On energy-momentum spectrum of stationary states with nonvanishing current on 1-d lattice systems

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Abstract

On one-dimensional two-way infinite quantum lattice system, a property of translationally invariant stationary states with nonvanishing current expectation is investigated. We consider GNS representation with respect to such a state, on which we have a group of space-time translation unitary operators. We show that spectrum of the unitary operators, energy-momentum spectrum with respect to the state, has a singularity at the origin.

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1 Introduction

Recently a lot of researchers get interested in nonequilibrium states \([1, 2, 3, 4, 5, 6]\). In spite of their efforts, few things have been known rigorously. The situation is contrast to equilibrium state. In equilibrium state business, for instance, properties of energy momentum spectrum have been well understood \([7, 8]\). We, in the present paper, study rigorously a property of energy-momentum spectrum with respect to nonequilibrium steady states. To put it concretely, we consider a one-dimensional lattice system with nearest neighbor interaction, and translationally invariant stationary states with nonvanishing current on it. Because of the space-time translational invariance of the state, on its GNS representation there exists a group of unitary operators whose spectrum is called energy-momentum spectrum. We show that the spectrum has singularity at the origin thanks to the nonvanishing nature of the current. Our discussion is model independent and general.

The paper is organized as follows. In the next section we briefly introduce one-dimensional lattice system and define a state with nonvanishing current what we are interested in. In section 3, we show our main theorem.

2 States with nonvanishing current on 1-d lattice systems

We deal with a one-dimensional two-way infinite quantum lattice system. To each site \(x \in \mathbb{Z}\) a Hilbert space \(\mathcal{H}_x\) which is isomorphic to \(\mathbb{C}^{N+1}\) is attached and observable algebra at site \(x\) is a matrix algebra on \(\mathcal{H}_x\) which is denoted by \(\mathcal{A}(\{x\})\). The observable algebra on a finite set \(\Lambda \subset \mathbb{Z}\) is a matrix algebra on \(\otimes_{x \in \Lambda} \mathcal{H}_x\) and denoted by \(\mathcal{A}(\Lambda)\). Natural identification can be used to derive an inclusion property \(\mathcal{A}(\Lambda_1) \subset \mathcal{A}(\Lambda_2)\) for \(\Lambda_1 \subset \Lambda_2\).

The total observable algebra is a norm completion of sum of the finite region observable algebra, \(\mathcal{A} := \bigcup_{\Lambda: \text{finite}} \mathcal{A}(\Lambda)\), which becomes a \(C^*\) algebra. (For detail, see \([8]\).)

To discuss the dynamics, we need \(\{\alpha_t\}_{t \in \mathbb{R}}\), a one-parameter \(*\)-automorphism group on \(\mathcal{A}\), which we assume is induced by a local interaction. In the present paper, for simplicity, we assume that the interaction is nearest neighbor one. That is, for each \(x \in \mathbb{Z}\) there exists a self adjoint element \(h_{x,x+1} \in \mathcal{A}([x, x+1])\), and the local Hamiltonian with respect to each finite region \(\Lambda\) is defined by

\[
H_{\Lambda} := \sum_{\{x,x+1\} \subset \Lambda} h_{x,x+1}.
\]

Moreover, we assume translational invariance of the interaction. That is,

\[
\tau_x(h_{y,y+1}) = h_{x+y,x+y+1},
\]

holds for each \(x, y \in \mathbb{Z}\) where \(\tau_x\) is a space translation \(*\)-automorphism. The Hamiltonian defines a one-parameter \(*\)-automorphism \(\alpha_t\) by

\[
\frac{d\alpha_t(A)}{dt} := -i \lim_{\Lambda \to \mathbb{Z}} [\alpha_t(A), H_{\Lambda}]
\]
for each local element $A \in \mathcal{A}$. Hereafter, for each local element $A \in \mathcal{A}$, we employ the notation $A(t) := \alpha_t(A)$.

