Penrose Limits and Spacetime Singularities

M. Blau\textsuperscript{a}, M. Borunda\textsuperscript{b}, M. O’Loughlin\textsuperscript{b}, G. Papadopoulos\textsuperscript{c}

\textsuperscript{a} Institut de Physique, Université de Neuchâtel, Rue Breguet 1
CH-2000 Neuchâtel, Switzerland
\textsuperscript{b} S.I.S.S.A. Scuola Internazionale Superiore di Studi Avanzati
Via Beirut 4, I-34014 Trieste, Italy
\textsuperscript{c} Department of Mathematics, King’s College London
London WC2R 2LS, U.K.

Abstract

We give a covariant characterisation of the Penrose plane wave limit: the plane wave profile matrix $A(u)$ is the restriction of the null geodesic deviation matrix (curvature tensor) of the original spacetime metric to the null geodesic, evaluated in a comoving frame. We also consider the Penrose limits of spacetime singularities and show that for a large class of black hole, cosmological and null singularities (of Szekeres-Iyers “power-law type”), including those of the FRW and Schwarzschild metrics, the result is a singular homogeneous plane wave with profile $A(u) \sim u^{-2}$, the scale invariance of the latter reflecting the power-law behaviour of the singularities.

\textsuperscript{1}e-mail: matthias.blau@unine.ch
\textsuperscript{2}e-mail: mborunda@he.sissa.it
\textsuperscript{3}e-mail: loughlin@sissa.it
\textsuperscript{4}e-mail: gpapas@mth.kcl.ac.uk
The Penrose limit construction [1] associates to every spacetime metric and choice of null geodesic in that spacetime a plane wave metric

\[ ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 . \]  

(1.1)

Here \( A_{ab}(u) \) is the plane wave profile matrix and the computation of the Penrose limit along a null geodesic \( \gamma \) amounts to determining the matrix \( A_{ab}(u) \) from the metric of the original spacetime.

This has recently been used to show [2] that the Penrose limit of the \( AdS_5 \times S^5 \) IIB superstring background is the BHFP maximally supersymmetric plane wave [3]. String theory in this RR background is exactly solvable [4, 5], giving rise to a novel explicit form of the AdS/CFT correspondence [6]. Following these developments, many Penrose limits have been computed for various supergravity backgrounds and their applications have been explored.

Despite these developments, the precise nature of the Penrose limit and the extent to which it encodes generally covariant properties of the original spacetime have remained somewhat elusive, also because the usual definition and practical implementations of the Penrose limit indeed require taking a limit and look rather non-covariant.

The primary purpose of this note is to provide a \textit{completely covariant characterisation and definition of the Penrose limit} wave profile matrix \( A_{ab}(u) \) which does not require taking any limit and which shows that \( A_{ab}(u) \) directly encodes diffeomorphism invariant information about the original spacetime metric. Specifically, we will show that

\[ A_{ab}(u) = - R_{aubu} \big|_{\gamma(u)} , \]  

(1.2)

where \( R \) is the curvature tensor of the original metric, and the components refer to a parallel-transported frame along the null geodesic with \( ds^2 = 2E^+E^- + \delta_{ab} E^a E^b \) and \( E_+ = \partial_u \). Thus \( A_{ab}(u) \), which is uniquely determined by these conditions up to constant orthogonal transformations, is nothing other than the standard [7, Section 4.2] \textit{transverse frequency matrix} of the null geodesic deviation equation of the original metric.

In particular, therefore, since lightcone gauge string theory on a plane wave becomes a two-dimensional theory of free fields with mass (frequency) matrix \((-A_{ab})\) [4, 5], we see that the components of the mass matrix of the fields are certain components of the curvature tensor of the spacetime before the Penrose limit is taken. Thus imaginary frequencies, which lead to tachyonic worldsheet modes in the lightcone gauge (but need
not signal an instability \([8, 9]\)), correspond to diverging null geodesics in the original spacetime.

Since singularities of \(A_{ab}(u)\) result from curvature singularities of the original spacetime, it is of interest to analyse the nature of Penrose limits of spacetime singularities, as they encode information about the rate of growth of curvature and geodesic deviation as one approaches the singularity of the original spacetime along a null geodesic.

What we will demonstrate (referring to \([10]\) for more details) is that in this case the Penrose limits exhibit a remarkably universal behaviour in the sense that for a large class of black hole, cosmological and null singularities, indeed all the Szekeres-Iyer metrics \([11, 12]\) with singularities of “power-law type”, one obtains plane wave metrics of the form

\[
ds^2 = 2dudv + A_{ab}x^a x^b \frac{du^2}{u^2} + d\vec{x}^2 ,
\]

with \(A_{ab}\) constant and eigenvalues bounded by \(1/4\). Due to the existence of the scale invariance \((u, v) \rightarrow (\lambda u, \lambda^{-1} v)\), these singular plane waves are *homogeneous* \([13, 14, 15]\), reflecting the scaling behaviour of the original power-law singularities.

