutions

$$\alpha = \frac{1}{2}(1 \pm \sqrt{1 + \frac{Q_2^2}{4r_h^2(A_2 + \gamma + 1)^2}}), \quad \beta = \frac{1}{2}(1 \pm \sqrt{1 + \frac{q_2^2}{4r_h^2(A_2 + \kappa + 1)^2}}).$$

It turns out to be convenient to rewrite these as

$$\alpha = \frac{1}{4r_h(A_2 + \gamma + 1)} \left(2r_h(A_2 + \gamma + 1) \pm \sqrt{Q_2^2 + 4r_h^2(A_2 + \gamma + 1)^2}\right),$$

$$\beta = \frac{1}{4r_h(A_2 + \kappa + 1)} \left(2r_h(A_2 + \kappa + 1) \pm \sqrt{q_2^2 + 4r_h^2(A_2 + \kappa + 1)^2}\right).$$

Thus, the most general asymptotically flat solution is

$$a = \ln \left(1 - \frac{r_h^2}{r^2}\right),$$

$$\Phi_4 = A_1 \ln \left(\frac{r + r_h}{r - r_h}\right),$$

$$c = 2 \ln \left(1 + \frac{r_h}{r}\right) + A_2 \ln \left(\frac{r + r_h}{r - r_h}\right),$$

$$x_6 = \ln \left(\alpha - (\alpha - 1) \left(\frac{r + r_h}{r - r_h}\right)^{-2(A_2 + \gamma + 1)}\right) + \gamma \ln \left(\frac{r + r_h}{r - r_h}\right),$$

$$x_2 = \ln \left(\beta - (\beta - 1) \left(\frac{r + r_h}{r - r_h}\right)^{-2(A_2 + \kappa + 1)}\right) + \kappa \ln \left(\frac{r + r_h}{r - r_h}\right),$$

with $\alpha$ and $\beta$ given by (61,62).

To begin to analyse the physics of these solutions, we note that there are two kinds of potential singularities in the solution (63-67). There is a singularity at $r = r_h$, where $a \to -\infty$, and other functions may diverge. Since $a \to -\infty$ gives $g_{00} \to 0$ in (27), this singularity could correspond to an event horizon, if we choose other constants of integration appropriately. However, there is another possible singularity; if we choose the lower sign in either (61) or (62), there will be a singularity in (66) or (67) respectively at some $r > r_h$. This type of singularity is the analogue of the repulson singularity in the original enhançon story [?]. We see that, as in the
discussion of nonextreme enhançons in [?], it arises from a discrete choice: there are

different branches of solutions. Henceforth, we will assume that we take the positive

sign in (61), and we will refer to the solution where we take the positive sign in (62)

as the horizon branch, and to the solution where we take the negative sign in (62) as

the shell branch. The shell branch solutions will only be valid outside of an enhançon

shell.\(^2\)

3.1 Uniqueness of the horizon branch

Addressing first the horizon branch, we will see that the only solution where the

coordinate singularity at \(r = r_h\) is a regular event horizon is the horizon branch

solution found previously in [?]. For \(r = r_h\) to be a regular horizon, we clearly need

the ten-dimensional dilaton \(\Phi_{10}\) to remain finite at \(r = r_h\). We should also require that

the volume of the two-sphere and K3 components of the metric remain finite there,

to avoid any diverging curvature invariants. Furthermore, we must require that the

factor in front of the \(dx^\parallel^2\) directions remain finite: as argued in [?], a divergence of

such a component may not lead to diverging curvature invariants, but it does cause

a divergence in components of the curvature in a suitable orthonormal frame. Taken

together, these conditions require that \(c, \Phi_4, x_2\) and \(x_6\) are finite at \(r = r_h\). That is,

they impose \(A_1 = A_2 = \gamma = \kappa = 0\).

