N=4 VERSUS N=2 PHASES
HYPERKÄHLER QUOTIENTS
and THE 2D TOPOLOGICAL TWIST

Marco Billó and Pietro Frè
International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2/34014 Trieste, Italy
and INFN/Sezione di Trieste.

Abstract

We consider the rheonomic construction of N=2 and N=4 supersymmetric gauge theories in two-dimensions, coupled to matter multiplets. In full analogy with the N=2 case studied by Witten, we show that also in the N=4 case one can introduce Fayet-Iliopoulos terms for each of the abelian factors of the gauge group. The three-parameters of the N=4 Fayet-Iliopoulos term have the meaning of momentum-map levels in a HyperKähler quotient construction just as the single parameter of the N=2 Fayet-Iliopoulos term has the meaning of momentum map level in a Kähler quotient construction. Differently from the N=2 case, however, the N=4 has a single phase corresponding to an effective \(\sigma\)-model. The Landau-Ginzburg phase possible in the N=2 case seems to be deleted in those N=2 theories that have an enhanced N=4 supersymmetry. The main application of our N=4 model is to an effective Lagrangian construction of a \(\sigma\)-model on ALE-manifolds or other gravitational instantons.

We discuss in detail the topological twists of these theories (A and B models) emphasizing the role of R-symmetries and clarifying some subtleties, not yet discussed in the literature, related with the redefinition of the ghost number and the identification of the topological systems after twisting. In the A-twist, we show that one obtains a topological matter system (of the topological \(\sigma\)-model type) coupled to a topological gauge theory. In the B-twist, instead, we show that the theory describes a topological matter system (of the topological Landau-Ginzburg type) coupled to an ordinary (non-topological) gauge theory: in addition one has a massive topological vector, which decouples from the other fields. Applying our results to the case of ALE-manifolds we indicate how one can use the topologically twisted theories to study the Kähler class and complex structure deformations of these gravitational instantons.

Our results are also preparatory for a study of matter coupled topological 2D-gravity as the twist of matter coupled N=2, D=2 supergravity.

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1 Introduction

In this paper we make a detailed comparison of two-dimensional N=2 and N=4 supersymmetric gauge theories coupled to matter multiplets, discussing their phase-structure and their relation both with the geometrical constructions known as Kähler or HyperKähler quotients and with two-dimensional topological field-theories [1]. Our interest in the subject was motivated by a recent paper by Witten [2] who analyzed the N=2 case and showed how by means of such a gauge theory one can interpolate between an N=2 Landau-Ginzburg model [3] and an N=2 σ-model on a compact Calabi-Yau manifold. This interpolation has important implications for the understanding of the corresponding topological versions of the two N=2 theories and for the structure of the N=2 superconformal model that emerges at their critical point.

We wanted to investigate the same problem in the N=4 case, having in particular in mind the N=4 σ-model on those HyperKähler manifolds that can be obtained as HyperKähler quotients of flat-manifolds. This situation was naturally suggested by our previous investigation [4] of the N=4 superconformal field theories associated with four-dimensional gravitational instantons in particular the ALE manifolds [5]. Specifically a natural question that arose in connection with these spaces was the following: what is the Landau-Ginzburg phase for a σ-model on such non-compact HyperKähler manifolds that can be described as affine algebraic varieties in $\mathbb{C}^N$, rather than projective algebraic varieties in $\mathbb{CP}^N$? The answer that we found is that there is no Landau-Ginzburg phase: indeed the N=4 theories are special instances of N=2 theories with such a superpotential that it admits only one type of extremum: the σ-model phase. The would-be Landau-Ginzburg phase disappears.

To obtain these results and make our detailed comparison between the general N=2 case and that which actually corresponds to an enhanced N=4 supersymmetry, we considered the formulation of the N=2 and N=4 theories in the set-up of the rheonomy framework [6]. This laborious technical work presented in the central sections of our paper had an additional motivation besides that of providing a unified framework for the N=4 and N=2 cases. This is our intention to study the coupling of matter systems to N=2 2D-supergravity and use this as a starting point for an approach to topological matter-coupled 2D gravity based on a systematic use of the topological twist in complete analogy with the results previously obtained in the D=4 case [7]. Indeed once a rigid supersymmetric theory is recast into the rheonomy framework its coupling to supergravity is already almost achieved since by construction the rheonomic action already contains all the couplings of the matter fields to the bare vielbein and gravitino fields. Just the possible couplings to the gravitino-curl and to the bosonic curvature are missing and they can be easily retrieved in a second step.

In the present paper we shall present the rheonomic curvature parametrizations and the rheonomic action for both the N=2 and the N=4 gauge theories coupled to chiral Wess-Zumino multiplets or hypermultiplets (in the N=4 case). In addition to our study of the phase structure we shall present a careful analysis of the R-symmetries and a critical discussion of the formal
structure for the A and B topological twists [8]. This discussion clarifies along the same lines of thought followed by us in the D=4 case [7] some points that were in our opinion not completely clear in the existing literature. It also provides the basis for the study of the topological twists in the gravitational case which is postponed to a future publication.

To get into the heart of our topic we begin with a review of the HyperKähler quotient construction.

**DEFINITION of HyperKähler manifolds**

On a HyperKähler manifold $\mathcal{M}$, which is necessarily 4n-dimensional, there exist three covariantly constant complex structures $J^i : T\mathcal{M} \to T\mathcal{M}$, $i = 1, 2, 3$, satisfying the quaternionic algebra constraint $J^i J^j = -\delta^i_j + \varepsilon^{ijk} J^k$. The matrices $J^i_{AB}$ are antisymmetric. By covariant constancy the three HyperKähler two-forms $\Omega^i = J^i_{AB} V^A \wedge V^B$ are closed: $d\Omega^i = 0$. In the four-dimensional case because of the quaternionic algebra constraint the $J^i_{AB}$ can be either selfdual or anti-selfdual; if we take them to be anti-selfdual: $J^i_{AB} = \varepsilon^{1234}_{ABC} J^i_{CD}$ then the integrability condition for the covariant constancy of $J^i$ forces the curvature two-form $R^{AB}$ to be selfdual: thus in the four-dimensional case HyperKähler manifolds are particular instances of gravitational instantons.

A HyperKähler manifold is a Kähler manifold with respect to each of its complex structures. Consider a compact Lie group $G$ acting on a HyperKähler manifold $\mathcal{S}$ of real dimension $4n$ by means of Killing vector fields $X$ that are holomorphic with respect to the three complex structures of $\mathcal{S}$; then these vector fields preserve also the HyperKähler forms:

$$\begin{align*}
\mathcal{L}_X g &= 0 \iff \nabla_{\mu} X^\mu = 0 \\
\mathcal{L}_X J^i &= 0, \quad i = 1, 2, 3
\end{align*}$$

This can be regarded as a map $D : \mathcal{S} \to \mathbb{R}^3 \otimes \hat{G}^* \Gamma$ where $\hat{G}^*$ denotes the dual of the Lie algebra $\hat{G}$ of the group $G$. Indeed let $x \in \hat{G}$ be the Lie algebra element corresponding to the Killing vector $X$; then for a given $m \in \mathcal{S}$ $D_i(m) : x \mapsto D^X_i(m) \in \mathbb{C}$ is a linear functional on $\hat{G}$.

The $\{D^X_i\}$ constitute then a *momentum map*. This can be regarded as a map $D : \mathcal{S} \to \mathbb{R}^3 \otimes \hat{G}^* \Gamma$ where $\hat{G}^*$ denotes the dual of the Lie algebra $\hat{G}$ of the group $G$. Indeed let $x \in \hat{G}$ be the Lie algebra element corresponding to the Killing vector $X$; then for a given $m \in \mathcal{S}$ $D_i(m) : x \mapsto D^X_i(m) \in \mathbb{C}$ is a linear functional on $\hat{G}$. In practice expanding $X = X_a k^a$ in a basis of Killing vectors $k^a$ such that $[k^a, k^b] = f^{abc} k^c$ where $f^{abc}$ are the structure constants of $\hat{G}$ we also have $D^X_i = X_a D^X_i \Gamma i = 1, 2, 3$; the $\{D^X_i\}$ are the components of the momentum map.

The *HyperKähler quotient* [9] is a procedure that provides a way to construct from $\mathcal{S}$ a lower-dimensional HyperKähler manifold $\mathcal{M}$ as follows. Let $\mathcal{Z}^* \subset \hat{G}^*$ be the dual of the centre.
of \(G\). For each \(\zeta \in \mathbb{R}^3 \otimes \mathbb{Z}^*\) the level set of the momentum map

\[
\mathcal{N} \equiv \bigcap_i D_i^{-1}(\zeta^i) \subset \mathcal{S},
\]

which has dimension \(\dim \mathcal{N} = \dim \mathcal{S} \Leftrightarrow 3 \dim G\) is invariant under the action of \(G\) due to the equivariance of \(D\). It is thus possible to take the quotient

\[
\mathcal{M} = \mathcal{N} / G.
\]

\(\mathcal{M}\) is a smooth manifold of dimension \(\dim \mathcal{M} = \dim \mathcal{S} \Leftrightarrow 4 \dim G\) as long as the action of \(G\) on \(\mathcal{N}\) has no fixed points. The three two-forms \(\omega^i\) on \(\mathcal{M}\) defined via the restriction to \(\mathcal{N} \subset \mathcal{S}\) of the \(\Omega^i\) and the quotient projection from \(\mathcal{N}\) to \(\mathcal{M}\) are closed and satisfy the quaternionic algebra thus providing \(\mathcal{M}\) with a HyperKähler structure.

For future use it is important to note that once \(J^3\) is chosen as the preferred complex structure the momentum maps \(D_{\pm} = D_1 \pm iD_2\) are holomorphic (resp. antiholomorphic) functions.