To define a current operator, we assume existence of local charge operators. Namely there exists a self-adjoint operator $n_x \in \mathcal{A}(\{x\})$ for each $x \in \mathbb{Z}$ with $\tau_x(n_0) = n_x$, and we put $N_\Lambda := \sum_{x \in \Lambda} n_x$ for each finite region $\Lambda$. The charge defines a one-parameter $*$-automorphism group on the observable algebra by

$$\frac{d\gamma_\theta(A)}{d\theta} = i \lim_{\Lambda \to \mathbb{Z}} [N_\Lambda, \gamma_\theta(A)].$$

We assume $N_\Lambda$ is conserved with respect to $H_\Lambda$, that is,

$$[N_\Lambda, H_\Lambda] = 0 \quad (1)$$

holds for each finite region $\Lambda$. In particular, putting $\Lambda = [x, x + 1]$, we obtain a commutator,

$$[h_{x,x+1}, n_x + n_{x+1}] = 0. \quad (2)$$

On the other hand, by letting $\Lambda \to \mathbb{Z}$ this relation derives a purely algebraic relation,

$$\alpha_t \circ \gamma_\theta = \gamma_\theta \circ \alpha_t.$$

With this algebraic relation, $\gamma_\theta$ is called a (continuous) symmetry transformation.

On this setting, electric current (hereafter we simply call it as current) between sites $x$ and $x + 1$ is defined by

$$j_{x,x+1} := -i[n_{x+1}, h_{x,x+1}] = i[n_x, h_{x,x+1}],$$

where the second equality is due to (2). If we consider the equation of motion for the charge contained in a finite region $\Lambda := [-L, 0]$, we obtain

$$\left. \frac{d\alpha_t(N_\Lambda)}{dt} \right|_{t=0} = j_{-L-1,-L} - j_{0,1}, \quad (3)$$

which corresponds to a continuity equation in continuum case. The following observation is significant to derive our main theorem. Thanks to (1), the current at the origin can be rewritten for any $L$ and $M$ satisfying $L \geq M > 0$ as

$$j_{0,1} := i[N_{[-L,0]}, H_{[-M,M+1]}].$$

The above seemingly abstract setting has physically interesting examples. For instance, interacting fermion system is on the list. For each $x \in \mathbb{Z}$, charge $n_x := c_x^* c_x$ and $h_{x,x+1} = -t(c_{x+1}^* c_x + c_x^* c_{x+1}) - \mu n_x + v(1)n_x n_{x+1}$ gives a nearest neighbor Hamiltonian. The current at the origin is calculated as $j_{0,1} = it(c_1^* c_0 - c_0^* c_1)$. Heisenberg model can be another example. $h_{x,x+1} := S_x^{(1)} S_{x+1}^{(1)} + S_x^{(2)} S_{x+1}^{(2)} + 3S_x^{(3)} S_{x+1}^{(3)}$ and $n_x := S_x^{(3)}$ leads the current $j_{0,1} = -S_0^{(2)} S_1^{(1)} + S_0^{(1)} S_1^{(2)}$.

Now we introduce states which we are interested in.
Definition 1 A state $\omega$ over two-way infinite lattice system $\mathcal{A}$ is called a translationally invariant stationary state with nonvanishing current (a state with nonvanishing current, for short) iff the following conditions are all satisfied:

1. $\omega$ is stationary, i.e., $\omega \circ \alpha_t = \omega$ for all $t$.
2. $\omega$ is translationally invariant, i.e., $\omega \circ \tau_x = \omega$ for all $x$.
3. $\omega$ gives non-vanishing expectation of the current, i.e., $\omega(j_{0,1}) \neq 0$.

Here we do not impose any other condition, stability for instance. Our definition hence might allow rather unphysical states which should be hardly realized. It, however, contains physically interesting states like nonequilibrium steady states obtained by inhomogeneous initial conditions which were discussed in [1, 2].

