Quarkonium from the Fifth Dimension

Sungho Hong\textsuperscript{1,2}, Sukjin Yoon\textsuperscript{1}, and Matthew J. Strassler\textsuperscript{1}

\textsuperscript{1} Department of Physics and Astronomy
P.O Box 351560, University of Washington
Seattle, WA 98195

\textsuperscript{2} Department of Physics and Astronomy
University of Pennsylvania
Philadelphia, PA 19104-6396

Abstract

Adding fundamental matter of mass $m_Q$ to $\mathcal{N} = 4$ Yang Mills theory, we study quarkonium, and “generalized quarkonium” containing light adjoint particles. At large 't Hooft coupling the states of spin $\leq 1$ are anomalously light (Kruczenski et al., hep-th/0304032). We examine their form factors, and show these hadrons are unlike any known in QCD. By a traditional yardstick they appear infinite in size (as with strings in flat space) but we show that this is a failure of the yardstick. All of the hadrons are actually of finite size $\sim \sqrt{g^2 N/m_Q}$, regardless of their radial excitation level and of how many valence adjoint particles they contain. Certain form factors for spin-1 quarkonia vanish in the large-$g^2 N$ limit; thus these hadrons resemble neither the observed $J/\Psi$ quarkonium states nor $\rho$ mesons.

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1 Introduction

The discovery that gauge theory at large ’t Hooft coupling can be described using string theory [?] has given rise to many interesting developments in both fields. The original idea has been extended to purely four-dimensional confining gauge theories [?, ?, ?]. These examples provide the first toy models of QCD whose kinematics is that of the real world. In particular, these theories share with QCD the property of four-dimensional approximate scale invariance in the ultraviolet, as well as confinement in the infrared. Consequently, the differences between these models and QCD are largely due to dynamical, rather than kinematical, issues. Indeed, these theories smoothly become QCD-like when the ’t Hooft coupling is small.

However, most of the theories studied up to now differ from QCD in a significant way: their matter comes in representations with of order $N^2$ fields, and they are all neutral under some portion of the center of the gauge group. These differences from QCD are much more important, both kinematically and dynamically, than the fact that many of these theories are supersymmetric. The reasons for this are clear. Quarks in the $N$ representation of $SU(N)$ color introduce new features into the $1/N$ expansion; quark pair-production makes confining flux tubes unstable; baryons now appear in the spectrum; and there are new flavor symmetries which may or may not be explicitly or spontaneously broken. Thus the theory without quarks in the $N$ representation, supersymmetric or not, differs significantly from the theory which contains them. It is thus important to the development of these toy models of QCD to introduce matter in the fundamental representation. In the limit where the number of flavors $N_f$ is much less than $N$, this was considered in [?, ?, ?]; a simpler method was invented in [?] and studied further in [?, ?]. Still more recently, there has been additional work in other related contexts [?].

One of the obvious objects to study in a theory with quarks is quarkonium. At small ’t Hooft coupling this is just a hydrogenic atom, but at large ’t Hooft coupling the system is highly relativistic. The quarkonium spectrum at large ’t Hooft coupling, in the model studied in [?], is remarkable from the field-theoretic point of view. Its details could not have been guessed from any known theoretical argument or from any aspect of the observed QCD spectrum, and it is profoundly tied up with the representation of the four-dimensional gauge theory as a higher-dimensional string theory. Still more puzzles emerge in states built from a quark, an antiquark, and one or more particles of the much lighter adjoint matter. These states would naively be expected to be qualitatively different in size and structure from pure quarkonium. Instead, it has been found [?] that these states are quite similar to pure quarkonium at large ’t Hooft coupling.
In order to gain better insight into the structure and couplings of these bound states, we have computed some of their form factors and transition matrix elements with respect to various conserved currents. Fourier transforming the form factors, we find that all the hadrons are more or less the same size — larger than one would expect for quarkonium, and smaller than one would expect when adjoint matter is bound to the quark-antiquark system. Our results do not solve any mysteries, but they do raise interesting questions and suggest other calculations to do in future. We are also led to a conjecture about the substructure of the hadrons.

We review the results of [?] and [?] in the following section. Sections 3, 4 and 5 contain our methodology, a summary of our results, and the detailed computations. The physical implications of our results are discussed in Section 6.

2 Preliminaries

The introduction of matter in the fundamental representation into theories with gravitational dual descriptions has been considered by a number of authors [?, ?, ?]. However, theoretical prejudices about the appropriate systems, and technical difficulties with those that were investigated, delayed progress for some time. Recently, Karch and Katz [?] cut the Gordion knot, pointing out that many interesting questions could be addressed in the simplest possible brane construction with fundamental matter: a small number $N_f$ of D7 branes in the vicinity of a large number $N$ of D3 branes.

2.1 The theory in question

The field theory corresponding to this arrangement of branes consists of $\mathcal{N} = 4$ $SU(N)$ Yang-Mills coupled in an $\mathcal{N} = 2$ supersymmetric fashion to $N_f$ hypermultiplets in the fundamental representation. We will write the $\mathcal{N} = 4$ vector multiplet as an $\mathcal{N} = 1$ vector multiplet $W_\alpha$ and three chiral multiplets $\Phi_1, \Phi_2, \Phi_3$ in the adjoint representation. The hypermultiplets are given as $\mathcal{N} = 1$ chiral multiplets $Q^r, \tilde{Q}^r$ ($r = 1, \ldots, N_f$) in the fundamental and antifundamental representation respectively, and we will call their scalars “squarks” and “antisquarks” and their fermions “quarks” and “antiquarks.” Written in $\mathcal{N} = 1$ language, the theory consists of kinetic terms for all the fields (along with $\mathcal{N} = 1$ superpartners of the kinetic terms) and a superpotential of the form

$$W = \sqrt{2} \text{tr} \left( [\Phi_1, \Phi_2] \Phi_3 \right) + \sum_{r=1}^{N_f} Q^r \Phi_3 \tilde{Q}^r + m_r Q^r \tilde{Q}^r$$
where $m_r$ is the mass of hypermultiplet $r$ and the trace is over color indices.

The theory has an $SO(4) \approx SU(2) \times SU(2)$ symmetry, consisting of an $SU(2)_\Phi$ symmetry rotating $\Phi_1$ and $\Phi_2$ and an $SU(2)_R \mathcal{N} = 2$ R-symmetry. The charges of the fields under these symmetries are shown in the table, where we write $\Phi_1 = X^4 + iX^5$, $\Phi_2 = X^6 + iX^7$, and $\Phi_3 = X^8 + iX^9$, the adjoint fermions as $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$, and the quarks as $\psi$, $\tilde{\psi}$.