The standard use of the HyperKähler quotient is that of obtaining non trivial HyperKähler manifolds starting from a flat \(4n\) real-dimensional manifold \(\mathbb{R}^{4n}\) acted on by a suitable group \(G\) generating triholomorphic isometries [9Γ10]. For instance this is the way it was utilized by Kronheimer [11] in its exhaustive construction of all self-dual asymptotically locally Euclidean four-spaces (ALE manifolds). We reviewed this construction in the already quoted discussion of the \(N=4\) conformal field theories describing string propagation on gravitational instantons [4]. Indeed the manifold \(\mathbb{R}^{4n}\) can be given a quaternionic structure and the corresponding quaternionic notation is sometimes convenient. For \(n = 1\) one has the flat quaternionic space \(\mathbb{H} \overset{\text{def}}{=} (\mathbb{R}^{4}, \{J^i\})\). We represent its elements

\[
q \in \mathbb{H} = x + iy + jz + kt = x^0 + x^iJ^i, \quad x, y, z, t \in \mathbb{R}
\]

realizing the quaternionic structures \(J^i\) by means of Pauli matrices: \(J^i = i (\sigma^i)^T\). Thus

\[
q = \begin{pmatrix} u & iv^* \\ iv & u^* \end{pmatrix} \quad \Leftrightarrow \quad q^\dagger = \begin{pmatrix} u^* & \leftrightarrow v^* \\ \leftrightarrow v & u \end{pmatrix}
\]

where \(u = x^0 + ix^3\) and \(v = x^1 + ix^2\). The euclidean metric on \(\mathbb{R}^{4}\) is retrieved as \(dq^\dagger \otimes dq = ds^2\mathbf{1}\). The HyperKähler forms are grouped into a quaternionic two-form

\[
\Theta = dq^\dagger \wedge dq \overset{\text{def}}{=} \Omega^iJ^i = \begin{pmatrix} i\Omega^3 & i\Omega^+ \\ i\Omega^- & \leftrightarrow \Omega^3 \end{pmatrix}.
\]

For generic \(n\) we have the space \(\mathbb{H}^n\) of elements

\[
q = \begin{pmatrix} u^A & iv^A \\ iv^A & u^{A*} \end{pmatrix} \quad \Leftrightarrow \quad q^\dagger = \begin{pmatrix} u^{A*} & \leftrightarrow v^{A*} \\ \leftrightarrow v^A & u^A \end{pmatrix}, \quad u^A, v^A \in \mathbb{C}^n, \quad A = 1, \ldots, n
\]
Thus \( dq^i \otimes dq = ds^2 \mathbf{1} \) gives \( ds^2 = dw^A \otimes dw^A + dv^4 \otimes dv^4 \) and the HyperKähler forms are grouped into the obvious generalization of the quaternionic two-form in eq.(4): 
\[
\Theta = \sum_{A=1}^n dq^A \wedge dq^A = \Omega^i J^i J^i \Omega^3 = 2i \partial \bar{\partial} K \text{ where the Kähler potential } K = \frac{1}{2} \left( \langle u^A, u^A + v^A \rangle v^4 \right) \Gamma \text{ and to } \Omega^i = 2i du^A \wedge dv^A \Gamma \Omega^- = (\Omega^i)^* .
\]

Let \((T_a)_B^A\) be the antihermitean generators of a compact Lie group \(G\) in its \(n \times n\) representation. A triholomorphic action of \(G\) on \(\mathbb{H}^n\) is realized by the Killing vectors of components

\[
X_a = \left( \hat{T}_a \right)_B^A q^B \frac{\partial}{\partial q^A} + q^B \left( \hat{T}_a \right)_A^B \frac{\partial}{\partial q^A} ; \quad \left( \hat{T}_a \right)_B^A = \begin{pmatrix} (T_a)_B^A & 0 \\ 0 & (T_a)_B^A \end{pmatrix} .
\]

Indeed one has \( \mathcal{L}_X \Theta = 0 \). The corresponding components of the momentum map are:

\[
D^a = q^{A^i} \begin{pmatrix} (T_a)_B^A & 0 \\ 0 & (T_a)_B^A \end{pmatrix} q_B + \begin{pmatrix} c \\ b \end{pmatrix} \leftrightarrow c \}
\]

where \( c \in \mathbb{R}, b \in \mathbb{C} \) are constants.

As we have already anticipated the geometrical HyperKähler quotient construction is intimately related with \(N=2\) supersymmetry in four-dimensions or with \(N=4\) supersymmetry in two-dimensions. The relation occurs through the auxiliary-field structure of the \(N=2\) vector multiplet in \(D=4\) or of the \(N=4\) vector multiplet in \(D=2\). In both cases in addition to the physical field \( \Gamma \) the vector multiplet contains a triplet of auxiliary scalars \( \mathcal{P} \) specifically a real scalar \( \mathcal{P} = \mathcal{P}^* \) and a complex scalar \( \mathcal{Q} \neq \mathcal{Q}^* \). \( \mathcal{Q} \) is utilized to gauge an isometry of an \(N=2\) \( \sigma \)-model in \(D=4\) or of an \(N=4\) \( \sigma \)-model in \(D=2\) the auxiliary fields \{\( \mathcal{P}, \mathcal{Q} \)\} are identified with the momentum-map functions \{\( \mathcal{D}^3 (m), \mathcal{D}^3 (m) \)\} of the \( \sigma \)-model target-space \( \mathcal{S} \). Indeed in both cases \( \mathcal{M}_4 \leftrightarrow \mathcal{S} \leftrightarrow \mathcal{M}_2 \leftrightarrow \mathcal{S} \) the condition for the \( \sigma \)-model to possess either \(N=2\) or \(N=4\) supersymmetry is that the target space \( \mathcal{S} \) be endowed with a HyperKähler structure.

In view of this fundamental property the HyperKähler quotient offers a natural way to construct an \(N=2\) \(D=4\) or \(N=4\) \(D=2\) \( \sigma \)-model on a non-trivial manifold \( \mathcal{M} \) starting from a free \( \sigma \)-model on a trivial flat-manifold \( \mathcal{S} = \mathbb{H}^n \). It suffices to gauge appropriate triholomorphic isometries by means of non-propagating gauge multiplets. Omitting the kinetic term of these gauge multiplets and performing the gaussian integration of the corresponding fields one realizes the HyperKähler quotient in a Lagrangian way. In the four-dimensional case this fact was fully exploited long time ago by Hitchin [9], Kärhede [10] Lindstrom and Rocek [11] further discussed by Galicki [12] and was applied in the context of string theory by Ferrara [13] and Girardello [14]. Actually the HyperKähler quotient is a generalization of a similar Pahler quotient procedure where the momentum map \( \mathcal{D} : \mathcal{S} \rightarrow \mathbb{R} \otimes \mathbb{G}^* \) consists just of one hamiltonian function \( \mathcal{I} \) rather than three. The Kähler quotient is related with either \(N=1D=4\) or \(N=2D=2\) supersymmetry, the reason being that in these cases the vector multiplet contains just one real auxiliary field \( \mathcal{P} \).
Recently, Witten has reconsidered the Kähler quotient construction of an N=2 two-dimensional σ-model in [2]. His point of view was that of regarding the Kähler quotient as an effective low-energy phenomenon rather than as a mere trick to implement the geometrical quotient construction in a Lagrangian field-theory language. In other words, he included the kinetic terms of the vector multiplet and also a Fayet-Iliopoulos term for each of the abelian factors in the gauge group; then he considered the whole system as a \textit{bona fide} gauge-theory spontaneously broken via an ordinary Higgs mechanism by the extrema of the scalar potential. Integrating on the massive modes that include the gauge vectors, the effective Lagrangian of the massless modes turns out to be that of an N=2 σ-model on a Kähler target manifold that is obtained as a hypersurface in a Kähler quotient. This, however, happens only in one phase, namely in a certain range of the parameters contained by the superpotential. When the parameters are in another range, we fall in a Landau-Ginzburg phase, namely the low-energy effective theory of the massless modes is a Landau-Ginzburg model with superpotential equal to the polynomial constraint \( W(X) \) that defines the target-manifold as a hypersurface in the σ-model phase.

Following the same line of thought, after reconstructing Witten’s theory in a rheonomic framework, we construct the N=4 analogue of this model. We introduce N=4 gauge multiplets and the N=4 analogues of the Wess-Zumino multiplets, namely the quaternionic hypermultiplets. We show that an N=4 counterpart of the Fayet-Iliopoulos term does indeed exist and involves three real parameters. After coupling to the hypermultiplets, these parameters play the role of triholomorphic momentum-map levels in the same way as in Witten’s case, the single parameter introduced by the N=2 Fayet-Iliopoulos term plays the role of momentum-map level for the holomorphic isometry. What is different in the N=4 case is the absence of auxiliary fields for the hypermultiplets. This implies that besides the interaction introduced by the gauge coupling, no other arbitrary quaternionic superpotential can be introduced. This is the essential reason why, at the end of the day, we do not find any analogue of the Landau-Ginzburg phase. It must be noted, however, that when we apply our construction to the ALE manifolds, the Fayet-Iliopoulos parameters have a deep geometrical meaning: they are the moduli of the self-dual metric.

In the last part of the paper, after an analysis of the R-symmetries that in the N=4 case from \( U(1)_L \otimes U(1)_R \) are promoted to \( U(2)_L \otimes U(2)_R \), we discuss the A and B topological twists, clarifying as we have already anticipated some delicate formal aspects of the procedure. In particular, we discuss the subtleties related with the redefinition of the ghost number, which has to be performed simultaneously with the redefinition of the spin.
2 The N=2 and N=4 rheonomic set up for globally supersymmetric field theories

The rheonomy approach to the construction of both locally and globally supersymmetric fieldtheories is almost fifteen years old and it has been extensively applied to all supergravity models in all space-time dimensions. A complete exposition of the method is contained in the book [6]: we refer to it for the basic concepts and we just begin with the specific definitions and conventions needed in our case.