We put a GNS representation with respect to a state with nonvanishing current $\omega$ as $(\mathcal{H}, \pi, \Omega)$. Since we fix a state $\omega$, indices showing the dependence on $\omega$ will be omitted hereafter. Moreover we identify $\mathcal{A}$ with $\pi(\mathcal{A})$ and will omit to write $\pi$.

Since the state with nonvanishing current $\omega$ is stationary and translationally invariant, one can define a unitary operator $U(x, t)$ for each $x \in \mathbb{Z}, t \in \mathbb{R}$ on $\mathcal{H}$ by

$$U(x, t)A\Omega := \alpha_t \circ \tau_x(A)\Omega$$

for each $A \in \mathcal{A}$. Thanks to commutativity of time and space translation, the unitary operators satisfy

$$U(x_1, t_1)U(x_2, t_2) = U(x_1 + x_2, t_1 + t_2)$$

and can be diagonalized into the form:

$$U(x, t) = \int_{k=-\pi}^{\pi} \int_{\epsilon=-\infty}^{\infty} e^{i(\epsilon t - kx)} E_{\omega}(dkd\epsilon),$$

where $E_{\omega}(dkd\epsilon)$ is a projection valued measure and called energy momentum spectrum. In the following section, we investigate a property of $E_{\omega}(dkd\epsilon)$.

3 Energy momentum spectrum

In this section we show a singular nature of energy-momentum spectrum $E_{\omega}(dkd\epsilon)$. The point of the proof is to estimate the following quantity:

$$\int dt (\Omega, i[N_{[-L,0]}, H_{[-M,M+1]}(t)]\Omega)f(t),$$

where $f$ is an arbitrary function with supp$f \subset [-T,T]$ satisfying $\int dt |f(t)|^2 < \infty$. Although $(\Omega, i[N_{[-L,0]}, H_{[-M,M+1]}(t)]\Omega)$ is not time invariant, it is almost time invariant for sufficiently large $L$ and $M$. To show it, we will employ repeatedly the following lemma.
Lemma 1 Let $V(h_{0,1})$ be a quantity which is determined by the interaction $h_{0,1}$ as

$$V(h_{0,1}) := 4(N+1)^4 e^2 ||h_{0,1}||.$$  

For each finite region $\Lambda$, we denote $d(\Lambda) := \max \{ |x-y| \mid x, y \in \Lambda \}$. Then for all $A \in \mathcal{A}(\Lambda_1)$ and $B \in \mathcal{A}(\Lambda_2)$ with $0 \in \Lambda_1$ and $0 \in \Lambda_2$ and $x$ satisfying $|x| - (d(\Lambda_1) + d(\Lambda_2)) > 0$, an inequality,

$$\|\tau_x \alpha_t(A), B\| \leq \exp \left\{ - |t| \left( \frac{|x| - (d(\Lambda_1) + d(\Lambda_2))}{|t|} - 2V(h_{0,1}) \right) \right\}$$

holds.

The proof is a direct application of theorem 6.2.11 of [8], and is omitted. This lemma guarantees the existence of a finite group velocity which is determined by form of the interaction. Now we show the following lemma:

Lemma 2 For an arbitrary $T > 0$ and an arbitrary function $f$ with the support $[-T, T]$ satisfying $\int dt |f(t)|^2 < \infty$, the following equation holds:

$$\lim_{M \to \infty} \lim_{L \to \infty} \int dt [\hat{N}_{[-L,0]}, \hat{H}_{[-M,M+1]}(t)] f(t) = \sqrt{2\pi} f_0, \hat{f}(0),$$

where $\hat{f}(\epsilon) := \frac{1}{\sqrt{2\pi}} \int dt f(t) e^{i\epsilon t}$ and $\hat{A} := \hat{A} - \omega(A)$ for $A \in \mathcal{A}$, and the limit is taken with respect to norm topology. Note that the order of the limiting procedures can not be exchanged.