Thus, we have a unique solution with a regular horizon. It has

\[
a = \ln \left(1 - \frac{r_h^2}{r^2}\right), \quad \Phi_4 = 0, \quad c = 2 \ln \left(1 + \frac{r_h}{r}\right),
\]

\[
x_6 = \ln \left(\alpha - (\alpha - 1) \left(\frac{r + r_h}{r - r_h}\right)^{-2}\right) = \ln \left(\frac{r^2 + (Q_2^2 + 4r_h^2)^{1/2}r + r_h^2}{(r + r_h)^2}\right),
\]

\[
x_2 = \ln \left(\beta - (\beta - 1) \left(\frac{r + r_h}{r - r_h}\right)^{-2}\right) = \ln \left(\frac{r^2 + (q_2^2 + 4r_h^2)^{1/2}r + r_h^2}{(r + r_h)^2}\right),
\]

where in the above we have used the values of \(\alpha, \beta\) from (61,62), taking the positive

sign in both equations. Using (27), this can be easily shown to be identical to the

horizon branch solution in [?] written in isotropic coordinates.

Thus, we find that the unique solution consistent with the symmetries we expect

the enhançon to have possessing a regular event horizon is the horizon-branch solution

found before. This is perhaps not a surprising result, but it is quite satisfying to be

able to extend the analysis of a particular ansatz undertaken in [?] to a consideration

of the most general form of nonextreme enhançon metric.

\(^2\)Solutions on the horizon branch do not have a repulson singularity, but they may nonetheless

have a non-trivial enhançon shell appearing in them, if the K3 volume in (27) reaches string-scale

outside the horizon (see [?] for details). We will ignore this issue in what follows; similar general

remarks to those we make for the nonextremal solutions on the shell branch will apply in this case.
3.2 Shell branch: Extremal solutions

We turn now to a discussion of the shell branch. As usual in discussions of the enhançon mechanism, it is useful to first consider the extreme case, and then extend this to nonextreme solutions. Let us therefore consider what happens to the general solution \(63-67\) if we take \(r_h = 0\).

This will depend on how we take the limit. If we take \(r_h \to 0\) with \(A_1, A_2, \kappa, \gamma\) held fixed, then we recover the usual extremal solution. We will get \(a = \Phi_4 = c = 0\),

\[
\alpha \approx \frac{|Q_2|}{4r_h(A_2 + \gamma + 1)}, \quad \beta \approx \frac{-|q_2|}{4r_h(A_2 + \kappa + 1)} \tag{71}
\]

(recalling that we are considering the shell branch, so we take the negative sign in (62)), which gives

\[
x_6 \approx \ln \left(1 + \alpha \frac{4(A_2 + \gamma + 1)r_h}{r}\right) \approx \ln \left(1 + \frac{|Q_2|}{r}\right), \tag{72}
\]

\[
x_2 \approx \ln \left(1 + \beta \frac{4(A_2 + \kappa + 1)r_h}{r}\right) \approx \ln \left(1 - \frac{|q_2|}{r}\right), \tag{73}
\]

which gives us the exterior metric of the BPS enhançon solution of [?].

On the other hand, we could take the limit \(r_h \to 0\) with \(\tilde{A}_1 = A_1 r_h\) etc held fixed, which will give a more general extremal solution. This still has \(a = 0\), but now

\[
c = \frac{2\tilde{A}_2}{r}, \tag{74}
\]

\[
\Phi_4 = \frac{2\tilde{A}_1}{r}, \tag{75}
\]

and

\[
x_6 = \ln \left(\alpha - (\alpha - 1)e^{-\frac{4(\tilde{A}_2 + \gamma)}{r}}\right) + \frac{2\tilde{\gamma}}{r}, \tag{76}
\]

\[
x_2 = \ln \left(\beta - (\beta - 1)e^{-\frac{4(\tilde{A}_2 + \kappa)}{r}}\right) + \frac{2\tilde{\kappa}}{r}. \tag{77}
\]

In this limit, (61,62) become

\[
\alpha = \frac{1}{4(\tilde{A}_2 + \tilde{\gamma})} \left(2(\tilde{A}_2 + \tilde{\gamma}) + \sqrt{Q_2^2 + 4(\tilde{A}_2 + \tilde{\gamma})^2}\right), \tag{78}
\]

\[
\beta = \frac{1}{4(\tilde{A}_2 + \tilde{\kappa})} \left(2(\tilde{A}_2 + \tilde{\kappa}) - \sqrt{q_2^2 + 4(\tilde{A}_2 + \tilde{\kappa})^2}\right). \tag{79}
\]

These additional solutions look similar to the exterior solution in the familiar BPS enhançon to some extent; they have a singularity at some \(r > 0\), where \(x_2 \to -\infty\), implying that the volume of the K3 goes to zero. We wish to ask if we can build a physical solution where this singularity is resolved. To resolve the singularity, we need to be able to consistently excise the region inside the radius where the K3 volume
reaches the self-dual point with flat space by introducing a shell of branes at this radius.