The starting point for the whole construction is the definition of the curvatures of the (2E) and (4M) extended 2D-superspace. We denote by $e^\pm$ the two components of the world-sheet zweibein (in the flat case $e^+=dz+\theta\leftrightarrow terms\Gamma e^-=d\bar{z}+\theta\leftrightarrow terms\Gamma$) by $\omega$ the world-sheet spin-connection 1-form (in the flat-case we can choose $\omega=0$) and by $\zeta^\pm$, $\bar{\zeta}^\pm$ the four fermionic one-forms gauging the (2E) supersymmetries namely the 4 components of the 2 gravitinos. In the flat case we have $\zeta^\pm = d\theta^\pm \Gamma \zeta^\pm = d\bar{\theta}^\pm \Gamma$. In the (4M) case in addition to $\zeta^\pm$, $\bar{\zeta}^\pm \Gamma$ we have four other fermionic 1-forms $\chi^\pm$, $\bar{\chi}^\pm \Gamma$ that complete the eight components of the four gravitinos. Furthermore in the N=2 case there is a bosonic 1-form $A^\bullet$ gauging the $U(1)$ central charge $\Gamma$ while in the N=4 case in addition to $A^\bullet \Gamma$ we have two others bosonic 1-forms $A^\pm$ gauging the other two central charges.

In terms of these 1-forms the superspace curvatures are:

\[
\begin{align*}
 de^+ + \omega \wedge e^+ &\leftrightarrow \frac{1}{2} \zeta^+ \wedge \zeta^- = T^+ \\
de^- \leftrightarrow \omega \wedge e^- &\leftrightarrow \frac{1}{2} \bar{\zeta}^+ \wedge \bar{\zeta}^- = T^- \\
d\zeta^+ + \frac{1}{2} \omega \wedge \zeta^+ &\leftrightarrow \rho^+ \\
d\bar{\zeta}^+ \leftrightarrow \frac{1}{2} \omega \wedge \bar{\zeta}^+ &\leftrightarrow \bar{\rho}^+ \\
d\zeta^- + \frac{1}{2} \omega \wedge \zeta^- &\leftrightarrow \rho^- \\
d\bar{\zeta}^- \leftrightarrow \frac{1}{2} \omega \wedge \bar{\zeta}^- &\leftrightarrow \bar{\rho}^- \\
d\omega &\leftrightarrow R \\
dA^\bullet &\leftrightarrow \zeta^- \wedge \bar{\zeta}^+ + \bar{\zeta}^+ \wedge \zeta^- = F^\bullet
\end{align*}
\]

Flat superspace is described by the equations

\[
T^\pm = \rho^\pm = \bar{\rho}^\pm = R = F^\bullet = 0
\]

In the background of these flat superspace 1-forms we are supposed to solve the Bianchi identities for the matter fields spanning the various matter multiplets and to construct the associated rheonomic actions. In this way we determine the SUSY rules and the world-sheet
supersymmetric actions for all the theories under consideration. If we remove eq.s (9) and we introduce a rheonomic parametrization for the curvatures (8) then we are dealing with N=2 2D-supergravity and the solution of Bianchi identities in this curved background constitutes the coupling of matter to supergravity. This programme is left for a future publication; in this paper we concentrate on the flat case.

For convenience we also recall the rule for complex conjugation. Let $\psi_1, \psi_2$ be two forms of degree $p_1, p_2$ and statistics $F_1, F_2$ ($F = 0, 1$ for bosons or fermions) so that $\psi_1 \psi_2 = (\Leftrightarrow 1)^{p_1 p_2 + F_1 F_2} \psi_2 \psi_1$ then we have:

$$(\psi_1 \psi_2)^* = (\Leftrightarrow 1)^{F_1 F_2} \psi_1^* \psi_2^* = (\Leftrightarrow 1)^{p_1 p_2} \psi_2^* \psi_1^*$$

Thus for example for the gravitinos we have:

$$(\zeta^+ \wedge \zeta^-)^* = \Leftrightarrow (\zeta^+)^* \wedge (\zeta^-)^* = \Leftrightarrow \zeta^- \wedge \zeta^+ = \Leftrightarrow \zeta^+ \wedge \zeta^-$$

We proceed next to write the curvatures of the N=4 extended two-dimensional superspace namely:

$$d\tau^+ + \omega \wedge \zeta^- \Leftrightarrow \frac{i}{2} \zeta^+ \wedge \zeta^- \Leftrightarrow \frac{i}{2} \zeta^+ \wedge \zeta^- = T^+$$

$$d\tau^- \Leftrightarrow \omega \wedge \zeta^- \Leftrightarrow \frac{i}{2} \zeta^+ \wedge \zeta^- \Leftrightarrow \frac{i}{2} \zeta^+ \wedge \zeta^- = T^-$$

$$d\chi^+ + \frac{1}{2} \omega \wedge \chi^- = \tau^+$$

$$d\bar{\chi}^+ \Leftrightarrow \frac{1}{2} \omega \wedge \bar{\chi}^- = \bar{\tau}^+$$

$$d\chi^- + \frac{1}{2} \omega \wedge \chi^- = \tau^-$$

$$d\bar{\chi}^- \Leftrightarrow \frac{1}{2} \omega \wedge \bar{\chi}^- = \bar{\tau}^-$$

$$d\omega = R$$

$$d\zeta^+ + \frac{1}{2} \omega \wedge \zeta^- = \rho^+$$

$$d\bar{\zeta}^+ \Leftrightarrow \frac{1}{2} \omega \wedge \bar{\zeta}^- = \bar{\rho}^+$$

$$d\zeta^- + \frac{1}{2} \omega \wedge \zeta^- = \rho^-$$

$$d\bar{\zeta}^- \Leftrightarrow \frac{1}{2} \omega \wedge \bar{\zeta}^- = \bar{\rho}^-$$

$$dA^\bullet \Leftrightarrow \zeta^- \wedge \bar{\zeta}^+ + \bar{\zeta}^+ \wedge \zeta^- + \chi^- \wedge \bar{\chi}^+ \Leftrightarrow \bar{\chi}^+ \wedge \chi^- = F^\bullet$$

$$dA^+ \Leftrightarrow \chi^- \wedge \bar{\zeta}^+ + \bar{\chi}^+ \wedge \zeta^- = F^+$$

$$dA^- \Leftrightarrow \zeta^- \wedge \bar{\chi}^+ + \bar{\zeta}^+ \wedge \chi^- = F^-$$

(12)
Also in this case flat superspace is described by

\[ T^\pm = \rho^\pm = \bar{\rho}^\pm = R = F^* = F^\pm = 0 \]  

(13)

For both the N=2 and the N=4 case the determination of the globally supersymmetric field theories is done by solving the Bianchi identities of the matter fields in the background of the flat superspace 1-forms and then by constructing the associated rheonomic actions. In this way for each matter multiplet we can determine the SUSY rules and the world-sheet supersymmetric actions. The convention for complex conjugation is the same in the N=4 and in the N=2 case.

3 The N=2 abelian gauge multiplet

In this section we discuss the rheonomic construction of an N=2 abelian gauge theory in two-dimensions. This study will provide a basis for our subsequent coupling of the N=2 gauge multiplet to an N=2 Landau-Ginzburg system invariant under the action of one or several U(1) gauge-groups or even of some non abelian gauge group \( G \).

In the N=2 case a vector multiplet is composed of a gauge boson \( \mathcal{A} \) namely a world-sheet 1-form\( \Gamma \) two spin 1/2 gauginos\( \Gamma \) whose four components we denote by \( \lambda^+ \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \) a complex physical scalar \( M \neq M^* \) and a real auxiliary scalar \( \mathcal{P}^* = \mathcal{P} \). Each of these fields is in the adjoint representation of the gauge group \( G \) and carries an index of that representation that we have not written.

In the abelian case defining the field strength

\[ F = d\mathcal{A} \]  

(14)

the rheonomic parametrizations that solve the Bianchi identities:

\[ dF = d^2\lambda^- = d^2\lambda^+ = d^2\lambda^+ = d^2\lambda^- = d^2M = d^2\mathcal{P} = 0 \]  

(15)

are given by

\[
F = \mathcal{F} e^+ e^- \Leftrightarrow \frac{i}{2}(\lambda^+ \zeta^- + \bar{\lambda}^- \zeta^+) e^- + \frac{i}{2}(\lambda^+ \bar{\zeta}^- + \bar{\lambda}^- \zeta^+) e^+ + M \zeta^- \bar{\zeta}^+ \Leftrightarrow M^* \zeta^+ \bar{\zeta}^- \\
dM = \partial_+ M e^+ + \partial_- M e^- \Leftrightarrow \frac{1}{4}(\lambda^- \zeta^+ \Leftrightarrow \bar{\lambda}^+ \bar{\zeta}^-) \\
d\lambda^+ = \partial_+ \lambda^+ e^+ + \partial_- \lambda^+ e^- + \left( \frac{\mathcal{F}}{2} + i \mathcal{P} \right) \zeta^+ \Leftrightarrow 2i \partial_+ M \bar{\zeta}^+ \\
d\bar{\lambda}^- = \partial_+ \bar{\lambda}^- e^+ + \partial_- \bar{\lambda}^- e^- + \left( \frac{\mathcal{F}}{2} \Leftrightarrow i \mathcal{P} \right) \zeta^- \Leftrightarrow 2i \partial_- M^* \bar{\zeta}^- \\
d\lambda^- = \partial_+ \lambda^- e^+ + \partial_- \lambda^- e^- + \left( \frac{\mathcal{F}}{2} + i \mathcal{P} \right) \bar{\zeta}^- \Leftrightarrow 2i \partial_+ M^* \zeta^+ \\
d\bar{\lambda}^+ = \partial_+ \bar{\lambda}^+ e^+ + \partial_- \bar{\lambda}^+ e^- + \left( \frac{\mathcal{F}}{2} + i \mathcal{P} \right) \bar{\zeta}^+ \Leftrightarrow 2i \partial_- M \zeta^- \\
d\mathcal{P} = \partial_+ \mathcal{P} e^+ + \partial_- \mathcal{P} e^- \Leftrightarrow \frac{1}{4}(\partial_+ \bar{\lambda}^+ \zeta^- \Leftrightarrow \partial_+ \bar{\lambda}^- \zeta^+ \Leftrightarrow \partial_- \lambda^+ \bar{\zeta}^- \Leftrightarrow \partial_- \lambda^- \bar{\zeta}^+) \]  