Proof: To estimate the equation, let us first consider the following quantity.

$$[\hat{N}_{[-L,0]}, \hat{H}_{[-M,M+1]}(t)] - [\hat{N}_{[-L,0]}, \hat{H}_{[-M,M+1]}(0)]$$

$$= \int_0^t ds [\hat{N}_{[-L,0]}, \alpha_s \left( \frac{d\hat{H}_{[-M,M+1]}(u)}{du} \right)_{u=0} \right]$$

$$= -i \int_0^t ds [\hat{N}_{[-L,0]}, \alpha_s \left( [\hat{H}_{[-M,M+1]}, \hat{H}_{[-M-1,M+2]}] \right) \right] \tag{4}$$

The term $i[\hat{H}_{[-M,M+1]}, \hat{H}_{[-M-1,M+2]}]$ expresses time derivative of energy contained in $[-M, M+1]$ and can be decomposed into in-going and out-going energy current. That is, in a similar manner with electric current, we define energy current at a site $x$ by $J_x := i[h_{x-1,x}, h_{x,x+1}] \in \mathcal{A}([x-1, x+1])$, then the above term is written as

$$i[\hat{H}_{[-M,M+1]}, \hat{H}_{[-M-1,M+2]}] = -J_M + J_{M+1},$$

and

$$\tag{4} = \int_0^t ds [\hat{N}_{[-L,0]} - J_{M+1}(s) + J_M(s)]$$

(5)
holds. Now thanks to spacelike commutativity, \([N_{[-L,0]}, J_{M+1}] = 0\) holds, and we obtain also for \(J_{-M}\),

\[
[N_{[-L,0]}, J_{-M}] = -i[N_{[-L,0]}, [H_{[-M,M+1]}, H_{[-M-1,M+2]}]] = i([H_{[-M,M+1]}, [H_{[-M-1,M+2]}, N_{[-L,0]}] + [H_{[-M-1,M+2]}, [N_{[-L,0]}, H_{[-M,M+1]}]]) = [H_{[-M,M+1]}, -\hat{J}_{0,1}] + [H_{[-M-1,M+2]}, \hat{J}_{0,1}] = 0,
\]

where we used Jacobi identity for commutators. To estimate (5), we bound the deviation for finite \(s\) by use of lemma 1 as

\[
\|[[N_{[-L,0]}, J_{M+1}(s)]\| \leq \sum_{-L-M \leq z \leq -M-1} \|n_z, J_0(s)\| \leq 2(N + 1)^4 \|n_0\| \|J_0\| \sum_{-L-M \leq z \leq -M-1} \exp\{-|s|\left(|z| - 4 |s| - 2 V(h_{0,1})\right)\} \leq 6(N + 1)^4 \|n_0\| \|J_0\| \frac{e^{-M}}{1 - e^{-1}} e^{2|s|V(h_{0,1})}.
\]

Next we estimate the other term of (5),

\[
\|[[N_{[-L,0]}, J_{-M}(s)]\| = \|[[\alpha_u(N_{[-L,0]}), J_{-M}]\| \leq \|[[N_{[-L,0]}, J_{-M}]\| + \int_0^s du \|\alpha_u\left(\frac{d\alpha_{-t}(N_{[-L,0]})}{dt}\right)_{t=0}, J_{-M}\| \leq \int_0^s du \|\alpha_u\left(\frac{d\alpha_{-t}(N_{[-L,0]})}{dt}\right)_{t=0}, J_{-M}\| \leq \int_0^s du \|\alpha_u(\hat{J}_{0,1}), J_{-M}\| + \int_0^s du \|\alpha_u(j_{-L-1, -L}), J_{-M}\| .
\]

The last line of (7) is thanks to (3). By translating \(J_{-M}\) to neighborhood of the origin, \(J_0 := \tau_M(J_{-M}) \in \mathcal{A}([-1, 1])\), we can use lemma 1 to estimate the first term of (7) as

\[
\|\alpha_u(\hat{J}_{0,1}), J_{-M}\| = \||\tau_M \circ \alpha_u(\hat{J}_{0,1}), J_0\| \leq 2\|\hat{J}_{0,1}\| \|J_{-M}\|(N + 1)^5 6 \exp\{-|u| \left(\frac{M - 5}{|u|} - 2 V(h_{0,1})\right)\}
\]