<table>
<thead>
<tr>
<th>Field</th>
<th>$SU(2)_\Phi \times SU(2)_R$</th>
<th>$U(1)_R$</th>
<th>$U(N_f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^4$, $\ldots$, $X^9$</td>
<td>$(\frac{1}{2}, \frac{1}{2})$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$</td>
<td>$(0, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$</td>
<td>$\pm 2$, $-1$, $+1$</td>
<td>1</td>
</tr>
<tr>
<td>$Q, \tilde{Q}$</td>
<td>$(0, \frac{1}{2})$</td>
<td>0</td>
<td>$N_f$</td>
</tr>
<tr>
<td>$\psi, \tilde{\psi}$</td>
<td>$(0, 0)$</td>
<td>$\mp 1$</td>
<td>$N_f$</td>
</tr>
</tbody>
</table>

The masses $m_r$ are the eigenvalues of a matrix which transforms in the adjoint of $U(N_f)$. If all masses $m_r$ are zero, the $U(1)_R \mathcal{N} = 1$ R-symmetry is preserved by the superpotential. Nonzero masses break this $U(1)$ R-symmetry, but leave invariant the $SO(4)$. The $U(N_f)$ is also generally broken by the masses, but if all the masses are equal, as we will assume throughout, $U(N_f)$ is preserved.

The string dual description of this field theory \cite{footnote1}, for large ’t Hooft coupling\footnote{We use the following notation: $g_{YM}$ is the $\mathcal{N} = 4$ gauge coupling, $\alpha = g_{YM}^2/4\pi$, $\lambda = g^2N$ is the ’t Hooft coupling, $g_s$ is the type IIB closed string coupling, and the couplings are related under gauge/string duality by $g_s = \alpha$.}, is simply given as $N_f$ D7 branes placed as probes inside $AdS_5 \times S^5$. Each D7 brane introduces a hypermultiplet of mass $m$, where $m$ is proportional to the minimum value of the $AdS_5$ radial coordinate $r$ to which the D7 brane descends. At any larger and fixed value of $r$, the D7 brane fills out an $S^3$ subspace of the $S^5$. We will give more details on this brane construction below.

The theory’s reduced supersymmetry is of little concern, but it does seem at first to have a serious dynamical problem. The gauge coupling now has a positive beta function, making the ultraviolet definition of the theory problematic, and potentially destroying any hope of making sense of the theory using gauge-string duality. In particular, the dilaton is no longer constant. Where the dilaton (equivalently, the gauge coupling of the gauge theory) becomes of order 1, one ought to do an S-duality transformation; but this would turn the D7-branes into magnetic 7-branes, which are
very difficult to handle.

However, the gauge beta function is very small. In particular, the beta function for $g_{YM}$ is of order $g^3_{YM}N_f$, so the beta function for the 't Hooft coupling $\lambda$ is of order $\lambda^2 N_f/N$. Another way to say this is that at large $N$ and small $N_f$ this is a naturally “quenched” theory: all quantum effects due to the fundamental matter are parametrically suppressed by $N_f/N$. The squarks and quarks are simply probes of the dynamics of the $\mathcal{N} = 4$ sector. From the supergravity point of view, this suppression is essentially the statement that the backreaction, through the variation of the dilaton, metric and other fields, caused by the D7-branes is negligible out to exponentially large $AdS$ radius. Consequently, as Karch and Katz argued, we can study to leading order in $N_f/N$ all of the issues that can normally be investigated in quenched QCD, including the spectrum of hadrons.

2.2 The (s)quark-anti(s)quark hadrons

Karch, Katz and Weiner [?] considered the formation of heavy-light mesons in this theory, working at large 't Hooft coupling $\lambda \gg 1$. They found a number of surprising results, including confinement of quarks without formation of flux tubes (what one might call “Gribov confinement.” [?])

Another natural issue for study is quarkonium, namely bound states of a quark and antiquark of equal mass. At weak coupling (i.e., small $\lambda$) this is simply the problem of the hydrogen atom, but at strong coupling we do not a priori know what to expect.

A seemingly different problem is that of a quark and antiquark bound to one or more of the $\Phi_i$ particles. In the simplest D3-D7 system, these particles are massless. An easier system to think about is obtained by giving the $\Phi_i$ small bare masses, so that (as in $\mathcal{N} = 1^*$[?]) confinement of flux occurs at a distance very long compared to the quarkonium mass and length scales. Then we can imagine stable bound states of a quark, antiquark and some number of the much lighter $\Phi_i$ particles. This problem has not been much studied at small 't Hooft coupling, but one would expect these bound states, in which the quark and antiquark combined must be in the adjoint representation, would be very different in size and shape from the simple quarkonium bound states.

Surprisingly, at large 't Hooft coupling both problems can be addressed simultaneously, and on the same footing. This was done by Kruczenski, Mateos, Myers and Winters [?]. They derived the spectrum and higher-dimensional wave functions of mesons consisting of one (s)quark and one anti(s)quark of equal mass, along with some number of massless adjoint particles. The spectrum for states of large spin
was found to be the same as that of hydrogen; this is to be expected, since in this limit the (s)quarks move nonrelativistically even though the coupling is strong. For states of lower spin it was shown that the spectrum has a Regge-like relation between mass and spin: \( m^2 \propto s \). And states of spin 1, \( \frac{1}{2} \), and 0 were found to be extraordinarily deeply bound — so deep that the mass of the hadron is a tiny fraction of the mass of its (s)quark constituents. It is these anomalously light states of low spin — states that are described in the string theory variables by modes of massless higher-dimensional fields — that are the focus of this article. We now review their properties in detail.

We generally follow the setup in [?]. The near-horizon geometry of \( N \) D3-branes filling the 0123 directions of the ten-dimensional space, and placed at the origin of the 456789 coordinates, is given by \( AdS_5 \times S^5 \). We write the metric on the Poincare’ patch variously as

\[
d s^2 = r^2 R^2 (d\eta^\mu d\eta_{\nu} + \sum_{c=4}^{9} R^2 r^2 (dx^c)^2) \]

where \( c = 4, 5, 6, 7, 8, 9, \mu = 0, 1, 2, 3, r^2 = \sum_{c} (x^c)^2, \) and \( R^2 = \alpha' \sqrt{4\pi g_s N} \). Note \( r = 0 \) is the horizon of \( AdS \) and \( r = \infty \) is its boundary. The D7-branes fill the 01234567 directions; each is placed at some position \( x^8 + ix^9 = m_Q \alpha' \). We will make all the masses equal in this paper, so without loss of generality we can take the masses to be real \( (x^8 = m_Q \alpha' \equiv L, x^9 = 0.) \) The induced metric on the D7-branes is then

\[
d s^2 = r^2 R^2 \eta_{\mu\nu} dx^\mu dx^\nu + \sum_{c=4}^{7} R^2 r^2 (dx^c)^2
= \frac{L^2}{R^2} (g^2 + 1) \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{L^2} \frac{1}{g^2 + 1} d\varrho^2 + R^2 \frac{g^2}{g^2 + 1} d\Omega^2_3, \tag{1}
\]

where \( g^2 = \frac{L^2}{R^2} - 1 \), and the \( S^3 \) involves the angular coordinates in the four-dimensional space spanned by \( x^4, x^5, x^6, x^7 \).