(16)
Given these parametrizations we next write the rheonomic action whose variation yields the above parametrizations as field equations in superspace together with the world-sheet equations of motion.

\[ \mathcal{L}_{\text{rheon}}^{(\text{gauge})} = \mathcal{F} \left[ F + \frac{i}{2} \left( \bar{\lambda}^+ \zeta^- + \bar{\lambda}^- \zeta^+ \right) e^- \right] \equiv \frac{i}{2} \left( \bar{\lambda}^+ \zeta^+ + \lambda^- \zeta^+ \right) e^+ \\
\equiv M \zeta^- \zeta^- \equiv M^* \zeta^+ \zeta^- \] \equiv \frac{1}{2} \mathcal{F} e^+ e^- \\
\equiv \frac{1}{2} \left( \bar{\lambda}^+ d\lambda^- + \bar{\lambda}^- d\lambda^+ \right) e^- + \frac{i}{2} \left( \bar{\lambda}^+ d\lambda^- + \lambda^- d\lambda^+ \right) e^+ \\
\equiv 4 \left[ dM^* \equiv \frac{1}{4} (\bar{\lambda}^+ \zeta^- \equiv \lambda^- \zeta^+) \right] (M_+ e^+ \equiv M_- e^-) \\
\equiv 4 \left[ dM + \frac{1}{4} (\lambda^- \zeta^+ \equiv \lambda^+ \zeta^-) \right] (M_+ e^+ \equiv M_- e^-) \right) \\
\equiv d(M_+ M_- + M_- M_+) e^+ e^- \equiv dM (\bar{\lambda}^- \zeta^+ + \lambda^+ \zeta^-) + dM^* (\bar{\lambda}^+ \zeta^- + \lambda^- \zeta^+) \\
\equiv \frac{1}{4} \left( \bar{\lambda}^+ \lambda^+ \zeta^- \zeta^+ + \bar{\lambda}^- \lambda^- \zeta^- \zeta^+ \right) + 2 \mathcal{P} \ e^+ e^- + 4i \frac{\partial \mathcal{U}^*}{\partial M} \left( \frac{F}{2} + i \mathcal{P} e^+ e^- \right) \\
\equiv 4i \left( \frac{\partial \mathcal{U}^*}{\partial M} \left( \frac{F}{2} + i \mathcal{P} e^+ e^- \right) \right) \equiv i \left( \frac{\partial^2 \mathcal{U}^*}{\partial M^2} \left( \lambda^- \zeta^+ \equiv \lambda^+ \zeta^- \right) e^+ e^- \right) \\
+ \left( \frac{\partial \mathcal{U}^*}{\partial M} + \frac{\partial \mathcal{U}^*}{\partial M^*} \right) \left( \lambda^- \zeta^+ \equiv \lambda^+ \zeta^- \right) e^+ e^- \equiv 2i \mathcal{U}^* \equiv M^* \left( \frac{\partial \mathcal{U}}{\partial M} \equiv \frac{\partial \mathcal{U}^*}{\partial M^*} \right) \zeta^- \zeta^+ \right] \equiv 2i \left( \frac{\partial \mathcal{U}}{\partial M} \equiv \frac{\partial \mathcal{U}^*}{\partial M^*} \right) \zeta^- \zeta^+ \right] \right) \right) \\
\right) (18)

The symbol \( \mathcal{U} \) denotes a holomorphic function \( \mathcal{U}(M) \) of the physical scalar \( M \) that is named the superpotential. It induces a self interaction of the scalar \( M \) field and an interaction of this field with the gauge-vector. The existence of an arbitrariness in the choice of the vector multiplet dynamics is a consequence of the existence of the auxiliary field \( \mathcal{P} \) in the solution of the Bianchi identities (15) and hence in the determination of the SUSY rules for this type of N=2 multiplet. In the superspace formalism the inclusion in the action of the terms containing the superpotential is effected by means of the use of the so called twisted chiral superfields. In the rheonomic framework there is no need of these distinctions: we just have an interaction codified by an arbitrary holomorphic superpotential.

Note that in eqs (16) and (18) we have suppressed the wedge product symbols for differential forms. This convention will be often adopted also in the sequel to avoid clumsiness. From the rheonomic action (16) we easily obtain the world-sheet action of the N=2 globally supersymmetric abelian vector multiplet by deleting all the terms containing the gravitino 1-forms replacing the first order fields \( \mathcal{F}, M \pm \) with their values following from their own field equations namely \( \mathcal{F} = \frac{1}{2} (\partial_+ A_- \equiv \partial_- A_+) \frac{\mathcal{F}}{2} t \mathcal{M}_\pm = \partial_\pm \mathcal{M} \) and by replacing \( e^+ \wedge e^- \) with \( d^2 z \)
that is factored out. In this way we get:

\[ \mathcal{L}_{\text{gauge}}^{(ws)} = \frac{1}{2} \mathcal{F}^2 \Leftrightarrow i (\dot{\lambda}^+ \partial_+ \lambda^- + \lambda^+ \partial_- \lambda^-) \Leftrightarrow 4 (\partial_+ M^* \partial_- M + \partial_- M^* \partial_+ M) + 2 \mathcal{P}^2 + 4 i \frac{\partial U}{\partial M} \left( \mathcal{F} \frac{i}{2} \Leftrightarrow i \mathcal{P} \right) \Rightarrow i \left( \frac{\partial^2 U}{\partial M^2} \dot{\lambda}^+ \lambda^- + \frac{\partial^2 U^*}{\partial M^2} \dot{\lambda}^- \lambda^+ \right) \]

In the particular case of a linear superpotential

\[ \mathcal{U} = \frac{t}{4} M , \; t \in \mathbb{C} \]  

setting

\[ t = r \Leftrightarrow i \theta / 2 \pi , \; r \in \mathbb{R} , \; \theta \in [0, 2 \pi] \]

the above expression reduces to

\[ \mathcal{L}_{ws} = \frac{1}{2} \mathcal{F}^2 \Leftrightarrow i (\dot{\lambda}^+ \partial_+ \lambda^- + \lambda^+ \partial_- \lambda^-) \Leftrightarrow 4 (\partial_+ M^* \partial_- M + \partial_- M^* \partial_+ M) + 2 \mathcal{P}^2 \Leftrightarrow 2r \mathcal{P} + \frac{\theta}{2 \pi} \mathcal{F} \]

The meaning of the parameters \( r \) and \( \theta \) introduced in the above lagrangian is clear. Indeed \( r \) giving a vacuum expectation value \( \mathcal{P} = \frac{r}{2} \) to the auxiliary field \( \mathcal{P} \) induces a spontaneous breaking of supersymmetry and shows that the choice \( \mathcal{U} = \frac{s}{4} M \) corresponds to the insertion of a Fayet-Iliopoulos term into the action. On the other hand the parameter \( \theta \) is clearly a theta-angle multiplying the first Chern class \( \frac{1}{2 \pi} \mathcal{F} \) of the gauge connection.

### 4 \( N = 2 \) Landau Ginzburg models with an abelian gauge symmetry

As stated above our interest in the \( N=2 \) vector multiplet was instrumental to the study of an \( N=2 \) Landau-Ginzburg system possessing in addition to its own self interaction a minimal coupling to a gauge theory. This is the system studied by Witten in [2] using superspace techniques rather than the rheonomy framework. By definition a Landau Ginzburg system is a collection of \( N=2 \) chiral multiplets self-interacting via an analytic superpotential \( \mathcal{W}(X) \). Each chiral multiplet is composed of a complex scalar field \( (X^i)^* = X^{i*} \) and two spin \( 1/2 \) fermions whose four components we denote by \( \psi^i \), \( \bar{\psi}^i \) and \( \psi^{i*} = (\psi^i)^* \), \( \bar{\psi}^{i*} = (\bar{\psi}^i)^* \). Together with a complex auxiliary field \( \mathcal{H} \) which is identified with the derivative of the holomorphic superpotential \( \mathcal{W}(X) \) namely \( \mathcal{H}^i = \bar{\eta}^{i*} \partial_j W^* \Gamma \eta^{j*} \) being the flat Kählerian metric on the complex manifold \( \mathbb{C}^n \) of which the complex scalar fields \( X^i \) are interpreted as the coordinates. Using this system of fields we could construct a rheonomic solution of the superspace Bianchi
identities a rheonomic action and a world-sheet action invariant under the supersymmetry transformations induced by the rheonomic parametrizations. In this action the kinetic terms are the canonical ones of a free field theory and the only interaction is that induced by the superpotential. Rather than doing this we prefer to study the same system in presence of a minimal coupling to the gauge system studied in the previous section. In practice this amounts to solve the Bianchi identities for the gauge covariant derivatives rather than for the ordinary derivatives using as a background the rheonomic parametrizations of the gauge multiplet determined above. At the end of the construction by setting the gauge coupling constant to zero we can also recover the formulation of the ordinary Landau-Ginzburg theory later referred to as the rigid Landau-Ginzburg theory.

Indeed the coupling of the chiral multiplets to the gauge multiplet is defined through the covariant derivative

$$\nabla X^i \equiv dX^i + i A q_j^i X^j$$

where the hermitean matrix $q_j^i$ is the generator of the $U(1)$ action on the chiral matter. As a consequence the Bianchi identities are of the form $\nabla^2 X^i = i F q_j^i X^j$.