In the same manner we obtain the bound for second term of (7) as

\[
\|\alpha_u(j_{-L-1, -L}), J_{-M}\| = \||j_{-L-1, -L}, \alpha_u \circ \tau_{L-M}(J_0)\| \leq 2\|\hat{J}_{0,1}\| \|J_0\|(N + 1)^5 6 \exp\{-|u| \left(\frac{L - M - 5}{|u|} - 2 V(h_{0,1})\right)\}.
\]

Combination of the above estimates leads

\[
(7) \leq \frac{e^{2V(h_{0,1})|s|}}{2V(h_{0,1})} \|\hat{J}_{0,1}\| \|J_0\|(N + 1)^5 6(e^{-M} + e^{-(L-M)})e^5.
\]
Therefore, from (6) and (8), we obtain
\[
||\mathcal{N}_{[-L,0], H_{[-M,M+1]}(t)} - \mathcal{N}_{[-L,0], H_{[-M,M+1]}}|| \\
\leq \int_0^T ds(||\mathcal{N}_{[-L,0], J_{M+1}(s)}|| + ||\mathcal{N}_{[-L,0], J_{-M}(s)}||) \leq Z_{M,L}(t),
\]
where
\[
Z_{M,L}(t) := 6(N + 1)^4 ||n_0|| ||J_0|| \left( \frac{e^{-M}}{1 - e^{-1}} e^{3e^{2V(h_0,1)|t|}} - 1 \right) \\
+ 2 ||j_{0,1}|| ||J_0||(N + 1)^5 6e^5 \left( e^{-M} + e^{-(L-M)} \right) \left( \frac{1}{2V(h_0,1)} \left( e^{2V(h_0,1)|t|} - 1 \right) - |t| \right).
\]

The integration of (9) with the function \( f \) derives
\[
||\int dt [\mathcal{N}_{[-L,0], H_{[-M,M+1]}(t)}] f(t) - i[\mathcal{N}_{[-L,0], H_{[-M,M+1]}}] f(t)|| \\
\leq \int dt ||\mathcal{N}_{[-L,0], H_{[-M,M+1]}(t)} - [\mathcal{N}_{[-L,0], H_{[-M,M+1]}}]|| f(t) \\
= \int_0^T dt ||\mathcal{N}_{[-L,0], H_{[-M,M+1]}(t)} - [\mathcal{N}_{[-L,0], H_{[-M,M+1]}}]|| f(t) \\
\leq \left( \int dt |f(t)|^2 \right)^{1/2} \left( \int_0^T dt ||\mathcal{N}_{[-L,0], H_{[-M,M+1]}(t)} - [\mathcal{N}_{[-L,0], H_{[-M,M+1]}}]||^2 \right)^{1/2} \\
\leq \left( \int dt |f(t)|^2 \right)^{1/2} \left( 2 \int_0^T dt Z_{M,L}(t)^2 \right)^{1/2} \\
\leq \left( \int dt |f(t)|^2 \right)^{1/2} \left( A(T)e^{-2M} + B(T)(e^{-2M} + e^{-2(L-M)} + 2e^{-L}) + C(T)(e^{-2M} + e^{-L}) \right)^{1/2}
\]
where \( A(T), B(T) \) and \( C(T) \) do not depend upon \( M \) and \( L \). Consequently we obtain the following:
\[
\lim_{M \to \infty} \lim_{L \to \infty} \int dt [\mathcal{N}_{[-L,0], \hat{H}_{[-M,M+1]}(t)}] f(t) = \sqrt{2\pi} j_{0,1} \tilde{f}(0).
\]
Thus the proof is completed. \quad \textbf{Q.E.D.}

This lemma gives a starting point for our discussion. Note that the ordering of limiting procedures, \( L \to \infty \) and \( M \to \infty \), cannot be changed. In fact one can easily see that if one takes \( M \to \infty \) first, the left hand side of the above lemma vanishes.