We will use various coordinates for our calculations which we summarize here:

It is useful to define an overall mass scale

\[
m_h \equiv \frac{L}{R^2} = \frac{m_Q}{\sqrt{4\pi g_s N}} = \frac{m_Q}{\sqrt{\lambda}}.
\]

\(^2\)We use \(- + ++\) signature.
Barred quantities will be defined relative to $m_h$ — for instance, we will often use dimensionless momenta $\bar{q} \equiv \sqrt{|q^2|}/m_h$ and dimensionless masses $\bar{m} = m/m_h$.

There are two real scalar fields and one gauge field on the D7 brane worldvolume;theses are the massless modes of the open strings whose ends are on the D7-branes.\footnote{Note these 7-7 strings are not dynamical in the gauge theory! In the Maldacena limit of the D3-D7 system, the 3-3 strings (the $\mathcal{N} = 4$ sector) and the 3-7 strings (the hypermultiplets) are dynamical, whereas the closed strings and 7-7 strings act as background fields. In the dual string description, the 3-3 and 3-7 strings are absent, and the closed and 7-7 strings are the dynamical degrees of freedom.}

If there are $N_f$ D7-branes, there are $N_f^2$ such fields, transforming in the adjoint representation of $U(N_f)$. The modes of the gauge field, when reduced to five dimensions, break up into spin-one and spin-zero modes in five dimensions. The spin-one modes are associated with the $U(N_f)$ flavor current, and operators built by adding $\Phi$ fields to the current:

When these operators act on the vacuum they can create a spin-one bound state of a $Q$, a $\bar{Q}$, and $\ell$ $\Phi$ particles (plus any number of gluons, gluino pairs, and $\Phi$–anti-$\Phi$ pairs, of course.)\footnote{They cannot, in this theory, create a spin-zero state; this is not a general feature of the large $\text{'t Hooft}$ coupling limit, however.} In particular, for $\ell = 0$ this is a spin-one quarkonium state, much the same as the $J/\psi$ or the $\Upsilon$ in QCD. The bound state spectrum, and the bulk wave functions of the individual bound states, were computed in [?], where they were termed “type I $I$”. These hadrons are modes of the 8-dimensional gauge boson $A_M$, of the form

Another mode which is easy to identify is the mode $I_-$ in [?], whose conformal dimension is $\Delta_0 \equiv \ell + 1$ ($\ell \geq 1$), and which transforms as $\left(\frac{\ell-1}{2}, \frac{\ell+1}{2}\right)$ under $SO(4) \cong SU(2)_\Phi \times SU(2)_R$. For $\ell = 1$ it has dimension 2 and is a triplet under $SU(2)_R$. This uniquely picks it out as a multiplet containing the $\mathcal{N} = 1$ chiral operator:

These are the two classes of (s)quark-containing operators and states that we will use for calculations in the rest of this paper. We will also consider matrix elements of the flavor current, which is the operator appearing in Eq. (??) with $\ell = 0$. This will require us to know the nonnormalizable mode of the corresponding five-dimensional gauge boson at spacelike $q^2 = \bar{q}^2 m_h^2$, which is of the form...
For completeness we briefly comment about the other classes of hadrons appearing in $\mathcal{N}$. There is a set of complex operators $\psi_\tilde{Q} \Phi^\ell-1 \psi_Q + \cdots$ which are obtained from the operators (??) by the action of two supersymmetry generators; these complex modes create the two “scalar modes” in $\mathcal{N}$. The type $III$ and type $I+$ hadrons are harder to identify, as there are a number of candidate operators, of which only particular linear combinations are chiral.\(^5\)

2.3 The $\mathcal{N} = 4$ sector

We will also consider certain modes of the $\mathcal{N} = 4$ sector, which appear as states of the ten-dimensional supergravity on $AdS_5 \times S^5$. We will only need the spin-one states created by acting on the vacuum with the $SO(4)$ current $j_4^\mu \sim \text{tr} \Phi_i^\dagger \partial_\mu \Phi_j (T_{SO(4)})_{ij} + \cdots$ and the spin-two states given by acting on the vacuum with $T^{\mu\nu} \sim \text{tr} \ F_\mu^\rho F^\rho_\nu + \Phi_i^\dagger \partial_\mu \partial_\nu \Phi + \cdots$. (The traces are over color indices.) However, it will be convenient to consider also a spin zero operators which has a close relationship with the spin one and two operators. The spin zero operator corresponds to the ten-dimensional metric element which is a singlet under the transformation on the seven brane. Under the breaking $SO(6) \to SO(4) \times SO(2)$, the traceless symmetric rank two tensor representation $20$ branches to $(1,1) \oplus (9,1) \oplus (1,2) \oplus (4,2)$. The singlet corresponds to the dimension-two $SO(4)$–singlet operator $S \sim \Phi^2_1 + \Phi^2_2 - 2\Phi^2_3$.

2.3.1 The continuous spectrum of the conformal theory

We will consider the operators whose spins range from zero to two. The non-normalizable mode for the operator with spin $S$ is $[?]