Let $W(X^i)$ be the holomorphic the superpotential: then the rheonomic solution of the Bianchi identities is given by the following parametrizations:

$$\nabla X^i = \nabla_+ X^i e^+ + \nabla_- X^i e^- + \psi^i \zeta^- + \bar{\psi}^i \bar{\zeta}^-$$

$$\nabla X^i = \nabla_+ X^i e^+ + \nabla_- X^i e^- \Leftrightarrow \psi^i \zeta^+ \Leftrightarrow \bar{\psi}^i \bar{\zeta}^+$$

$$\nabla \psi^i = \nabla_+ \psi^i e^+ + \nabla_- \psi^i e^- \Leftrightarrow \frac{i}{2} \nabla_+ X^i \zeta^+ + \eta^{ij} \partial_j W \zeta^+ + i M q_j^i X^j \zeta^+$$

$$\nabla \bar{\psi}^i = \nabla_+ \bar{\psi}^i e^+ + \nabla_- \bar{\psi}^i e^- \Leftrightarrow \frac{i}{2} \nabla_+ X^i \bar{\zeta}^+ \eta^{ij} \partial_j W \bar{\zeta}^+ \Leftrightarrow i M^* q_j^i X^j \bar{\zeta}^+$$

$$\nabla \bar{\psi}^i = \nabla_+ \bar{\psi}^i e^+ + \nabla_- \bar{\psi}^i e^- \Leftrightarrow \frac{i}{2} \nabla_+ X^i \bar{\zeta}^- \eta^{ij} \partial_j W \bar{\zeta}^- \Leftrightarrow i M^* q_j^i X^j \bar{\zeta}^-$$

From the consistency of the above parametrizations with the Bianchi identities one also gets the following fermionic world-sheet equations of motion:

$$\frac{i}{2} \nabla_- \psi^i \Leftrightarrow \eta^{ij} \partial_j W^* \bar{\psi}^j \Leftrightarrow \frac{i}{4} \lambda^* q_j^i X^j + i M q_j^i \bar{\psi}^j = 0$$

$$\frac{i}{2} \nabla_+ \bar{\psi}^i \eta^{ij} \partial_j W^* \psi^j \Leftrightarrow \frac{i}{4} \lambda^* q_j^i X^j \Rightarrow M^* q_j^i \bar{\psi}^j = 0$$

and their complex conjugates for the other two fermions. Applying to eq. (25) a supersymmetry transformation as it is determined by the parametrizations (24) we obtain the bosonic field equation:

$$\frac{1}{8} (\nabla_+ \nabla_- X^i + \nabla_- \nabla_+ X^i) \Leftrightarrow \eta^{ik} \partial_k \partial_\nu W^* \psi^j \psi^j \bar{\psi}^j \bar{\psi}^j + \eta^{ij} \partial_k \partial_\nu W^* \eta^{\bar{j} \bar{j}} \partial_{\bar{j}} W + \bar{\psi}^j \partial_{\bar{j}} \partial_{\bar{j}} W \Rightarrow \frac{i}{4} \lambda q_j^i \bar{\psi}^j \Leftrightarrow \frac{1}{4} P q_j^i X^j = 0$$

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Equipped with this information we can easily derive the rheonomic action from which the parametrizations (24) and the field equations (25) (26) follow as variational equations: it is the following one:

\[
\mathcal{L}^{(r,\text{chiral})}_{\text{chiral}} = \eta_{ij} \left( \nabla X_i^{\cdot} \leftrightarrow \psi^i \bar{\zeta} \leftrightarrow \bar{\psi}^i \zeta \right) \left( \Pi^+ e^+ \leftrightarrow \Pi^\cdot e^\cdot \right) \\
+ \eta_{ij} \left( \nabla X_j^{\cdot} \leftrightarrow \psi^j \bar{\zeta} \leftrightarrow \bar{\psi}^j \zeta \right) \left( \Pi^_+ e^+ \leftrightarrow \Pi^\cdot_\cdot e^\cdot \right) \\
+ 4i \eta_{ij} \left( \psi^i \nabla \psi^j e^+ \leftrightarrow \bar{\psi}^i \nabla \bar{\psi}^j e^- \right) \\
+ 4i \left( \psi^k \partial_k W \zeta^+ e^+ \leftrightarrow \bar{\psi}^k \partial_k \bar{W} \bar{\zeta}^+ e^- \right) \\
+ \eta_{ij} \left( \psi^i \psi^j \zeta^+ \zeta^- \leftrightarrow \bar{\psi}^i \bar{\psi}^j \bar{\zeta}^+ \bar{\zeta}^- \right) \\
+ 8 \left( \partial_i \partial_j W \phi^i \phi^j + \text{c.c.} \right) e^+ e^- \\
\right) e^\cdot e^- \\
\right) (\psi^j \eta_{ij} \phi^k X^k \zeta^+ e^+ + \text{c.c.}) \\
\right) (\psi \bar{\psi}^j \eta_{ij} \phi^k X^k \zeta^- e^- + \text{c.c.}) \\
\right) \left( 2P \eta_{ij} X^i \phi^j X^k + 8M \eta_{ij} X^i \phi^j X^k \right) e^+ e^- \\
\right)
\]

(27)

The world-sheet lagrangian for this system is now easily obtained through the same steps applied in the previous case. To write it we introduce the following simplifications in our notation: a) we use a diagonal form for the flat \( C^n \) metric \( \eta_{ij}, X^i X^j = X^i X^i \Gamma \) we diagonalize the \( U(1) \) generator \( \eta \) by setting \( \phi^i \equiv \phi \delta^i_j \) (\( \phi \) being the charge of the field \( X^i \)). Then we have:

\[
\mathcal{L}^{(w,\text{chiral})}_{\text{chiral}} = \left( \nabla_+ X^i \nabla_- X^i + \nabla_- X^i \nabla_+ X^i \right) + 4i \left( \psi^i \nabla_+ \psi^i + \bar{\psi}^i \nabla_- \bar{\psi}^i \right) \\
+ 8 \left( \psi^i \bar{\psi}^j \partial_i \partial_j W + \text{c.c.} \right) + \partial_i W \partial_i W \right) + 2i \sum_i q^i \left( \psi^i \bar{\psi}^i \zeta^- e^- \right) \\
+ 8i \left( M \sum_i q^i \psi^i \zeta^+ e^+ \right) \right) + 8M \sum_i (q^i)^2 X^i \zeta^+ e^+ \\
+ 2P \sum_i q^i X^i X^i \\
\right)
\]

(28)

5 Structure of the scalar potential in the N=2 Landau-Ginzburg model with an abelian gauge symmetry

We consider next the coupled system whose lagrangian with our conventions is the difference of the two lagrangians we have just described:

\[
\mathcal{L} = \mathcal{L}_{\text{gauge}} \leftrightarrow \mathcal{L}_{\text{chiral}} \\
\]

(29)

the relative sign being fixed by the requirement of positivity of the energy. The world-sheet form of the action (29) is the same modulo trivial notation differences as the action
(2.19)+(2.23)+(2.27) in Witten’s paper [2]. We focus our attention on the potential energy of the bosonic fields: it is given by the following expression

\[
\mathcal{U} = 2\mathcal{P}^2 \leftrightarrow 4\mathcal{P} \left( \frac{\partial \mathcal{U}}{\partial M} + \frac{\partial \mathcal{U}^*}{\partial M^*} \right) + 2\mathcal{P} \sum_i q_i|X^i|^2
\]

\[
\leftrightarrow 8\partial_i W \partial_i W^* \leftrightarrow 8|M|^2 \sum_i (q_i^2)^2|X^i|^2
\]

The variation in the auxiliary field \(\mathcal{P}\) yields the expression of \(\mathcal{P}\) itself in terms of the physical scalars:

\[
\mathcal{P} = \frac{\partial \mathcal{U}}{\partial M} + \frac{\partial \mathcal{U}^*}{\partial M^*} \leftrightarrow \frac{1}{2} \sum_i q_i^2|X^i|^2
\]

In the above equation the expression \(\mathcal{D}^X(X, X^*) = \sum_i q_i^2|X^i|^2\) is the momentum map function for the holomorphic action of the gauge group on the matter multiplets. Indeed if we denote by \(X = i \sum_i q_i^2 (X^i \partial_i \Leftrightarrow X^i \partial_i, a)\) the killing vector and by \(\Omega = \sum_i dX^i \wedge dX^i^*\) then we have \(id\mathcal{D}^X = i_x \Omega\). As anticipated the auxiliary field \(\mathcal{P}\) is identified with the momentum-map function \(\Gamma\) plus the term \(\frac{\partial \mathcal{U}}{\partial M} + \frac{\partial \mathcal{U}^*}{\partial M^*}\) due to the self interaction of the vector-multiplet. In the case of the linear superpotential of eq.s (20) and (21) the auxiliary field is identified with:

\[
\mathcal{P} = \frac{1}{2}(D^X(X, X^*) \leftrightarrow r)
\]

Eliminating \(\mathcal{P}\) through eq. (31) we obtain the final form for the scalar field potential in this kind of models namely:

\[
U = 2 \left[ \frac{\partial \mathcal{U}}{\partial M} + \frac{\partial \mathcal{U}^*}{\partial M^*} \right] \leftrightarrow \frac{1}{2} \sum_i q^2|X^i|^2 \right]^2 + |\partial_i W|^2 + 8|M|^2 \sum_i (q_i^2)^2|X^i|^2
\]

In the case of the linear superpotential this reduces to

\[
U = \frac{1}{2} \left[ r \leftrightarrow \sum_i q^2|X^i|^2 \right]^2 + 8|\partial_i W|^2 + 8|M|^2 \sum_i (q_i^2)^2|X^i|^2
\]

The theory characterized by the above scalar potential exhibits a two phase structure as the parameter \(r\) varies on the right line. This is the essential point in Witten’s paper that allows an interpolation between an \(N=2\) \(\sigma\)-model on a Calabi-Yau manifold and a rigid Landau-Ginzburg theory. The review of these two regimes is postponed to later sections. Here we note that the above results can be generalized to the case of a non abelian vector-multiplet or to the case of several abelian gauge multiplets.

6 Extension to the case where the gauge symmetry of the \(N=2\) Landau-Ginzburg model is non abelian

We fix our notations and conventions.
Consider a Lie algebra $\mathcal{G}$ with structure constants $f^{abc}$:

$$[t^a, t^b] = if^{abc} t^c$$

(35)

In every representation the hermitean generators $t^a = (t^a)^\dagger$ are normalized in such a way that $\text{Tr}(t^at^b) = \delta^{ab}$. Let us name $T^a$ the generators of the adjoint representation defined by $f^{abc} = i(T^a)^{bc}$.