If we consider the junction between this solution and flat space, we can define the shell stress tensor in terms of the discontinuity in the extrinsic curvature [?, ?],

\[ S_{AB} \equiv \frac{1}{\kappa^2} (\gamma_{AB} - G_{AB} \gamma_{C}^C) \]  

(80)

where \( \gamma_{AB} \equiv K_{AB}^+ + K_{AB}^- \) is the jump in the extrinsic curvature

\[ K_{AB}^\pm = \pm \frac{1}{2 \sqrt{G_{rr}}} \frac{\partial}{\partial r} (G_{AB}). \]  

(81)

Assuming the interior metric is flat, \( K_{AB}^- = 0 \), so \( \gamma_{AB} = K_{AB}^+ \). The components of the stress tensor for a general metric of the form (27) are then

\[ S_{tt} = \frac{1}{\kappa^2 \sqrt{G_{rr}}} (4c' + x_2' + x_6') G_{tt}, \]  

(82)

\[ S_{\mu\nu} = \frac{1}{\kappa^2 \sqrt{G_{rr}}} (2a' + 2c' + \Phi'_4 + x_2' + x_6') G_{\mu\nu}, \]  

(83)

\[ S_{ij} = \frac{1}{\kappa^2 \sqrt{G_{rr}}} 2a' G_{ij}, \]  

(84)

\[ S_{ab} = \frac{1}{\kappa^2 \sqrt{G_{rr}}} (2a' + 2c' + \Phi'_4 + x_6') G_{ab}, \]  

(85)

where indices \( \mu, \nu \) run over the non-compact longitudinal directions, \( i, j \) run over the \( S^2 \) directions, and \( a, b \) run over the K3 directions. We thus see that \( S_{ij} = 0 \) for any solution with \( r_h = 0 \), as we would expect for an extremal solution.

Since the stress tensor in the sphere directions vanishes, it is natural to see what happens if we try to model the source for this shell by a collection of fundamental branes, generalising the BPS enhançon solution. The DBI action for wrapped D6-branes is

\[ S = - \int_{\mathcal{M}_2} d^3 \xi \, e^{-\Phi_4} (\mu_6 V(r) - \mu_2) (- \det G_{\mu\nu})^{1/2} \]  

(86)

where \( \mathcal{M}_2 \) is the unwrapped part of the worldvolume, which lies in six non-compact dimensions, \( V(r) \) is the running volume of the K3, and \( G_{\mu\nu} \) is the induced (string frame) metric. Plugging in the metric (27), we obtain

\[ S = - \int d^3 \xi e^{a-c} (\mu_6 e^{-x_6} - \mu_2 e^{-x_2}). \]  

(87)

Since the action does not couple to the 4d dilaton \( \Phi_4 \), it cannot source a discontinuity in this field; thus, we must set \( \bar{A}_1 = 0 \). The action has a Lorentz symmetry relating the time direction and the non-compact spatial directions; we can therefore only use it as the source if the shell stress tensor also respects this symmetry, which forces us to set \( \bar{A}_2 = 0 \). We are then just left with the terms coming from \( x_2' \) and \( x_6' \) in the stress-energy. If these are to be sourced by the brane action, these functions
need to satisfy $x'_2 e^{x_2} = \text{constant}$, $x'_6 e^{x_6} = \text{constant}$. These constraints force us to set $\tilde{\gamma} = \tilde{\kappa} = 0$. This gives us back the usual BPS enhançon solution.

Thus, while we have found additional solutions with $r_h = 0$, these are not physical extreme enhançon solutions, in the sense that they do not correspond to the geometry sourced by a collection of BPS branes. Requiring that the shell stress tensor have the appropriate form to correspond to the brane sources completely fixes the constants of integration in the solution. That is, in the extreme case at least, our usual no-hair intuition continues to hold. The additional parameters do not actually correspond to a family of generalised physical solutions; the only truly physical solution is the usual one.