To study the property of energy momentum spectrum, a proper correlation function should be investigated.

**Definition 2** To investigate the property of \( E_\omega dkde \) we define a “function” \( \tilde{\rho}(k, \epsilon) \) as
\[
\tilde{\rho}(k, \epsilon)dkde = (\Omega, i\hbar_0 E_\omega dkde) \hat{h}_{0,1}\Omega).
\]
Precisely \( \tilde{\rho} \) is a distribution over infinitely differentiable function of \( k \) and \( \epsilon \). To get rid of an effect of a product of expectations \( \omega(n_0)\omega(h_{0,1}) \), we again use the notation \( A := A - \omega(A) \).
The following is the main theorem.

**Theorem 1** For \( \omega \), a state with nonvanishing current, the energy spectrum has singularity at the origin. i.e.,

\[
-2\pi i \left( \frac{\partial}{\partial k} \tilde{\rho}(k, \epsilon) + \frac{\partial}{\partial k} \tilde{\rho}(-k, -\epsilon)^* \right) \bigg|_{k=0} = \omega(j_{0,1})\delta(\epsilon)
\]

holds.

**Proof**
Since what we are interested in is the spectrum property with respect to \( \omega \), we take an expectation value for \( \omega \) of the above lemma 2.

\[
\lim_{M \to \infty} \lim_{L \to \infty} \int dt \langle \Omega, i[\hat{N}_{-L,0}, \hat{H}_{-M,M+1}(t)] f(t) \rangle \Omega = \sqrt{2\pi} \omega(j_{0,1}) \hat{f}(0),
\]

(10)

The information with respect to the energy-momentum spectrum is encoded in the left hand side of the above equation. To draw it we define functions \( r_L \) and \( s_M \) as

\[
r_L(x) := 1 \text{ for } -L \leq x \leq 0, \text{ otherwise } 0
\]

\[
s_M(x) := 1 \text{ for } -M \leq x \leq M, \text{ otherwise } 0.
\]

These objects are used to derive

\[
\int dt \langle \Omega, i[\hat{N}_{-L,0}, \hat{H}_{-M,M+1}(t)] \rangle f(t) = \int dt \sum_x \sum_y r_L(x)s_M(y)(\langle \Omega, i[\hat{n}_x(0), \hat{h}_{y,y+1}(t)] \rangle f(t).
\]

(11)

By use of the spectrum decomposition of the space-time translation unitary operator \( U(z,t) := \int e^{i(\epsilon t - k z)} E_\omega(dk d\epsilon) \), we denote Fourier transform of \( \tilde{\rho}(k, \epsilon) \) as

\[
\rho(z,t) := \frac{1}{2\pi \sqrt{2\pi}} \int de \int_{-\pi}^{\pi} dk \tilde{\rho}(k, \epsilon)e^{i(kz - \epsilon t)} = \frac{1}{2\pi \sqrt{2\pi}} \langle \Omega, i\hat{n}_0 \hat{h}_{-z,-z+1}(-t) \rangle,
\]

(12)

then we can write the equation (11) as

\[
(11) = 4\pi \sqrt{2\pi} \int dt \sum_z \text{Re} \left( \rho(z,-t) (\sum_x r_L(x)s_M(x-z)) \right) f(t)
\]

(13)

Now the relation

\[
\lim_{L \to \infty} \sum_x r_L(x)s_M(x-z) = \begin{cases} 
2M + 1, & z < -M \\
M + 1 - z, & -M \leq z \leq M \\
0, & M < z
\end{cases}
\]

(14)
is used to show the limiting value for $L$ to infinity as

$$
\lim_{L \to \infty} \int dt(\Omega, i[\hat{N}_{[-L,0]}, \hat{H}_{[-M,M+1]}(t)]\Omega)f(t) = 4\pi \sqrt{2\pi} \int dt f(t) \text{Re}((\sum_{z<-M} \rho(z,-t)(2M+1))
$$