The $SO(4)$ current is associated under AdS/CFT duality with a five-dimensional gauge boson $\hat{A}$, which descends from the off-diagonal elements of the ten-dimensional metric. In particular, let $v^a$ be a Killing vector on the three-sphere which points purely parallel to the D7-brane world-volume (and thus leaves it invariant.) Then the metric elements $h_{\mu a} = \hat{A}_\mu v_a, h_{ra} = \hat{A}_r v_a$ (where $ds^2 = ds^2_{AdS_5 \times S^5} + h_{\mu a} dx^\mu dx^a$) define the $SO(4)$ gauge boson on $AdS_5$. In radial gauge $\hat{A}_r = 0$, the non-normalizable modes of $\hat{A}$ at spacelike $q^2$ are

\(^5\)Note there are no chiral operators of protected dimension containing $\Phi_3$ and $Q$ together, because the form of the superpotential implies $\Phi_3 Q$ and $Q \Phi_3$ are simply proportional to $Q$ and $Q$, respectively, within the chiral ring.
For the spin-two current we need the modes of the traceless $AdS_5$ part of the metric. We consider traceless fluctuations $ds^2 = ds^2_{AdS_5 \times S^5} + h_{\mu\nu}dx^\mu dx^\nu$, $h_\mu^\mu = 0$, in radial gauge $h_{rr} = 0$, $h_{r\mu} = 0$. The normalizable modes take the form

$$ds^2 = ds^2_{AdS_5 \times S^5} + h_{\mu\nu}dx^\mu dx^\nu,$$

Since the $\mathcal{N} = 4$ theory is conformal in the infrared, its spectrum of normalizable modes is continuous: any timelike $q^2$ is allowed. We should not think of the normalizable modes as hadrons, but simply as a spectral decomposition of a conformal field.

### 2.3.2 The discrete spectrum of hadrons of a confining model

However, it will often be useful to consider instead a confining theory, in which the conformal invariance of the $\mathcal{N} = 4$ sector is broken in the deep infrared. For instance, we can imagine the $\mathcal{N} = 4$ sector is deformed into a theory similar to $\mathcal{N} = 1^*$ [?], where the $\mathcal{N} = 1$ chiral superfields $\Phi_1, \Phi_2, \Phi_3$ are given a small mass. This causes confinement to set in at a low scale $\Lambda$. In all known models of this type, the $AdS$ radial coordinate is effectively cut off at $r = r_{min} = \Lambda R^2$. Our results will be insensitive to the details of this cutoff, as will become clear, so a crude model will suffice to capture the essence of the physics. In any such model, the flavorless $\mathcal{N} = 4$ sector of the theory will have a discrete spectrum of spin-$s$ ($s \leq 2$) hadrons, with masses of order $n\pi\Lambda$ ($0 < n \in \mathbb{Z}$), and with mode functions given at large $r$ by the normalizable modes of the corresponding bulk fields.

To be definite, we will model confinement through the (arbitrarily chosen) boundary condition that the ten-dimensional wave function of each hadron vanishes at $r = r_{min}$, that is, at $v_{max} = \frac{m_s^2}{\Lambda^2}$. In this simple model, the wave functions for the hadrons created by the $SO(4)$ current are precisely those of (??) for $v < v_{max}$, with a quantization condition that only $q^\mu$ satisfying $J_1(\bar{m}\sqrt{v_{max}}) = 0$, where again $\bar{m} = \sqrt{|q^2|}/m_h$, are allowed. Thus we have a countably infinite series of modes, with mass $m_n = \bar{m}\sqrt{v_{max}} = \zeta_{1,n}$, $n \geq 1$, where $\zeta_{1,n}$ is the $n^{th}$ zero of $J_1$. The normalization is obtained from [?]

Similarly, the spin-two hadrons created by $T^{\mu\nu}$ have wave functions given by
3 Methodology

3.1 Definitions and Notation

We will concern ourselves mainly with the form factors from the hadronic matrix elements of the \( U(N_f) \) flavor current \( J_f^\mu \), the \( SO(4) \approx SU(2)_\Phi \times SU(2)_R \) current \( J_4^\mu \), and the energy-momentum tensor \( T^{\mu\nu} \). We will also consider matrix elements of the \( SO(4) \)-singlet spin-zero dimension-two operator \( S \propto \Phi^2_1 + \Phi^2_2 - 2\Phi^2_3 \), which is part of the conformal \( \mathcal{N} = 4 \) sector.

We will compute a large number of form factors, so it is important that our notation be clear. When the maximal information needs to be displayed, we will use the following notation for a form factor. Suppose we have a spin-zero (type \( I \)) or a spin-one (type \( II \)) initial state with \( \ell, n_1 \) quantum numbers. In our calculations the final state will always be of the same type and share the same \( \ell \), though \( n_2 \neq n_1 \) in general. The operator whose matrix element we are computing will be referred to by an index \( S = 0, 1, 2 \) for the spin-0,1,2 operators \( S, \; J_4^\mu, \) and \( T^{\mu\nu} \), and by an index \( f \) for the flavor current \( J_f^\mu \). The maximal notation will therefore be

For scalar hadrons, the matrix elements of the spin-one \( SO(4) \) current can be written

\[
\langle n_2, \ell, p', \zeta' | J_4^\mu | n_1, \ell, p, \zeta \rangle = \frac{1}{m^2} (p \cdot \zeta')(p' \cdot \zeta)(p' + p)\nu (\eta^{\mu\nu} - q^{\mu}q^{\nu}/q^2)F_q(q^2) \times (2\pi)^4 \delta^4(\sum_i p_i),
\]

where \( \epsilon, \zeta, \) and \( \zeta' \) are the polarization vectors of the current, in-state, and out-state respectively, and \( F_e, F_m, \) and \( F_q \) are electric, magnetic, and quadrupole form factors.
with all other indices suppressed. For reasons to be discussed later, in the supergravity limit \( F_e = F_m \) and \( F_q = 0 \), so our notation will usually be to drop the subscript except when necessary, writing \( F_{n_1,n_2}^{(1)} \) for the \( SO(4) \) electric form factor and replacing 1 with \( f \) for the flavor current.

The formulas for the spin-two current are similar, although more complicated. Again we will find there is only one non-vanishing form factor both for spin-zero and spin-one hadrons, which we will label \( F_{n_1,n_2}^{(2)} \) when necessary.

### 3.2 Determining the shape of the hadrons

Initially we will compute the form factors in momentum space, where they are Lorentz-invariant functions of \( q^2 \). To obtain information that is easier to interpret intuitively, we would like to reexpress the form factors in position space. There are ambiguities in how this is to be done, and problems which might arise at timelike \( q^2 \) where there are poles.

A four-dimensional Fourier transform of the form factor has the feature that it is Lorentz invariant. However this function does not have a well-known physical interpretation. Moreover one must consider large timelike \( q^2 \) where certain difficulties with the supergravity approximation will arise.

A three-dimensional Fourier transform of the form factor at spacelike \( q^2 \) is useful for nonrelativistic systems. For a two-body nonrelativistic bound state, this quantity is the square of the wave function, and a similarly simple interpretation applies for many body system: for a spin-one current, it gives the three-dimensional distribution of the corresponding charge. But our bound states are highly relativistic (since their binding energy is so large), and (for small \( \ell, n \)) their form factors are large even when \( \sqrt{|q^2|} \) is of order the hadron mass. We are therefore not confident that this interpretation extends straightforwardly to our case.