In passing, it is interesting to note the effect of the deformations in the more general solution on the asymptotics of the solution—in particular, on the ADM mass. If we just consider turning the $\tilde{\kappa}$ parameter on slightly, modifying the behaviour of $x_2$, its asymptotics will be

$$e^{x_2} \approx \left(1 + \frac{4(\beta - 1)\tilde{\kappa}}{r}\right) \left(1 + \frac{2\tilde{\kappa}}{r}\right). \quad (88)$$

Assuming $\tilde{\kappa} \ll q_2^2$,

$$\beta \approx -\frac{|q_2|}{4\kappa} \left(1 - 2\frac{\tilde{\kappa}}{|q_2|}\right), \quad (89)$$

so

$$e^{x_2} \approx 1 - \frac{|q_2|}{r} + \frac{4\tilde{\kappa}}{r}. \quad (90)$$

The effect of this will be that positive values of $\tilde{\kappa}$ increase the ADM mass. This teaches us two things: first, the solutions with $\tilde{\kappa} \neq 0$ are clearly not supersymmetric, since they do not saturate the BPS bound. Second, this suggests a potentially useful way to correct the problem with the WEC in the nonextreme case.

### 3.3 Shell branch: Nonextremal solutions

Let us now consider the nonextremal shell branch, where we take $r_h \neq 0$. We have the freedom to consider any solution in the general family (63-67). However, in this section, we will focus just on the effects of turning on the parameter $\kappa$ which modifies the behaviour of $x_2$. The philosophy underlying this approach is that we need to focus on a subset of the possible deformations to keep the formulae arising in the discussion of manageable complexity, and this seems to be the most natural deformation to consider, since it is $x_2$ which already has ‘unusual’ behaviour in any shell branch solution. We will show that turning on this deformation is sufficient to produce solutions which do not violate the WEC.

Let us first review the argument that the WEC condition is violated in the usual nonextremal shell branch solution [?]. The nonextremal solution of [?] is the special case of our general asymptotically flat solution (63-67) where $A_1 = A_2 = \gamma = \kappa = 0$, and we take the negative sign in (62). This metric then has a repulson singularity at some $r = r_r$, where $x_2 \to -\infty$. As in the extremal case, the shell branch solution can
apply only outside of some enhançon shell, located at the radius where the volume of the K3 reaches the self-dual point, \( V = V_\ast = (2\pi\sqrt{\alpha'})^4. \) From the metric (27) we see that the enhançon radius is given by

\[
e^{x_2-x_6} = \frac{V_\ast}{V}.
\]  

(91)

For the nonextreme shell branch solution of (\[?\]) in our coordinates, this becomes

\[
\frac{(\beta - (\beta - 1)e^{2z})}{(\alpha - (\alpha - 1)e^{2z})} = \frac{V_\ast}{V}.
\]  

(92)

We assume that we excise the portion of the solution inside this radius and replace it with either flat space or a horizon branch solution. There is then a discontinuity at this radius, corresponding to a shell whose stress tensor is calculated as in the extremal case in the previous subsection. Assuming the interior solution is still flat (which maximises the shell’s contribution to the overall ADM mass), we see from (82) that the shell energy density is

\[
\rho \propto -x'_2 - x'_6 - 4c'.
\]  

(93)

The \( x'_6 \) and \( c' \) terms make positive contributions to the energy density. However, the choice of the negative sign in (62) implies that \( \beta < 0 \), and as a consequence the first term is negative;

\[
x'_2 = -\frac{2(1 - \beta)e^{2z}}{(\beta - (\beta - 1)e^{-2z})}\partial_r z < 0. 
\]  

(94)

To see that this negative term dominates, we first write the first two terms together, using (92),

\[
x'_2 - x'_6 = -\frac{2(1 - \beta)e^{2z_0}}{(\beta - (\beta - 1)e^{2z_0})}\left(1 - \frac{(\alpha - 1)\sqrt{V_\ast}}{(1 - \beta)\sqrt{V}}\right)\partial_r z.
\]  

(95)

This expression is valid only at the enhançon radius \( z = z_e \), where (92) is satisfied. Now

\[
\frac{(\alpha - 1)}{(1 - \beta)} = \frac{\sqrt{Q^2_2 + 4r^2_h - 2r_h}}{\sqrt{Q^2_2 + 4r^2_h + 2r_h}} \leq \frac{V}{V_\ast},
\]  