String theory in homogeneous plane wave backgrounds is exactly solvable \([14, 16]\), and it has been shown that string oscillator modes in the above singular background can even be analytically continued through the singularity \([14]\). Since the Penrose limit can be considered as the origin of a string expansion around the original background \([13]\) (see \([17]\) for the AdS case), the above observations about the relation of these backgrounds to interesting spacetime singularities provide additional impetus for understanding string theory in an expansion around such metrics.

## 2 A Covariant Characterisation of the Penrose Limit

### 2.1 The Penrose Limit

The Penrose limit associates to every spacetime metric \(g_{\mu\nu}\) and choice of null geodesic \(\gamma\) in that spacetime a (limiting) plane wave metric. The first step is to rewrite the metric in coordinates adapted to \(\gamma\), Penrose coordinates \([1]\), as

\[
ds^2 = 2dUdV + a(U, V, Y^k) dV^2 + 2b_i(U, V, Y^k) dV dY^i + g_{ij}(U, V, Y^k) dY^i dY^j
\]

This corresponds to an embedding of \(\gamma\) into a twist-free congruence of null geodesics, given by \(V\) and \(Y^k\) constant, with \(U\) playing the role of the affine parameter and \(\gamma(U)\) coinciding with the geodesic at \(V = Y^k = 0\).
The next step is to perform the change of coordinates \((\Omega \in \mathbb{R})\)

\[(U, V, Y^k) = (u, \Omega^2 \bar{v}, \Omega y_k) .\]  

The Penrose limit metric \(\bar{g}_{\mu\nu}\) is then defined by

\[ds^2 = \lim_{\Omega \to 0} \Omega^{-2} d\bar{s}^2 = 2dud\bar{v} + \bar{g}_{ij}(U)dy^i dy^j ,\]  

where \(d\bar{s}^2\) is the metric in the coordinates \((u, \bar{v}, y^i)\) and \(\bar{g}_{ij}(U) = g_{ij}(U, 0, 0)\). This is the metric of a plane wave in Rosen coordinates. Pragmatically speaking, the Penrose limit metric is obtained by setting the components \(a\) and \(b_i\) of the metric to zero and restricting \(g_{ij}\) to the null geodesic \(\gamma\).

A coordinate transformation \((u, \bar{v}, y^k) \rightarrow (u, v, x^a)\) puts the metric into the standard Brinkmann form

\[ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\bar{x}^2 .\]  

Here

\[A_{ab}(u) = -\bar{R}_{ab}(u) = -\bar{R}_{iab}(u)\bar{E}_i^a(u)\bar{E}_b^a(u) ,\]  

with \(\bar{R}_{ab}(u)\) the only non-vanishing component of the Riemann curvature tensor of \(\bar{g}_{\mu\nu}\). \(\bar{E}_i^a\) is an orthonormal coframe for the transverse metric \(\bar{g}_{ij}\), satisfying the symmetry condition

\[\dot{\bar{E}}_a^i \bar{E}_b^j = \dot{\bar{E}}_b^i \bar{E}_a^j .\]  

2.2 Curvature and Penrose Limits

We now establish the relation between the wave profile \(A_{ab}(u)\) of the Penrose limit metric and certain components of the curvature tensor of the original metric.

We consider the components \(R_{ijU}^i\) of the curvature tensor of the metric (2.1) which enter into the geodesic deviation equation of the corresponding null geodesic congruence. The first observation is that

\[R_{ijU}^i = -\partial U \Gamma_{ijU} + \Gamma_{kU}^i \Gamma_{jU}^k\]  

does not depend on the coefficients \(a\) and \(b_i\) of the metric and only involves \(U\)-derivatives of \(g_{ij}\). It follows that these components of the curvature tensor are related to those of the Penrose limit metric by

\[\bar{R}_{ijU}^i = R_{ijU}^i|_{\gamma} .\]  

Next we introduce a pseudo-orthonormal frame \(E^M_\mu\), \(M = (+, -, a)\) for the metric (2.1),

\[ds^2 = 2E^+ E^- + \delta_{ab}E^a E^b ,\]  