By contrast, the two-dimensional Fourier transform of \( F(q^2) \) (for spacelike \( q^2 \)) does have an interpretation which is applicable for relativistic systems. For a hadron moving with extremely high momentum in the \( z \) direction, and choosing \( q^a = (0, \vec{q}_\perp, 0) \) in the \( x - y \) plane, we define

\[
\tilde{F}(\vec{x}_\perp) = 12\pi \int d^2q_\perp \ e^{i\vec{q}_\perp\cdot\vec{x}_\perp} F(q^2)
\]

This function is the hadron’s two-dimensional “transverse charge distribution” \( \rho_\perp(\vec{x}_\perp) \) in the \( x - y \) plane, times \( 2\pi \) in our conventions. The usefulness of this interpretation stems from its connection with generalized parton distributions [?, ?, ?], as shown
with considerable care and rigor in [?]. This applies in our case even though the (s)quarks have large masses, as long as the hadron is ultrarelativistic compared to the quark mass scale.\(^6\)

In some cases we will find elegant closed-form expressions for \(\tilde{F}(x_\perp^2)\). In others we can still compute many of their general properties: the large- and small-\(q^2\) behavior of \(F(q^2)\), the large- and small-\(x_\perp^2\) behavior of its Fourier transform, and some characteristic measures of hadron shape and size.

One classic measure of the size of a hadron is given by the second moment \(\langle r^2 \rangle\) of the transverse charge distribution. For any given form-factor, we can compute this moment using

\[3.3 \quad \text{Calculational techniques}\]

The required calculational techniques are well-established and straightforward; see for example [?]. Each hadron we will consider is a mode of a particular five-dimensional field, which itself is a mode of an eight-dimensional field on the D7 brane or of a ten-dimensional field in the bulk. The hadrons containing a (s)quark and anti(s)quark will be of the so-called type I or type II class described in the previous section, both of which descend from the gauge bosons on the D7-brane. To compute the matrix element of a current, we need to examine the five-dimensional field whose boundary value couples to that current. For \(J_f\), this is the mode of the gauge boson on the D7-brane which has its index in spacetime and is constant on the \(S^3\). (Acting on the vacuum it creates the type II hadrons Eq. (??) with \(\ell = 0\).) For \(J_4\) the gauge boson in question is the dimensional reduction of a ten-dimensional supergravity mode; as in [?] it is associated with a Killing vector on the \(S^3 \subset S^5\). For \(T^{\mu\nu}\) we need the five-dimensional massless graviton. To compute the matrix element \(\langle \text{out}|\mathcal{O}|\text{in} \rangle\) requires knowledge of the trilinear interaction between the three five-dimensional modes corresponding to the initial hadron, the final hadron, and the operator \(\mathcal{O}\). This interaction can be derived from the Born-Infeld action on the D7 brane. In the supergravity limit (large \(\lambda\)), all of the interactions that we will require are obtained from the single term

\(^{6}\)Since the form factors are functions of \(q^2\) only, one can convert from any one of these transforms to any other, at least when restricting to spacelike \(q^2\). Our choice of the two-dimensional transform is therefore somewhat arbitrary; however it gives unambiguous and interpretable information about the structure of the hadrons.
4 Summary of our main results

In this section we summarize our results, and in the following section we present the detailed computations.

4.1 Form Factors

4.1.1 General results for all theories

We begin with some observations which apparently follow from conformal invariance and large-$N$ alone. We have not derived them from general arguments, but it should be possible, and would be interesting, to do so. In particular, it is not yet known whether the following properties are true only in the supergravity limit and thus apply only at large 't Hooft coupling.

The results below apply to any hadrons with the property that (1) their mass scale is large compared to any other scale which breaks conformal symmetry; (2) the mass scale which sets their masses breaks conformal symmetry only at order $1/N$. Such hadrons will have wave functions which solve an equation in the background of a conformal theory. If the conformal sector has conserved currents in addition to the energy-momentum tensor, then we find the leading form factors of the energy-momentum tensor are related to those of the currents, which are in turn related to those of operators of spin zero and dimension two. In our case, for scalar hadrons, the form factors for $S, J_4^\mu, T^{\mu\nu}$ satisfy

$$F_{n,n'}(1) \propto \frac{1}{q^2}.$$}

In contrast, the flavor current does not couple to the conformal sector, and is sensitive to the masses of the quarks. Its form factors fall off exponentially at large $x$. Conformal invariance at large $q^2$ requires $F^{(f)\ell}_{n,n'} \propto F^{(1)\ell}_{n,n'} \propto 1/q^{2\ell}$; but supergravity imposes a stronger condition, namely that the two form factors are actually equal at large $q$ (when they are both normalized to 1 at $q^2 = 0$.)

In general, spin-one hadrons can have three form factors under spin-one currents: electric monopole, magnetic dipole, and electric quadrupole. We find (similarly to [?]) that the anomalous magnetic dipole ($F_m - F_e$) and the quadrupole form factor ($F_q$) are zero in the large-$\lambda$ limit. The nonvanishing form factor $F_e = F_m$ has the same large-$x$ behavior as in the spin-zero case, though its large $q^2$ behavior is $1/q^{2\ell+4}$. This is true for both $SO(4)$ and flavor currents, and follows from properties of the couplings between hadrons and currents in the supergravity limit. This will be discussed elsewhere [?].
4.1.2 General results for this theory

In our particular theory, considerable simplification is obtained from the fact that the wave functions in the bulk are given by powers of $v$ and $(1 - v)$ times Jacobi polynomials. The form factors are easily analyzed in position space using the two-dimensional Fourier transform, where they are given by integrals of the form (??).

In general, we find

At large $x$, the coefficient of the potentially-leading $x^{-2(1+S)}$ term is

Some of these facts have natural momentum-space counterparts. The behavior at small $x$ — in particular the absence of a logarithm multiplying $x^{2j}$ for $j < \ell - 1$ — is associated with the requirement of conformal invariance that the momentum space form factor fall as $1/q^{2\ell}$. Similarly, the $(x^2)^{-(S+1)}$ behavior of $\hat{F}^{(S)}$ at large $x$ follows from the fact that the expansion of $F(q^2)$ near zero is as a polynomial plus $(q^2)^S \log q$.