(96)

since \( |Q_2|/|q_2| = V/V_\ast \). Thus, the first two terms together give a negative answer. Furthermore, for this supergravity analysis to be relevant, we need to assume that \( V_\ast \gg V \), so that higher-order corrections involving the K3 curvature are suppressed. This implies by (92) that \( (\beta - (\beta - 1)e^{2z}) \ll 1 \), so these terms will dominate over the remaining positive term, \(-4c' \propto \frac{8\pi_3}{\sqrt{r_{d_1}(r_{d_2} + r_{d_3})}}\). Thus, \( \rho < 0 \), and the shell violates the WEC. The usual nonextreme enhançon solution thus cannot correspond to the geometry sourced by a physical collection of branes.

A primary motivation for looking for more general solutions was to see how general this problem is. We will now show that we can produce solutions where the shell
satisfies the WEC by generalising to non-zero values of $\kappa$. First, we note that changing $\kappa$ will change the enhançon radius; (91) now implies
\[
\frac{(\beta - (\beta - 1)e^{2(\kappa+1)z})}{(\alpha - (\alpha - 1)e^{2z})} e^{-\kappa z} = \frac{V^*}{V}.
\] (97)
The first two terms in the energy density are then
\[
-x'_2 - x'_6 = \left[2(\kappa + 1)(\beta - (\beta - 1)e^{2(\kappa+1)z}) + \kappa + \frac{2(\alpha - 1)e^{2z}}{(\alpha - (\alpha - 1)e^{2z})}\right] \partial_r z.
\] (98)
Using (97), we can rewrite this as
\[
-x'_2 - x'_6 = -2(\kappa + 1)(1 - \beta)e^{2(\kappa+1)z} \left[1 - \frac{(\alpha - 1)}{(\kappa + 1)(1 - \beta) V^* e^{-\kappa z}}\right] \partial_r z + \kappa \partial_r z.
\] (99)
In this generalisation, it is still true that
\[
\frac{(\alpha - 1)}{(\kappa + 1)(1 - \beta)} = \frac{\sqrt{Q_2^2 + 4r_h^2 - 2r_h}}{\sqrt{q^2_2 + 4(\kappa + 1)^2r_h^2 + 2(\kappa + 1)r_h}} < \frac{V}{V^*}.
\] (100)
However, this does not imply that the factor in square brackets in (98) is positive. For positive $\kappa$, the factor of $e^{-\kappa z} > 1$, and it can easily be made sufficiently large to make this factor negative, at least for small values of $r_h$. Note also that the additional $\kappa \partial_r z$ term is also acting in the same direction for positive $\kappa$. Thus, the contribution of the $x'_6$ term can dominate over that of the $x'_2$ term for suitable values of $\kappa$, leading to a shell stress energy which satisfies the WEC.\(^3\)

However, we still have the problem that the solution depends on constants of integration, which seem to represent an unphysical freedom to modify the geometry. Simply imposing the WEC cannot completely fix the constants of integration in the solution. These parameters are best thought of as parameterising the shell stress tensor, and are not wholly fixed at the supergravity level, because supergravity on its own cannot completely determine the shell stress tensor. At the fundamental level, there should be a definite form for this stress tensor, which will fix these parameters (possibly up to some discrete choices). However, this will require some input from physics beyond supergravity, which provides a real microphysical model for the shell stress tensor, as the DBI action did in the BPS case.

Thus, we have a complete description of the solutions at the supergravity level which satisfy the appropriate symmetry assumptions, and we can see that some of them will satisfy the WEC, which is our primary physics constraint on them at this level. However, since we do not have a microphysical model for the shells in the nonextremal cases, we cannot determine which (if any) of this family of solutions actually correspond to physical nonextreme generalisations of the enhançon mechanism.

\(^3\)This seems a natural way to modify the solution to satisfy the WEC; however, other possibilities certainly exist. For example, turning on a positive $\gamma$ will modify the stress-energy in a very similar way, and can also lead to solutions which satisfy the WEC.
4 Conclusions

We have been studying the extension of the enhançon mechanism [?] to nonextremal, finite temperature geometries. In [?, ?], it was found that there are nonextremal versions of the enhançon geometry, and it was noted that there are two different branches of solutions: the horizon branch, which always has a regular event horizon, and the shell branch, which always has an enhançon shell outside of the horizon (if any).