Let us introduce the gauge vector field as a $\mathcal{G}$-valued one-form:

$$A = A_\mu^a T^a dx^\mu$$

(36)

In the case we are interested the index $\mu$ takes two values and we can write $A = A_a^\pm e^+ + A_a^- e^-$. Note that $A_\dagger = A$. The field strength is defined as the two-form

$$F = dA + iA \wedge A$$

(37)

The Bianchi Identities read

$$\nabla F \overset{\text{def}}{=} dF + i(A \wedge F \Leftrightarrow F \wedge A) = 0$$

(38)

The component expression of the field strength and of its associated Bianchi identity is:

$$F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]} \leftrightarrow \frac{1}{2} f^{abc} A^b_\mu A^c_\nu$$

$$\partial_{[\mu} F^a_{\nu]} \leftrightarrow f^{abc} A^b_{[\mu} F^c_{\nu]} = 0$$

(39)

Note that the Bianchi identity for a field $M = M^a T^a$ transforming in the adjoint representation is:

$$\nabla^2 M = i[F, M]$$

(40)

The non-abelian analogue of the rheonomic parametrizations (16) is obtained in the following way: first we write the $\mathcal{G}$-valued parametrization of $F$:

$$F = \mathcal{F} e^+ e^- \leftrightarrow \frac{i}{2} (\lambda^+ \zeta^- + \lambda^- \zeta^+) e^- + \frac{i}{2} (\lambda^+ \bar{\zeta}^+ + \lambda^- \bar{\zeta}^-) e^+ + M \zeta^- \bar{\zeta}^+ \Leftrightarrow M^* \zeta^+ \bar{\zeta}^-$$

(41)

In this way we have introduced the gauge scalars $M = M^a T^a$ and the gauginos $\lambda^\pm = \lambda^\pm T^a \Gamma \lambda^\pm = \bar{\lambda}^{\pm T^a}$; their parametrizations are obtained by implementing the Bianchis for $F \Gamma \nabla F = 0$. One must also take into account the Bianchi identities for these fields: $\nabla^2 M = i[F, M]$ and $\nabla^2 \lambda^\pm = i[F, \lambda^\pm]$ (analogously for the tilded gauginos). The rheonomic parametrizations fulfilling all these constraints turn out to be the following ones:

$$F = \mathcal{F} e^+ e^- \leftrightarrow \frac{i}{2} (\lambda^+ \zeta^- + \lambda^- \zeta^+) e^- + \frac{i}{2} (\lambda^+ \bar{\zeta}^+ + \lambda^- \bar{\zeta}^-) e^+ + M \zeta^- \bar{\zeta}^+ \Leftrightarrow M^* \zeta^+ \bar{\zeta}^-$$

$$\nabla M = \nabla_+ M e^+ + \nabla_- M e^- \leftrightarrow \frac{1}{4} (\lambda^- \zeta^+ \Leftrightarrow \tilde{\lambda}^+ \tilde{\zeta}^-)$$
\[ \nabla \lambda^+ = \nabla_+ \lambda^+ e^+ + \nabla_- \lambda^+ e^- \left( \frac{\mathcal{F}}{2} \Leftrightarrow 2i[M^+, M] + i\mathcal{P} \right) \zeta^+ \Leftrightarrow 2i \nabla_- M \tilde{\zeta}^+ \]
\[ \nabla \lambda^- = \nabla_+ \lambda^- e^+ + \nabla_- \lambda^- e^- \left( \frac{\mathcal{F}}{2} \Leftrightarrow 2i[M^+, M] \Leftrightarrow i\mathcal{P} \right) \tilde{\zeta}^+ \Leftrightarrow 2i \nabla_+ M^+ \zeta^+ \]
\[ \nabla \mathcal{P} = \nabla_+ \mathcal{P} e^+ + \nabla_- \mathcal{P} e^- \Leftrightarrow \frac{1}{4} \left( \nabla_+ \lambda^+ \Leftrightarrow 2[\lambda^+, M] \right) \zeta^- \Leftrightarrow \left( \nabla_+ \lambda^- + 2[\lambda^-, M^+] \right) \tilde{\zeta}^+ \]
\[ \Leftrightarrow \left( \nabla_- \lambda^+ + 2[\lambda^-, M^+] \right) \zeta^- + \left( \nabla_- \lambda^- \Leftrightarrow 2[\lambda^-, M] \right) \tilde{\zeta}^+ \]
\[ (42) \]

We obtain the rheonomic action for the N=2 non-abelian gauge multiplet in two steps by setting:
\[ \mathcal{L}_{\text{non-abelian}}^{(\text{rheoan})} = \mathcal{L}_0 + \Delta \mathcal{L}_{\text{int}} \] (43)

where \( \mathcal{L}_0 \) is the free part of the Lagrangian whose associated equations of motion would set the auxiliary fields to zero: \( \mathcal{P} = \mathcal{P}_a T^a = 0 \) The insertion of the interaction term \( \Delta \mathcal{L}_{\text{int}} \) corrects the equation of motion of the auxiliary fields depending on a holomorphic function \( \mathcal{U}(M) \) of the physical gauge scalars \( M^a \) just as in the abelian case. The form of \( \mathcal{L}_0 \) is given below where the trace is performed over the indices of the adjoint representation:

\[ \mathcal{L} = \text{Tr} \left\{ \mathcal{F} \left[ F + \frac{i}{2}(\lambda^+ \zeta^- + \lambda^- \zeta^+) e^- \Leftrightarrow \frac{i}{2}(\lambda^+ \tilde{\zeta}^+ + \lambda^- \tilde{\zeta}^+) e^+ \Leftrightarrow M \zeta^- \tilde{\zeta}^+ + M^+ \zeta^+ \tilde{\zeta}^- \right] \right\} \]
\[ \Leftrightarrow \frac{1}{2} \mathcal{F}^2 e^+ e^- \Leftrightarrow \frac{i}{2}(\lambda^+ \nabla \lambda^- + \lambda^- \nabla \lambda^+) e^- + \frac{i}{2}(\lambda^+ \nabla \lambda^- + \lambda^- \nabla \lambda^+) e^+ \]
\[ \Leftrightarrow \frac{1}{4} \left[ \nabla M^+ \Leftrightarrow \frac{1}{4}(\lambda^+ \zeta^- \Leftrightarrow \lambda^- \zeta^+) \right] \left( M_+ e^+ \Leftrightarrow M_- e^- \right) \]
\[ \Leftrightarrow \frac{1}{4} \left[ \nabla M + \frac{1}{4}(\lambda^- \zeta^+ \Leftrightarrow \lambda^+ \zeta^-) \right] \left( M_+^1 e^+ \Leftrightarrow M_-^1 e^- \right) \]
\[ \Leftrightarrow 4(M_+^1, M_- + M_-^1, M_+) e^+ e^- \Leftrightarrow \nabla M(\lambda^- \tilde{\zeta}^+ + \lambda^+ \zeta^-) + \nabla M^+(\lambda^+ \tilde{\zeta}^- + \lambda^- \zeta^+) \]
\[ + 2[M^+, M] \left( (\lambda^+ \zeta^- \Leftrightarrow \lambda^- \zeta^+) e^+ \Leftrightarrow (\lambda^+ \tilde{\zeta}^- \Leftrightarrow \lambda^- \tilde{\zeta}^+) e^- \right) \]
\[ \Leftrightarrow \frac{1}{4}(\lambda^+ \zeta^- \zeta^- + \lambda^- \lambda^+ \zeta^+ \zeta^-) + 2\mathcal{F}^2 e^+ e^- \] (44)

As stated above, the variational equations associated with this action yield the rheonomic parametrizations (42) for the particular value \( \mathcal{P}_a = 0 \) of the auxiliary field. Furthermore they also imply \( \mathcal{P} = 0 \) as a field equation.

To determine the form of \( \Delta \mathcal{L}_{\text{int}} \) we suppose that in presence of this interaction the new field equation of \( \mathcal{P}_a \) yields

\[ \mathcal{P}_a = \frac{\partial \mathcal{U}(M)}{\partial M^a} + \left( \frac{\partial \mathcal{U}(M)}{\partial M^a} \right)^* = \frac{\partial \mathcal{U}(M)}{\partial M^a} + \frac{\partial \mathcal{U}^*(M^*)}{\partial M^a}. \] (45)

\( \mathcal{U} \) is a holomorphic function of the scalars \( M^a \) that characterizes their self-interaction. Then we can express \( \nabla \mathcal{P}_a \) through the chain rule: \( \nabla \mathcal{P}_a = \frac{\partial \mathcal{U}}{\partial M^a} \nabla M^a + \frac{\partial \mathcal{U}^*}{\partial M^a} \nabla M^a. \) Using
the rheonomic parametrizations (42) for $\nabla M^a$ and comparing with the parametrization of $\nabla P^a$ in the same eq.(42) we get the fermionic equations of motion that the complete interacting lagrangian should imply as variational equations:

$$\nabla_+ \bar{\lambda}_a^+ \Leftrightarrow 2i f^{abc} \lambda_b^+ M_c = \Leftrightarrow \frac{\partial^2 U}{\partial M^{i^a} \partial M^{i^b}} \lambda_b^+$$

$$\nabla_- \lambda_a^+ \Leftrightarrow 2i f^{abc} \bar{\lambda}_b^+ M_c^* = \Leftrightarrow \frac{\partial^2 U}{\partial M^a \partial M^b} \bar{\lambda}_b^+$$

(46)

plus of course the complex conjugate equations. Furthermore also the parametrization of $\nabla F^a$ is affected by having $P^a$ a non-zero function of $M$. This can be seen from the parametrizations (42). Taking the covariant derivative of $\nabla \lambda_a^+$ and focusing on the $\zeta^+ \zeta^+$ sector one can extract $\nabla_{\Phi^a} \bar{\mathcal{F}}^a$ the component of $\nabla \mathcal{F}^a$ along $\zeta^+$:

$$\nabla_{\Phi^a} \bar{\mathcal{F}}^a = f^{abc} M_b^i \nabla_{\Phi^a} M_c + \frac{i}{2} \frac{\partial^2 U}{\partial M^a \partial M^b} \lambda_b^-$$

(47)

Analogously one gets the other fermionic components of $\nabla \mathcal{F}^a$.