For spin-one hadrons, Eq. (??) is unmodified, but Eq. (??) becomes

It is useful to evaluate the $SO(4)$ form factors at $x = 0$:

The flavor form factors are less amenable to such a description due to their mathematical complexity. We do not have general results beyond those of the previous section, except that all such form factors can be written as a sum over a finite number of spin-one $\ell = 0$ hadron poles:

$$F_{n_1,n_2}^{(f)\ell}(q^2) = \sum_{n=0}^{n_1+n_2+2\ell+1} \frac{c_{n,n_1,n_2}^{\ell}}{q^2 + \bar{m}_n^2},$$

where $\bar{m}_n^2 = 4(n + 1)(n + 2)$. The $-$ (+) sign in the upper limit of the summation applies for spin-zero (spin-one) external hadron states. In position space this form factor can be written as a corresponding sum of $K_0(m_nx)$ Bessel functions.
4.1.3 Ground state form factors for each $\ell$

Since $P_0^{(\alpha,\beta)} = 1$, the ground states have form factors proportional to

For spin-one, the electric form factor has a similar form, with

For the flavor case, $c_{n,0,0}^\ell$ in Eq. (3) is given in Eq. (8) for spin zero hadrons and in Eq. (??) for spin one hadrons. The small-$x$ behavior of $\tilde{F}^{(f)}$ is similar to that of $\tilde{F}^{(1)}$.

4.1.4 Diagonal form factors at large $\ell$

Remarkably, although these formulas become complicated at large $\ell$, an alternative and surprisingly simple formula can be used instead. We find that for both spin zero and spin one hadrons,

For the flavor case, the large $\ell$ limit of $c_{n,0,0}^\ell$ is given in Eqs. (9) and (??) for spin-zero and spin-one hadrons respectively.

4.1.5 Diagonal form factors at large $n$

At large $n$, a formula can be obtained which is valid for sufficiently large $x$.

For the flavor case, $c_{n,\infty,\infty}^\ell$ for both spin zero and spin one hadrons is given in Eq. (10).

4.1.6 Results for some off-diagonal matrix elements

The conformal invariance of the $\mathcal{N} = 4$ sector and the derivative relations between the Jacobi polynomials imply some additional relations for off-diagonal matrix elements between the ground state and an excited state at a given $\ell$.

$$\tilde{F}^{(0)}_{0,n_2}(x) = \frac{\tilde{C}_{\ell n_2}^I \tilde{C}_{\ell 0}^I}{n_2!(\tilde{C}_{\ell+n_2,0}^I)^2} \left( \frac{d}{dx^2} \right)^{n_2} \tilde{F}^{(0)\ell+n_2}_{0,0}(x).$$

From this expression the form factors for $S = 1, 2$ can be obtained from our general result (??). The same result holds for spin-one hadrons with $\tilde{C}^I$ replaced with $\tilde{C}^{II}$. 

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4.2 The transverse sizes of the hadrons

Armed with this information we can compute certain moments, in particular \(<r> = m_h^{-1}\langle x\rangle\) and \(<r^2> = m_h^{-2}\langle x^2\rangle\), with respect to the flavor, \(SO(4)\) and energy-momentum distributions of these hadrons; we use subscripts \(f, 1, 2\) for moments of the corresponding form factors \(F^{(f)}, F^{(1)}, F^{(2)}\). Where these moments are infinite we regulate them using finite \(\Lambda\) (see Sec. 2.3.2); where they are finite we set \(\Lambda \to 0\).

4.2.1 The ground states \(n = 0\) at general \(\ell\)

For every \(\ell\), we can compute properties of the ground state \((n = 0)\). For the spin-zero states, we find

\[
\langle r_1 \rangle = 2\langle r_2 \rangle = \frac{\pi}{2m_h} \cdot \frac{\Gamma(\ell + \frac{1}{2})\Gamma(2\ell + 2)}{\Gamma(\ell)\Gamma(2\ell + \frac{3}{2})};
\]

\[
\langle r_1^2 \rangle = \frac{1}{m_h^2} \cdot \frac{\ell}{\ell + 1} \left\{ \log \left( \frac{m_h}{\Lambda} \right) \right\};
\]

\[
\langle r_2 \rangle = \frac{1}{2m_h} \cdot \frac{\ell}{\ell + 1};
\]

\[
\frac{0.496\ldots}{m_h} \approx \langle r_f \rangle_{\ell=1} \leq \langle r_f \rangle < \langle r_f \rangle_{\ell=\infty} \approx \frac{0.697\ldots}{m_h};
\]

\[
\langle r_f^2 \rangle = \frac{1}{m_h^2} \cdot \frac{\ell}{\ell + 1} \left[ H_{2\ell+1} - H_{\ell} \right],
\]

where \(H_{\ell} = 1 + 1/2 + 1/3 + \cdots + 1/\ell\) is the \(\ell\)-th harmonic number.

For the spin-one states, the formulas are similar:

\[
\langle r_1 \rangle = 2\langle r_2 \rangle = \frac{\pi}{2m_h} \cdot \frac{\Gamma(\ell + \frac{3}{2})\Gamma(2\ell + 4)}{\Gamma(\ell + 2)\Gamma(2\ell + \frac{7}{2})};
\]

\[
\langle r_2 \rangle = \frac{1}{2m_h} \cdot \frac{\ell}{\ell + 1};
\]
\[ \langle r_1^2 \rangle = \frac{1}{m_h^2} \left\{ \log \left( \frac{m_h}{\Lambda} \right) \right\}; \]

\[ \langle r_2^2 \rangle = \frac{1}{2m_h^2}; \]

\[ \frac{0.672 \ldots}{m_h} \approx \langle r_f \rangle_{\ell=0} \leq \langle r_f \rangle_{\ell=\infty} \approx \frac{0.697 \ldots}{m_h}; \]

\[ \langle r_f^2 \rangle = \frac{1}{m_h^2} \frac{\ell + 2}{\ell + 1} \left[ H_{2\ell+3} - H_{\ell+2} \right]. \]

4.2.2 The ground states at large \( \ell \)

For \( n = 0, \ell \to \infty \), for both spin-zero and spin-one hadrons,

\[ \langle r_1 \rangle = 2\langle r_2 \rangle \longrightarrow \frac{\pi}{2\sqrt{2}m_h}; \]

\[ \langle r_1^2 \rangle \longrightarrow \frac{1}{m_h^2} \left\{ \log \left( \frac{m_h}{\Lambda} \right) \right\}; \]

\[ \langle r_2 \rangle \longrightarrow \frac{1}{2m_h^2}; \]

\[ \langle r_f \rangle \longrightarrow 0.697 \ldots m_h; \]

\[ \langle r_f^2 \rangle \longrightarrow \log 2m_h^2. \]
4.2.3 The small-\(\ell\) states at large \(n\)

For both spin-zero and spin-one hadrons, the limit \(n \to \infty\) at small and fixed \(\ell\) gives

\[
\langle r_1 \rangle = 2\langle r_2 \rangle \to \frac{1}{m_h};
\]

\[
\langle r_1^2 \rangle \to \frac{1}{m_h^2}\log\left(\frac{m_h}{\Lambda}\right);
\]

\[
\langle r_2^2 \rangle \to \frac{1}{2m_h^2};
\]

\[
\langle r_f \rangle \to \frac{0.61\ldots}{m_h};
\]

\[
\langle r_f^2 \rangle \to 2(1 - \log 2)m_h^2.
\]

5 Computations

In this section we derive the results outlined in the previous section.