(15)

Next consider what will occur when $M$ is made infinity in the above equation. In the following, we show that (15) and (16) approach zero as $M \to \infty$. Let us begin with (15),

$$
(15) = 2 \int dt f(t) \text{Re}(\sum_{z>M} (\Omega, [\hat{n}_0, \hat{h}_{z,z+1}(t)]\Omega))(2M+1)
$$

(15)

Thanks to Cauchy-Schwarz inequality, one can obtain

$$
|\langle (15) \rangle| \leq \int dt |f(t)| |(\Omega, [\hat{n}_0, \sum_{z>M} \hat{h}_{z,z+1}(t)]\Omega)|(2M+1)
$$

\leq (2M + 1) \left( \int dt |f(t)|^2 \right)^{1/2} \left( \int_{-T}^T dt \| [\hat{n}_0, \sum_{z>M} \hat{h}_{z,z+1}(t)]\Omega \|^2 \right)^{1/2}.\n$$

Since as for the integrand of the above equation, the group-velocity lemma 1 is used to show

$$
\| [\hat{n}_0, \hat{h}_{z,z+1}(t)] \| \leq 2(N + 1)^3 \| \hat{n}_0 \| \| \hat{h}_{0,1} \| 2\exp \left\{ -|t| \left( \frac{|z| - 3}{|t|} - 2V(h_{0,1}) \right) \right\} \n$$

(18)

and

$$
\| [\hat{n}_0, \sum_{z>M} \hat{h}_{z,z+1}(t)] \| \leq 2(N + 1)^3 \| \hat{n}_0 \| \| \hat{h}_{0,1} \| 2e^3 e^{2|t| V(h_{0,1})} e^{-M} \n$$

(19)

Thus finally we obtain

$$
|\langle (15) \rangle| \leq \left( \int dt |f(t)|^2 \right)^{1/2} (2M+1)2(N+1)^3 \| \hat{n}_0 \| \| \hat{h}_{0,1} \| 2e^3 e^{2|t| V(h_{0,1})} e^{-M} \n$$

(18)

which approaches zero as $M \to \infty$.

Next we estimate the equation(16),

$$
|\langle (16) \rangle| \leq (M + 1) \left( \int dt |f(t)|^2 \right)^{1/2} \left( \int_{-T}^T dt |(\Omega, [\hat{n}_0, \sum_{z=-M}^{M} \hat{h}_{z,z+1}(t)]\Omega)|^2 \right)^{1/2}.\n$$