Summarizing in order to obtain $P^a = \frac{\partial U(M)}{\partial M^a} + \frac{\partial U(M)}{\partial M^a} \Gamma$ to reproduce the fermionic field equations (46) and the last terms in the fermionic components of the parametrization (47) of $\nabla \mathcal{F}^a$ we have to set:

$$\Delta \mathcal{L}_0 = 4i \frac{\partial U}{\partial M^a} \left( \frac{F^a}{2} + i P^a e^+ e^- \right) \Leftrightarrow 4i \left[ \frac{\partial U^*}{\partial M^{i^a}} \left( \frac{F^a}{2} \right) e^+ e^- \right]$$

$$+ i \left( \frac{\partial U}{\partial M^a} \frac{\partial^2 U}{\partial M^b \partial M^c} \lambda_b^+ \lambda_c^+ \right) e^+ e^-$$

$$+ \left( \frac{\partial U}{\partial M^a} = \frac{\partial U^*}{\partial M^i} \right) \left[ \left( \lambda_a^+ \zeta^+ \Leftrightarrow \lambda_b^- \zeta^+ \right) e^+ \right. + \left. \left( \lambda_a^+ \zeta^- \Leftrightarrow \lambda_b^- \zeta^- \right) e^- \right]$$

$$2i \left[ 2U \Leftrightarrow M^a \left( \frac{\partial U}{\partial M^a} \Leftrightarrow \frac{\partial U^*}{\partial M^{i^a}} \right) \right] \zeta^+ \bar{\zeta}^- + 2i \left[ 2U^* \Leftrightarrow M^{i^a} \left( \frac{\partial U^*}{\partial M^a} \Leftrightarrow \frac{\partial U}{\partial M^a} \right) \right] \zeta^+ \bar{\zeta}^-$$

(48)

Note that $U$ must be a gauge singlet. A linear potential of the type $U = \sum_a c^a M^a$ with $c^a = \text{const}$ does not satisfy this requirement. Hence the "linear potential" of the abelian case corresponding to the insertion of a Fayet-Iliopoulos term has no non-abelian counterpart. Similarly a $\theta$-term is also ruled out in the non-abelian case. Indeed a term like $\frac{\theta}{2\pi} F^a$ would not be gauge-invariant with a constant $\theta^a$. Also in this case a term of this type would be implied by a linear superpotential $U$ which is therefore excluded. The problem is that no linear function of the gauge scalars $M^a$ can be gauge-invariant.

In conclusion if the Lie algebra $G$ is not semisimple then for each of its $U(1)$ factors we can introduce a Fayet-Iliopoulos and a $\theta$-term. As we are going to see the same property will occur in the $N=4$ case. Fayet-Iliopoulos terms are associated only with abelian factors of the gauge-group namely with the center $\mathcal{Z} \subset G$ of the gauge Lie-algebra. This yield of supersymmetry
perfectly matches with the properties of the Kähler or HyperKähler quotients. Indeed we recall from section I that the level set of the momentum map (see eq.(2) is well-defined only for \( \zeta \in R^3 \otimes \mathbb{Z}^* \) in the HyperKähler case and for \( \zeta \in R \otimes \mathbb{Z} \) in the Kähler case\( \mathbb{Z}^* \) being the center of the dual Lie-algebra \( \mathfrak{g}^* \). Now the level parameters \( \zeta \) are precisely identified with the parameters introduced into the Lagrangian by the Fayet-Iliopoulos terms.

7 R-symmetries of the N=2 Landau-Ginzburg model with and without local gauge symmetries

As we stated in the introduction a crucial role in the topological twist of the N=2 and N=4 theories is played by the so called R-symmetries. These are global symmetries of the rheonomic parametrizations (namely automorphisms of the supersymmetry algebra) and of the action (both the rheonomic one and that concentrated on the bosonic world-sheet) that have a non trivial action also on the gravitino one-forms (in the global theories this means on the supersymmetry parameters but when extending the analysis to the locally supersymmetric case this means also on the world-sheet gravitinos). In the N=2 theories the R-symmetry group is \( U(1)_L \otimes U(1)_R \) the first \( U(1)_L \) acting as a phase rotation \( \zeta^\pm \leftrightarrow \zeta^\pm e^{\pm i\alpha L} \) on the left-moving gravitinos and leaving the right-moving gravitinos invariant the second \( U(1)_R \) factor rotating in the same way the right-moving gravitinos \( \zeta^\pm \leftrightarrow \zeta^\pm e^{\pm i\alpha R} \) and leaving the left-moving ones invariant. In the N=4 case as we are going to see the R-symmetry extends to an \( U(2)_L \otimes U(2)_R \) group each \( U(2) \)-factor acting on a doublet of complex gravitinos \( (\zeta^+, \chi^+) \) with or without the tildas.

We begin by considering the R-symmetries of the N=2 Landau-Ginzburg model with abelian gauge symmetries discussed in the previous sections.

Let us assume that the superpotential \( W(X) \) of the gauge invariant Landau-Ginzburg model is quasi-homogeneous of degree \( d \in R \) with scaling weights \( \omega_i \in R \) for the chiral scalar fields \( X^i \). This means that if we rescale each \( X^i \) according to the rule:

\[
X^i \leftrightarrow \exp[\omega_i \lambda] X^i
\]

where \( \lambda \in C \) is some constant complex parameter then the superpotential rescales as follows:

\[
W \left( e^{\omega_i \lambda} X^i \right) = \exp[d \lambda] \ W \left( X^i \right)
\]

Under these assumption we can easily verify that the rheonomic parametrizations the rheonomic and world-sheet action of the N=2 locally gauge invariant Landau-Ginzburg model are also invariant under the following global \( U(1)_L \otimes U(1)_R \) transformations:
If we define the R-symmetry charges of a field $\varphi$ by means of the formula
\[
\varphi \leftrightarrow \exp\left[ i \left( q_L \alpha_L + q_R \alpha_R \right) \right] J \varphi \tag{52}
\]
then the charge assignments of the locally gauge invariant N=2 Landau-Ginzburg model are displayed in table I.

We can also consider a rigid N=2 Landau-Ginzburg model. By this we mean a Landau-Ginzburg theory of the type described in the previous sections where the coupling to the gauge fields has been suppressed. The structure of such a theory is easily retrieved from our general formulae (24) $\Gamma(27)\Gamma(28)$ by setting the gauge-coupling constant to zero: redefine $q_j^i \leftrightarrow g \bar{q}_j^i$ and then let $g \leftrightarrow 0$. In this limit the matter fields decouple from the gauge fields and we obtain the following world-sheet lagrangian:
\[
\mathcal{L}_{chiral}^{(wa)} = \left( \partial_+ X^i \partial_- X^i + \partial_- X^i \partial_+ X^i \right) \\
+ 2i (\psi^i \partial_- \psi^{i*} + \overline{\psi}^i \partial_- \overline{\psi}^{i*}) + 2i (\psi^{i*} \partial_- \psi^i + \overline{\psi}^{i*} \partial_- \overline{\psi}^i) \\
+ 8 \left\{ \left( \psi^i \overline{\psi}^{i*} \partial_\tau j \mathcal{W} + \text{c.c.} \right) + \partial_\tau \partial_j \mathcal{W}^* \right\} \tag{53}
\]
where to emphasize that we are discussing a different theory we have used a curly letter $\mathcal{W}(X)$ to denote the superpotential. The action (53) defines a model extensively studied in the literature both for its own sake [15] and in its topological version [3]. This action is invariant against the supersymmetry transformations that we derive from the rheonomic parametrizations (24) upon suppression of the gauge coupling ($g \leftrightarrow 0$) namely from:
\[
\nabla X^i = \partial_+ X^i e^+ + \partial_- X^i e^- + \psi^i \zeta^- + \overline{\psi}^i \overline{\zeta}^- \\
\nabla X^{i*} = \partial_+ X^{i*} e^+ + \partial_- X^{i*} e^- \leftrightarrow \psi^{i*} \zeta^+ \leftrightarrow \overline{\psi}^{i*} \overline{\zeta}^+ \\
\nabla \psi^i = \partial_+ \psi^i e^+ + \partial_- \psi^i e^- \leftrightarrow i \partial_+ X^i \zeta^+ + \eta_j^i \partial_j \mathcal{W}^* \zeta^- \\
\nabla \psi^{i*} = \partial_+ \psi^{i*} e^+ + \partial_- \psi^{i*} e^- + i \partial_- X^{i*} \zeta^- + \eta_j^{i*} \partial_j \mathcal{W} \overline{\zeta}^+ 
\]
\[
\n\nabla \tilde{\psi}^i = \partial_+ \psi^i e^+ + \partial_- \psi^i e^- \Leftrightarrow \frac{i}{2} \partial_+ X^i \tilde{\zeta}^+ \Leftrightarrow \eta \tilde{\psi}^i \partial_j W^* \zeta^-
\]
\[
\n\nabla \tilde{\psi}^* = \partial_+ \phi^i e^+ + \partial_- \phi^i e^- + \frac{i}{2} \partial_+ X^i \tilde{\zeta}^* \Leftrightarrow \eta \tilde{\psi}^i \partial_j W \zeta^*
\]

Assuming that under the rescalings (49) the superpotential \( W(X) \) has the scaling property (50) with an appropriate \( d = d_W \) then the rigid Landau-Ginzburg model admits a \( U(1)_L \otimes U(1)_R \) group of R-symmetries whose action on the fields is formally the restriction to the matter fields of the R-symmetries (51) namely:

\[
X^i \Leftrightarrow \exp[i \frac{\omega_i}{d_W} \alpha_+ \alpha_R] X^i \quad X^{i*} \Leftrightarrow \exp[i \frac{\omega_i}{d_W} \alpha_- \alpha_R] X^{i*} \\
\psi^i \Leftrightarrow \exp[i \frac{d_W - \omega_{ij} \alpha_+ \alpha_R}{d_W}] \psi^i \quad \tilde{\psi}^i \Leftrightarrow \exp[i \frac{d_W - \omega_{ji} \alpha_- \alpha_R}{d_W}] \tilde{\psi}^i \\
\psi^{i*} \Leftrightarrow \exp[i \frac{d_W - \omega_{ij} \alpha_+ \alpha_R}{d_W}] \psi^{i*} \quad \tilde{\psi}^{i*} \Leftrightarrow \exp[i \frac{d_W - \omega_{ji} \alpha_- \alpha_R}{d_W}] \tilde{\psi}^{i*}
\]