5.1 Spin zero hadrons

We now proceed to study the spin-zero (type \(I\)) hadrons created by the operators \(\tilde{Q}\Phi^{\ell-1}Q\) (and other members of the same \(SO(4)\) multiplet.)

5.1.1 Flavor current

In order to study the (s)quark-anti(s)quark hadrons using the matrix elements of the flavor current, we need to consider a situation with \(N_f > 1\). The hadrons in question transform in the adjoint of \(U(N_f)\), so for \(N_f = 1\) they are neutral under the flavor current. We will consider \(N_f\) hypermultiplets of equal mass, which leaves the \(U(N_f)\) unbroken. The modes described in [?], which only depend on the quadratic terms
in the D7-brane gauge fields, are unchanged for \( N_f > 1 \), except for transforming under a nontrivial representation of flavor. However, the cubic interactions among the D7-brane gauge bosons give the main contribution to the flavor-current matrix elements.

For the coupling of the flavor current to two spin-zero hadrons, the important term in the D7-brane Born-Infeld action is

\[
g_8 \int d^8x \sqrt{-g} g^{\alpha \beta} g^{\mu \nu} f_{abc} A_\mu^a A_\nu^b \partial_\alpha A_\beta^c,
\]

where \( f_{abc} \) is the structure constant of the group \( SU(N_f) \) and we have rescaled the gauge fields \( A^M \rightarrow g_8 A^M \) to obtain this form. In this form, the boundary \((r = \infty)\) value of the non-normalizable mode times the coupling \( g_8 \) is set to unity.

Substituting the non-normalizable mode for \( A_\mu \), and normalizable modes for the mesons, we obtain the matrix element:

\[
\langle a; \ell, n_2; p' | J^b_\mu(q) | c; \ell, n_1; p \rangle = i f_{abc} (p + p')_\nu (\eta^{\mu \nu} - q^\mu q^\nu / q^2) (2\pi)^4 \delta^4 \left( \sum_i p_i \right) F_{\ell n_1, n_2}^\ell (\bar{q}^2),
\]

\[
F_{n_1, n_2}^\ell (\bar{q}^2) = \frac{L^2}{2} \int_0^1 dw (1 - w)^{-2} \phi_{II, \text{non}}^\ell (w; \alpha) \phi_{\ell, n_1}^\ell (w) \phi_{\ell, n_2}^\ell (w).
\]  

The above integral can be done by partial integration, using the modes in Eq. (??) and Eq. (??). The normalizable mode is a polynomial in \( w \) and \( \int dw F(a + 1, b + 1; c + 1; w) = \frac{w}{ab} F(a, b; c; w) \). By integrating the non-normalizable mode and differentiating the product of normalizable modes repeatedly, this integral can be evaluated. The general form of the form factor obtained in this way is

\[
F_{n_1, n_2}^\ell (\bar{q}^2) = \sum_{n=0}^{n_1 + n_2 + 2\ell - 1} \frac{C_{n,n_1, n_2}^\ell}{\bar{q}^2 + \bar{m}_n^2},
\]

where \( \bar{m}_n^2 = 4(n+1)(n+2) \) is the squared mass of the \( n \)-th vector meson with \( \ell = 0 \), and \( C_{n,n_1, n_2}^\ell \) is a constant independent of \( \bar{q} \).

For the general case, we can get the large \( \bar{q}^2 \) behavior in the following way: Remembering that we evaluate the integral by repeated partial integration, the \( \ell \)-times integration of \( \phi_{II, \text{non}}^\ell (w; \alpha) \) gives a term with its leading behavior \( 4^\ell \ell! / (\bar{q}^2)^\ell \) for large \( \bar{q}^2 \) and the function we need to differentiate repeatedly has the structure

\[
L^2 (1 - w)^2 \phi_{\ell, n_1}^\ell (w) \phi_{\ell, n_2}^\ell (w) = \frac{\ell (\ell + 1) C_{\ell n_1}^\ell}{(n_1 + \ell)(n_1 + \ell + 1)} \frac{\ell (\ell + 1) C_{\ell n_2}^\ell}{(n_2 + \ell)(n_2 + \ell + 1)} \bar{q}^{\ell + 1} (1 - w)^{\ell - 1}
\]
plus corrections suppressed by additional factors of $(1 - w)$. Therefore, the first $\ell - 1$ terms obtained by partial integrations vanish and the leading large $\bar{q}^2$ behavior is given by

It is possible to compute the Fourier transformation of the form factor. Using the 2-d Fourier transformation

$$FT\left(\frac{1}{q^2 + m^2}\right) = K_0(mx),$$

we get

$$\tilde{F}_{n_1,n_2}(x) = m_h^2 \sum_{n=0}^{n_1 + n_2 + 2\ell - 1} c_{n,n_1,n_2}^\ell K_0(m_n x)$$

$$\rightarrow \quad m_h^2 c_{\hat{n},n_1,n_2}^\ell \frac{\sqrt{\pi}}{2m_{\hat{n}}x} e^{-m_{\hat{n}}x}, \quad (6)$$

$$\rightarrow \quad m_h^2 \sum_{n=0}^{n_1 + n_2 + 2\ell - 1} c_{n,n_1,n_2}^\ell \sum_{k=0}^{\infty} \left[-\log\left(\frac{m_n x}{2}\right) + \psi(k + 1)\right] \left(\frac{m_n x}{2}\right)^{2k} \left(\frac{2}{k!}\right)^2, \quad (7)$$

where $\hat{n}$ is the smallest value of $n$ with nonvanishing $c_{n,n_1,n_2}^\ell$. If $\sum c_{n,n_1,n_2}^\ell = 0$, the logarithmic divergence for $x \to 0$ vanishes; as can be seen from Eq. (5), this is also the condition that the coefficient of $1/\bar{q}^2$ at large $\bar{q}^2$ vanishes. We know this must occur for $\ell > 1$. In general, from the fact that the form factor falls as $1/\bar{q}^{2\ell}$, we can deduce that $\sum c_{n,n_1,n_2}^\ell (\bar{m}_n)^{2j} = 0$ for $j = 0, 1, \cdots, \ell - 2$, and thus the leading non-analytic term in the small-$x$ expansion is $-\left[x^{2(\ell - 1)} + \cdots\right] \log x$.