(19)
The integrand of the above equation can be written by use of stationarity of \(\omega\) as
\[
(\Omega, [\hat{n}_0, \sum_{z=-M}^M \hat{h}_{z,z+1}(t)]\Omega) = (\Omega, [\hat{n}_0, \sum_{z=-M}^M \hat{h}_{z,z+1}(0)]\Omega) + \int_0^t ds(\Omega, [\hat{n}_0, \frac{d}{ds} \hat{H}_{[-M,M+1]}(s)]\Omega)
\]
\[
= \int_0^t ds(\Omega, [\hat{n}_0, \alpha_s \left( \frac{d\hat{H}_{[-M,M+1]}(s)}{dt} \right)_{t=0}]\Omega)
\]
\[
= \int_0^t ds(\Omega, [\hat{n}_0, J_{-M}(s) - J_{M+1}(s)]\Omega).
\]
As before, decomposition into energy current terms
\[
\frac{d\hat{H}_{[-M,M+1]}}{dt} = -[H_{[-M,M+1]}, H_{[-M-1,M+2]}] = J_{-M} - J_{M+1},
\]
where \(J_{-M} \in A([-M - 1, -M + 1])\) and \(J_{M+1} \in A([M, M + 2])\) leads
\[
|\Omega, [\hat{n}_0, \sum_{z=-M}^M \hat{h}_{z,z+1}(t)]\Omega| \leq \left| \int_0^t ds\| [\hat{n}_0, J_{-M}(s)]\Omega \right| + \left| \int_0^t ds\| [\hat{n}_0, J_{M+1}(s)]\Omega \right|.
\]
And the following estimations which are obtained by direct use of group-velocity lemma
\[
\| [\hat{n}_0, J_{-M}(s)]\| \leq 2(N + 1)^4 \| n_0 \| \| J_0 \| 3\exp \left\{ -|s| \left( \frac{M - 4}{|s|} - 2V(h_{0,1}) \right) \right\}
\]
\[
\| [\hat{n}_0, J_{M+1}(s)]\| \leq 2(N + 1)^4 \| n_0 \| \| J_0 \| 3\exp \left\{ -|s| \left( \frac{M - 4}{|s|} - 2V(h_{0,1}) \right) \right\}
\]
thus it leads,
\[
|\Omega, [\hat{n}_0, \sum_{z=-M}^M \hat{h}_{z,z+1}(t)]\Omega| \leq 2(N + 1)^4 \| n_0 \| \| J_0 \| 3e^4 e^{-M} \frac{e^{2V(h_{0,1})T} - 1}{V(h_{0,1})}.
\]
Finally we obtain
\[
\left( \int_{-T}^T dt\| \Omega, [\hat{n}_0, \sum_{z=-M}^M \hat{h}_{z,z+1}(t)]\Omega \|^2 \right)^{1/2} \leq 2(N + 1)^4 \| n_0 \| \| J_0 \| 3e^4 e^{-M}
\]
\[
\times \frac{1}{V(h_{0,1})} \sqrt{\frac{1}{2V(h_{0,1})} (e^{4V(h_{0,1})T} - 4e^{2V(h_{0,1})T} + 4TV(h_{0,1}))}^{1/2}
\]
and can see
\[
\lim_{M \to \infty} (16) = 0
\]
holds. Now we estimate the equation (17) as

\[ (17) = 2\pi \sqrt{2\pi} \int df(t)(- \sum_{-M \leq z \leq M} z\rho(z,-t) - \sum_{-M \leq z \leq M} z\rho(z,-t)^*) \]

\[ = -i \int d\epsilon \int dt \int dk f(t) \sum_{-M \leq z \leq M} e^{it+zk} \frac{\partial}{\partial k}(\bar{\rho}(k,\epsilon) + \bar{\rho}(-k,-\epsilon)). \quad (21) \]

An equation, \(\lim_{M \to \infty} \sum_{-M \leq z \leq M} e^{ikz} = 2\pi \delta(k)\), derives the following relation for \(M \to \infty\),

\[ \lim_{M \to \infty} (17) = -i\sqrt{2\pi}2\pi \int d\epsilon f(\epsilon) \left( \frac{\partial}{\partial k}\bar{\rho}(k,\epsilon) + \frac{\partial}{\partial k}\bar{\rho}(-k,-\epsilon)^* \right) \bigg|_{k=0} \quad (22) \]

Finally we obtain the following equation,

\[ -2\pi i \left( \frac{\partial}{\partial k}\bar{\rho}(k,\epsilon) + \frac{\partial}{\partial k}\bar{\rho}(-k,-\epsilon)^* \right) \bigg|_{k=0} = \omega(j_{0,1})\delta(\epsilon). \quad (23) \]

The proof is completed. Q.E.D.

4 Conclusion and Outlook

We considered states over one-dimensional infinite lattice which are stationary, translationally invariant and have non-vanishing current expectations. The spectrum of spacetime translation unitary operator with respect to such a state was investigated and was shown to have singularity at the origin \((k,\epsilon) = (0,0)\). The theorem is a consequence of only the nonvanishingness of current expectation, and we do not know whether physically more natural conditions give more detail information of the spectrum. It is also interesting to investigate whether our result can be generalized to higher dimensional lattices.

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References