In this paper, we have extended the work of [?] by finding the most general solution consistent with the symmetries and charges associated with the enhançon. These solutions represent generalisations of the exterior geometry in the enhançon solution. One of the constants of integration, \( r_h \), can be interpreted as a nonextremality parameter, so these are generally nonextremal solutions. We find that the branch structure noted in [?] arises when we impose asymptotic flatness: this results in a quadratic equation for one of the constants of integration, with the two roots corresponding to the horizon branch and the shell branch.

Considering the horizon branch, and assuming that there is no shell outside of the horizon, we showed that imposing regularity of the solution at the event horizon fixes the remaining free parameters, showing that the unique solution with a regular horizon is, as expected, the horizon branch solution of [?]. This solution reduces to an uncharged black hole at large mass.

Considering the shell branch, we saw that we had a family of solutions at \( r_h = 0 \). On the shell branch, we are considering singular supergravity metrics (there is a delta-function singularity at the location of the shell), so we can no longer fix these constants of integration by imposing regularity of the solution. However, the only solution in this family for which the stress tensor of the shell inferred from the supergravity solution is of the form predicted for a collection of wrapped branes by the DBI action was the familiar BPS solution of [?]. Thus, we find that if we specify a particular form for the shell stress tensor, then as expected, there was no remaining freedom in the form of the solution; the solution is completely described by giving its conserved charges and ADM mass.

In the nonextreme case, the shell branch solution obtained in [?] is unphysical, as it violates the weak energy condition [?]. We have shown that this problem can be circumvented by considering more general solutions. This provides us with a multi-parameter family of solutions which satisfy all the constraints on physical solutions at the supergravity level. This freedom to add ‘hair’ to the exterior solution arises because the form of the shell stress tensor is not completely fixed. Indeed, the four free parameters in the exterior solution correspond precisely to the freedom to specify three components of the shell stress tensor and the discontinuity in the dilaton, although the translation between the parameters and the stress tensor is quite non-trivial. (The freedom to specify the shell stress in the sphere directions, which is not affected by these parameters, corresponds to the further ambiguity previously noted in [?], in the division of the energy above extremality between the shell itself and a black hole inside the shell.) Thus, if we had a microphysical model of the shell, we would expect to be able to fix all of this freedom. However, this requires further input
from physics beyond supergravity, which we do not have available for the nonextreme cases. The appearance of these parameters thus exposes the limits of the supergravity approach to enhançon physics.

Let us reiterate the essential difference between the two branches: on the horizon branch, we seek a smooth supergravity solution. We can then determine the solution uniquely without requiring additional input, as it does not involve explicit sources. On the shell branch, the singularity can never be clothed by a horizon; we want to describe its resolution by the expansion of the branes sourcing the geometry. We cannot determine the appropriate geometry uniquely, as it involves explicit sources, and we do not have a fundamental description of those sources for the nonextremal case.\(^4\)

In fact, our lack of understanding of the nonextremal physics goes deeper: we cannot exclude the possibility that none of these solutions provide an appropriate physical description of a nonextremal enhançon. It is possible that the shell thickens once we add some energy to it, invalidating the thin-shell approximation used here \(\text{[?]}\); alternatively, the non-abelian gauge fields which become light near the shell may become important (this may even lead to violations of spherical symmetry) \(\text{[?]}\).

It is also worth noting that our study of more general solutions has not resolved the issue of the branch structure and phase transitions. Assuming the near-extremal behaviour is described by some shell branch solution, while the behaviour at large masses should be described by the horizon branch, one expects that there will be some phase transition between the two branches as a function of mass. Unfortunately, since we are unable to identify the correct shell branch solution on the basis of supergravity information alone, we cannot even set up the problem of studying this phase transition.

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\(^4\)This suggests that the most interesting case in which to investigate the extension of singularity resolutions to near-extreme solutions will be the mechanism of \(\text{[?]}\), where the singular geometry is replaced by a smooth supergravity solution with no explicit sources. Some investigations of this system appear in \(\text{[?]}\). It is likely that attempts to extend other singularity resolutions that involve explicit branes, such as \(\text{[?]}\), to nonextreme cases will suffer from the difficulty we have encountered.