In principle, we can get all $c_{n,n_1,n_2}^\ell$ by partial integrations, but it’s difficult to get a closed form for $c_{n,n_1,n_2}^\ell$ in general. In the case of $n_1 = n_2 = 0$, we can get a closed
form\(^7\) which is given by

\[
c_{\ell,n,0,0} = C_{\ell,0,0}^2 C_{\ell,0,0}^2 B(\ell + 1, n + \ell + 2) F_2(-n, -n - 1, \ell + 1; 2, -n - \ell - 1; 1)
\]

\[
= \begin{cases} 
(1)^{\ell+1} \frac{4(2n+3)(n+1)!}{((n/2)!)^2} \frac{(1+2\ell)!((\ell+n)/2)!}{(\ell+1)(\ell-n/2-1)!} & n \text{ even} \\
(1)^{\ell+1} \frac{2(n+1)(2n+3)(n+2)!}{((n/2+1)!)((n/2)!)^2} \frac{(1+2\ell)!((\ell+n)/2)!}{(\ell+1)(\ell-n/2-1)!} & n \text{ odd}
\end{cases}
\]

\[
\rightarrow \begin{cases} 
(1)^{\ell+1} \frac{4(2n+3)(n+1)!}{2^{(n/2)!}(n/2)!^2} & n \text{ even}
\end{cases}
\]

\[
Order(1/\ell) \rightarrow 0 \quad n \text{ odd}
\]

(8)

where \(B(a,b)\) is the Beta function and we used the Stirling's formula \(z! \sim \sqrt{2\pi z} \frac{z^z}{e^z} \) in the last equation.

In the case of \(n_1 = n_2 \rightarrow \infty\), \(c_{\ell,n_1,n_2}\), becoming independent of \(\ell\), is given by\(^8\)

\[
c_{\ell,n,\infty,\infty} = C_{\ell,0,0}^2 \frac{(2n)!}{2^{2n}n!(n+1)!} 3F_2(-n, -n - 1, \frac{3}{2}; 2, -n; 1)
\]

\[
= \begin{cases} 
+ \frac{(2n+3)(n+1)!}{2^{(n/2)!}(n/2)!^2} & n \text{ even} \\
- \frac{(n+1)(2n+3)(n+2)!}{2^{(n+1)!}(n/2+1)!} & n \text{ odd}
\end{cases}
\]

\[
\rightarrow (1)^{\ell+1} \frac{16}{\pi} n^2, \quad n \rightarrow \infty
\]

(10)

We can use this information to get some measures of the size of the scalar meson. First, we can get an exact answer for \((\ell, n) = (\infty, 0)\),

\[
m_{h}^2 \langle \overline{r}_{f}^2 \rangle_{\ell \rightarrow \infty} = 4 \sum_{n=0}^{\infty} \frac{c_{\ell,0,0}}{m_{n}}
\]

\[
= \sum_{k=0}^{\infty} (-1)^{k} \frac{(2k)!}{2^{2k}(k!)^2} (2k + 1) - \sum_{k=0}^{\infty} (-1)^{k} \frac{(2k + 1)!}{2^{2k+1}(k+1)!}
\]

\[
= \sinh^{-1}(1) - [\log(1 + \sqrt{2}) - \log 2] = \log 2
\]

\(^7\)This form can be obtained by using the decomposition \(c_{\ell,n,0,0} = f_{n} f_{\ell,n,0,0}/m_{h}^2\), which is briefly discussed in the appendix.

\(^8\)Here we used the asymptotic representation of the Jacobi polynomial,

\[
P_{n}^{(\alpha,\beta)}(\cos \theta) = \cos \left(\left[n + (\alpha + \beta + 1)/2\right] \theta - (\alpha/2 + 1/4)\pi\right) \sqrt{\pi n} \frac{(\sin \frac{\theta}{2})^{\alpha + \frac{1}{2}} (\cos \frac{\theta}{2})^{\beta + \frac{1}{2}}}{(\cos \frac{\pi}{2})^{\beta + \frac{1}{2}}} + O(n^{-\frac{3}{2}}), \quad 0 < \theta < \pi
\]
Next, $\langle r_f \rangle_{\ell \to \infty}$ can be estimated well enough numerically by considering the first few terms in the summation because $c_{n,0,0}^{\ell=\infty}$ has alternating sign and the magnitude of $c_{n,0,0}^{\ell=\infty}/\bar{m}_n^3$ decreases for large $n$: for large even $n$,

$$
\frac{c_{n,0,0}^{\ell=\infty}}{\bar{m}_n^3} \sim (-1)^{n/2} \frac{2}{\sqrt{2\pi}} n^{-3/2}.
$$

This gives

$$
m_h \langle r_f \rangle_{\ell \to \infty} = \frac{\pi}{2} \sum_{n=0}^{\infty} c_{n,0,0}^{\ell=\infty} \frac{n}{(\bar{m}_n)^3} \approx 0.697.
$$

For general $\ell$,

$$
m_h \langle r_f \rangle_{\ell} = \frac{\pi}{2} \sum_{n=0}^{2\ell-1} c_{n,0,0}^{\ell} \frac{n}{(\bar{m}_n)^3},
$$

which increases with $\ell$ but is bounded by $\langle r_f \rangle_{\ell \to \infty}$. We can also estimate $\langle r_f \rangle_{n \to \infty}$ by using $c_{n,\infty,\infty}$ for $n_1 = n_2 = \infty$:

More exact results on $\langle r_f^2 \rangle_{\ell}$ can be obtained by differentiating Eq. (4) with respect to $q^2$ first and then integrating. For the scalar meson with $(\ell, n = 0)$,

**Examples:**

In the case of $\langle 1, 0 | J_f^\mu(q) | 1, 0 \rangle$,

$$
F^{\ell=1}_{0,0}(q^2) = \frac{6}{q^2 + \bar{m}_0^2} + \frac{6}{q^2 + \bar{m}_1^2} \sim \frac{12}{q^2}
$$

$$
\tilde{F}^{\ell=1}_{0,0}(x) = 6m_h^2 [K_0(m_0 x) + K_0(m_1 x)]
$$

$$
\rightarrow 6m_h^2 \sqrt{\frac{\pi}{2m_0 x}} e^{-m_0 x}, \quad x \to \infty
$$

$$
\rightarrow -12m_h^2 \log (m_h x), \quad x \to 0
$$

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