(2.9)
which is parallel along the null geodesic congruence, \( \nabla_u E^M_\mu = 0 \). We choose \( E_+ = \partial_u \)
to be tangent to the geodesics. Then it is not difficult to see that \( E_a \) has the form
\[
E_a = E^i_a \partial_i + E^U_a \partial_U ,
\]
where \( E^a_i \) is a vielbein for \( g_{ij}(U,V,Y^K) \) satisfying
\[
\dot{E}_{ai} E^a_b = \dot{E}_{bi} E^a_i .
\]
This condition is independent of \( a, b \) and only involves \( U \)-derivatives of \( E^a_i \). We can thus conclude that the vielbeins \( \dot{E}^a_i \) of the Penrose limit metric satisfying the symmetry condition (2.6) can be obtained from the parallel-propagated (2.11) vielbeins of the full metric by restriction to the null geodesic \( \gamma \),
\[
\dot{E}^a_i = E^a_i |_\gamma .
\]
Combining (2.5) with (2.8) and (2.12), and using (2.10) we thus obtain the key result that the frequency matrix (wave profile) \( A_{ab}(u) \) of the Penrose limit metric is the transverse null geodesic deviation matrix of the original metric, evaluated in a parallel propagated frame [7, Section 4.2],
\[
A_{ab}(u) = -(R^i_{r+j} E^r_a E^j_b) |_\gamma .
\]
As a consequence, even though we had to appeal to Penrose adapted coordinates (2.1) to implement the standard definition (2.3) of the Penrose limit, we now arrive at a fully covariant characterisation and definition of the Penrose limit. While this is implied by what we have already said, it may be worth reiterating it:

Given a null geodesic \( \gamma \), one constructs a pseudo-orthonormal parallel propagated coframe \((E_+, E_-, E_a)\) with \( E_+ = \partial_u \) tangent to the null geodesics and \( E_- \) characterised by \( g(E_-, E_-) = 0 \) and \( g(E_+, E_-) = 1 \). Then the Penrose limit is the plane wave metric characterised by the wave profile
\[
A_{ab}(u) = -R_{a+b+} |_\gamma ,
\]
which is determined uniquely up to \( u \)-independent orthogonal transformations.

3 Singular Homogeneous Plane Waves from Penrose Limits of Spacetime Singularities

Given the above results it is of interest to study Penrose limits of spacetime singularities, as they encode diffeomorphism invariant information about the rate of growth of curvature and geodesic deviation along a null geodesic.
We will see that for a large class of spacelike, timelike or null singularities the Penrose limit metric is that of a singular homogeneous plane wave \[14, 15\],

\[ds^2 = 2dudv + A_{ab}x^a x^b \frac{du^2}{u^2} + d\bar{x}^2\] (3.1)

exhibiting the scale invariance \((u, v) \rightarrow (\lambda u, \lambda^{-1}v)\). This has already been observed before in some special brane and cosmological backgrounds \[13, 18\]. Note that, because of the scale invariance, this metric is uniquely characterised by the eigenvalues of \(A_{ab}\) (up to permutations), in contrast to the case of symmetric plane waves (\(A_{ab}(u)\) constant) for which the eigenvalues are scaled by \(\lambda^2\) under the above transformation.

### 3.1 Examples: FRW and Schwarzschild Plane Waves

We briefly illustrate this with some examples which will be discussed in more detail in \[10\].

The \(D = (n + 1)\)-dimensional FRW metric

\[ds^2 = -dt^2 + a(t)^2(dr^2 + f_k(r)^2d\Omega_n^2)\] (3.2)

\((f_k(r) = r, \sin r, \sinh r\) for \(k = 0, +1, -1\) respectively) has a unique Penrose limit, characterised by the diagonal wave profile \(A_{ab} = A(u)\delta_{ab}\) with

\[A(u) = -\frac{8\pi G}{n-1} \left(\frac{\rho(u) + p(u)}{a(u)^2}\right)\] (3.3)

Here \(\rho\) and \(p\) are the energy-density and pressure of the perfect fluid, and \(a(u) = a(t(u))\) etc., with \(a(t)\) determined by the Friedmann equations and \(t(u)\) determined by the null geodesic condition with \(u\) the affine parameter. Specialising to the equation of state \(p = w\rho\), one finds that near the singularity \(A(u)\) behaves as\(^1\)

\[A(u) = -\frac{h}{(1 + h)^2 u^{-2}}\] (3.4)

where \(h = 2/n(1 + w)\). Thus this is a singular homogeneous plane wave, with frequency squares bounded by \(1/4\) (which is attained for \(h = 1\)). This is interesting because it is known \[14, 15\] that the behaviour of particles and strings in the background (3.1) is qualitatively different for frequency squares less than or greater than \(1/4\).

For the \(D = (d + 2) \geq 4\)-dimensional Schwarzschild metric

\[ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_d^2\] (3.5)

\(^1\)This is also the general behaviour of the \(k = 0\) metrics and generalises the result reported in \[13\].
where \( f(r) = 1 - 2m/r^{d-1} \), one finds that \( A_{ab}(u) \) is diagonal, with [19]

\[
A_{22}(u) = \ldots = A_{dd}(u) = \frac{(d+1)m\ell^2}{r(u)^{d+2}}
\]  

(3.6)

and (since this is a vacuum plane wave) \( A_{11} = -\sum_{a=2}^{d} A_{aa} \). Here \( r(u) \) is the solution to the null geodesic equation and \( \ell \neq 0 \) the angular momentum. Thus the resulting plane wave metric is singular iff the original null geodesic runs into the singularity, which will happen for sufficiently small values of \( \ell \). In this case, as \( r(u) \to 0 \) one has

\[
A_{22}(u) = \ldots = A_{dd}(u) = -\frac{2(d+1)}{(d+3)^2} u^{-2}. 
\]  

(3.7)

Again this is a singular homogeneous plane wave with frequencies bounded by \( 1/4 \).

Note that the behaviour near the singularity depends only on the spacetime dimension \( D = d + 2 \), neither on the mass parameter \( m \) of the black hole nor on the angular momentum \( \ell \) of the null geodesic used to approach the singularity. Curiously, the frequency squares obtained in the Penrose limit of the Schwarzschild metric are precisely those of a dust-filled FRW universe of the same dimension (for example, \( 6/25 \) for \( D = 4 \)).

Thus string and particle propagation in the near-singularity regime of the Schwarzschild plane wave is identical to that in the \( w = 0 \) FRW plane wave in all but one of the transverse directions. The appearance of an imaginary frequency in the remaining transverse direction is of course dictated by the fact that the Schwarzschild plane wave is, unlike the FRW plane wave, a vacuum solution.

### 3.2 A General Result: Penrose Limits of Szekeres-Iyer Power-Law Type Singularities

From a purely calculational point of view, the occurrence of \( u^{-2} \)-type plane waves with frequency squares bounded by \( 1/4 \) can be attributed to the fact that in Rosen coordinates (2.3) the leading behaviour in \( U \) of the transverse metric \( \bar{g}_{ij}(U) \) close to the singularity is of power-law type [14, 15], which is reflected in the scale invariance of the Penrose limit. Thus to assess the generality of this kind of result, one needs to enquire about the generality of spacetime singularities exhibiting such a power-law behaviour.

In [11] (see also [12]), in the context of investigations of the Cosmic Censorship Hypothesis, Szekeres and Iyer studied a large class of four-dimensional spherically symmetric metrics they dubbed “metrics with power-law type singularities”. Such metrics encompass all the FRW metrics, Lemaître-Tolman-Bondi dust solutions, cosmological singularities of the Lifshitz-Khalatnikov type, as well as other types of metrics with null singularities. In “double-null form”, these metrics take the form

\[
ds^2 = 2e^{A(U,V)} dUdV + e^{B(U,V)} d\Omega_2^2,
\]  

(3.8)
where $A(U, V)$ and $B(U, V)$ have expansions
\[
A(U, V) = p \ln X(U, V) + \text{regular terms}
\]
\[
B(U, V) = q \ln X(U, V) + \text{regular terms}
\] (3.9)
near the singularity surface $X(U, V) = 0$. The residual coordinate transformations $U \to U'(U)$, $V \to V'(V)$ preserving the form of the metric (3.8) can be used to make $X(U, V)$ linear in $U$ and $V$,
\[
X(U, V) = aU + bV, \quad a, b = \pm 1, 0 ,
\] (3.10)
with $ab = \pm 1, 0$ corresponding to timelike, spacelike and null singularities respectively. This choice of gauge essentially fixes the coordinates uniquely, and thus the “critical exponents” $p$ and $q$ contain diffeomorphism invariant information. The Schwarzschild metric, for example, has $p = -1/2$ and $q = 1$, as is readily seen by starting with the metric in Eddington-Finkelstein or Kruskal-Szekeres coordinates and transforming to the Szekeres-Iyer gauge.

It is now a simple matter to determine all the Penrose limits of the near-singularity metrics
\[
ds^2 = 2X^p dU dV + X^q d\Omega_2^2 ,
\] (3.11)
in order to study the genericity of the $u^{-2}$-behaviour with frequency squares bounded by $1/4$. One finds that for generic values of $p$ and $q$ within certain regions of the $(p, q)$-plane the behaviour as $X(u) \to 0$ is always of this type. For example, from this point of view the $\pm 6/25$ of the Schwarzschild metric arise as $q(p + 2)/(p + q + 2)^2$ and $p(q + 2)/(p + q + 2)^2$ for $p = -1/2, q = 1$.

Curiously, the resulting diagram in the $(p, q)$-plane delineating the regions with $u^{-2}$-behaviour from other possibilities bears a tantalising resemblance to the Szekeres-Iyer phase diagram [11, Fig.1] of near-singularity energy-momentum tensors.

**Acknowledgements**

This work has been supported by the European Community’s Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime and by the Swiss National Science Foundation. MBo thanks CONACyT (Mexico) and is grateful to the Institut de Physique, Neuchâtel, for hospitality during the final stages of this work. GP thanks the Abdus Salam ICTP for hospitality where part of this project was completed.

**References**