One however has to be careful that the parameter \( d_W \) in eqs (55) is the scale dimension of the superpotential \( W(X) \) and not \( d \) the scale dimension of the original \( W(X) \) of the gauge coupled model. This discussion is relevant in view of the N=2 gauge model considered by Witten [2] as an interpolation between a rigid N=2 Landau-Ginzburg theory and an N=2 \( \sigma \)-model that appear as the low energy effective actions in two different phases of the same gauge theory. In Witten’s case the superpotential of the locally gauge invariant Landau-Ginzburg theory is chosen as follows:

\[
W(X^i) = X^0 W(X^i)
\]

where the index \( i \) runs on \( n \) values \( i = 1, ..., n \) the index \( I \) runs on \( n + 1 \) values \( I = 0, 1, ..., n \) and \( W(X^i) \) is a quasi-homogeneous holomorphic function of degree \( d_W \) under the rescalings (49) with appropriate choices of the \( \omega_i \). Then choosing arbitrarily a scale weight \( \omega_0 \) for the field \( X^0 \) the complete superpotential \( W(X^i) \) becomes a quasi-homogeneous function of degree \( d = d_W + \omega_0 \). Now in Witten’s model as we are going to see later in our discussion of the N=2 phases there is a phase where the gauge multiplet becomes massive together with the multiplet of \( X^0 \) while all the \( X^i \)-multiplets are massless and have vanishing vacuum expectation values. In this phase the low energy effective action is a rigid Landau-Ginzburg model with superpotential \( W(X^i) \). In this case if we want to identify the R-symmetries of the effective action with those of the original theory something which is important in the discussion of the topological twists we have to be careful to choose \( \omega_0 = 0 \). Only in this case \( d = d_W \) and eqs (55) are truely the restriction of eqs (51).

An extremely opposite case occurs in the N=2 reinterpretations of the N=4 models. As we are going to see also there the superpotential of the gauge model has the structure (56) but in this case the holomorphic function is not quasi-homogeneous a fact that can be retold by saying that \( d_W = 0 \) with \( \omega_i = 0 \). In this case the R-symmetries of the rigid Landau-Ginzburg model (55) are undefined and loose meaning. However from the N=4 structure of the model we
deduce the existence of an R-symmetry where the fields $X^i$ have $q_L = q_R = 0$ and their fermionic partners $\psi^i$ and $\bar{\psi}^i$ have $(q_L = 1, q_R = 0)$ and $(q_L = 0, q_R = 1)$ respectively while $X^0$ has charges $(q_L = \mp 1, q_R = \mp 1)$ it's partners $\psi^0, \bar{\psi}^0$ being assigned the charges $(q_L = \mp 1, q_R = 0)$ and $(q_L = 0, q_R = \mp 1)$ respectively. This result is reconciled with general N=2 formulae if we declare that $\omega_0 = 1$ which implies $d = 1$. With this choice the above charge assignments are the same as those following from formulae (51). The reason why in this case the formulae of the rigid Landau-Ginzburg model (55) become meaningless is simple: in this case differently from Witten's case there is no rigid Landau-Ginzburg phase. For all value of the parameters we end up in a $\sigma$-model phase. Indeed the above assignments of the R-charges is just the one typical of the $\sigma$-model. This will become clear after we have discussed the N=2 $\sigma$-model and its global symmetries.

8 N=2 sigma-models

As a necessary term of comparison for our subsequent discussion of the effective low energy lagrangians of the N=2 matter coupled gauge models and of their topological twists in the present section we consider the rheonomic construction of the N=2 $\sigma$-model. By definition this is a theory of maps:

$$X : \Sigma \leftrightarrow M$$

(57)

from a two-dimensional world sheet $\Sigma$ to a Kähler manifold $M$ whose first Chern number $c_1(M)$ is not necessarily vanishing. In the specific case when $M$ is a Calabi-Yau n-fold ($c_1 = 0$) the $\sigma$-model leads to an N=2 superconformal field theory with central charge $c = 3n$ but as far as ordinary N=2 supersymmetry is concerned the Calabi-Yau condition is not required the only restriction on the target manifold being that it is Kählerian.

Our notation is as follows. The holomorphic coordinates of the Kählerian target manifold $M$ are denoted by $X^i$ ($i = 1, ..., n$) and their complex conjugates by $X^i$. The field content of the N=2 $\sigma$-model is identical with that of the $N = 2$ Landau-Ginzburg theory: in addition to the $X$-fields that transform as world-sheet scalars the spectrum contains four sets of of spin 1/2 fermions $\psi^i, \bar{\psi}^i, \psi^i, \bar{\psi}^i$ that appear in the N=2 rheonomic parametrizations of $dX^i$ and $dX^i*$:

$$dX^i = \Pi^i_+ e^+ + \Pi^i_- e^- + \psi^i \zeta^- + \bar{\psi}^i \bar{\zeta}^-$$

$$dX^i* = \Pi^{i*}_+ e^+ + \Pi^{i*}_- e^- \leftrightarrow \psi^{i*} \zeta^- \leftrightarrow \bar{\psi}^{i*} \bar{\zeta}^-$$

(58)

The equations above are identical with the homologous rheonomic parametrizations of the Landau-Ginzburg theory (the first two of eqs 54). The difference with the Landau-Ginzburg case appears at the level of the rheonomic parametrizations of the fermion differentials. Rather
than the last four of eq.s (54 we write:

\[\nabla \psi^i = \nabla^+_\psi^i e^+ + \nabla^- \psi^i e^- \equiv \frac{i}{2} \Pi^i_+ \zeta^+\]

\[\nabla \bar{\psi}^i = \nabla^+_\bar{\psi}^i e^+ + \nabla^- \bar{\psi}^i e^- \equiv \frac{i}{2} \Pi^i_- \bar{\zeta}^+\]

\[\nabla \psi^{i*} = \nabla^+_\psi^{i*} e^+ + \nabla^- \psi^{i*} e^- + \frac{i}{2} \Pi^{i*}_+ \zeta^+\]

\[\nabla \bar{\psi}^{i*} = \nabla^+_\bar{\psi}^{i*} e^+ + \nabla^- \bar{\psi}^{i*} e^- + \frac{i}{2} \Pi^{i*}_- \bar{\zeta}^+\]

(59)

where the symbol \(\nabla\) denotes the covariant derivative with respect to the target space Levi-Civita connection:

\[\nabla \psi^i = d\psi^i \Leftrightarrow \Gamma^i_{jk} dX^j \psi^k\]

\[\nabla \bar{\psi}^i = d\bar{\psi}^i \Leftrightarrow \Gamma^i_{jk} dX^j \bar{\psi}^k\]

\[\nabla \psi^{i*} = d\psi^{i*} \Leftrightarrow \Gamma^{i*}_{jk} dX^{j*} \psi^{k*}\]

\[\nabla \bar{\psi}^{i*} = d\bar{\psi}^{i*} \Leftrightarrow \Gamma^{i*}_{jk} dX^{j*} \bar{\psi}^{k*}\]

(60)

In agreement with standard conventions the metric, connection and curvature of the Kählerian target manifold are given by:

\[g^{i*} = \frac{\partial}{\partial X^i} \frac{\partial}{\partial X^{i*}} \mathcal{K}\]

\[\Gamma^i_{jk} = \Leftrightarrow g^{i*} \partial_j g^{k*}\]

\[\Gamma^{i*}_{jk} = \Leftrightarrow g^{i*} \partial_j g^{k*}\]

\[\Gamma^i_{j} = \Gamma^i_{jk} dX^k\]

\[R_{i,jk} = g_{i*} R^p_{jk}^{p* l}\]

\[R^p_{jk} = \partial_k \Gamma^p_{j}\]

\[R^i_j = R^i_{jk} dX^{k*} \wedge dX^j\]

(61)

where \(\mathcal{K}(X^*, X)\) denotes the Kähler potential. The parametrizations (58) and (59) are the unique solution to the Bianchi identities:

\[d^2 X^i = d^2 X^{i*} = 0\]

\[\nabla^2 \psi^i \Leftrightarrow R^i_j \psi^j\]

\[\nabla^2 \bar{\psi}^i \Leftrightarrow R^i_j \bar{\psi}^j\]

\[\nabla^2 \psi^{i*} \Leftrightarrow R^{i*}_{j} \psi^{j*}\]

\[\nabla^2 \bar{\psi}^{i*} \Leftrightarrow R^{i*}_{j} \bar{\psi}^{j*}\]

(62)
The complete rheonomic action that yields these parametrizations as outer field equations is given by the following expression:

\begin{align}
S_{\text{rheonomic}} &= \int \left[ g_{ij} \left( \partial^i X^j \right) \left( \partial^j X^i \right) + \phi_{ij} \left( \partial^i \phi_{j}^* \right) \right] d\sigma \\
&\quad+ 2i g_{ij} \left( \partial^i \phi_{j}^* - \partial^j \phi_{i}^* \right) d\sigma + 2i g_{ij} \left( \partial^i \phi_{j}^* - \partial^j \phi_{i}^* \right) d\sigma + 8 R_{ij} \psi_i \psi_j \psi_k \psi_l \right] d^2 z
\end{align}

where we have denoted by

\begin{align}
\nabla_{\pm} \phi^i &= \partial_{\pm} \phi^i \equiv \Gamma_{jk} \partial_{\pm} X^j \psi^k \\
\nabla_{\pm} \phi^* &= \partial_{\pm} \phi^* \equiv \Gamma_{jk} \partial_{\pm} X^* \psi^k
\end{align}

the world-sheet components of the target-space covariant derivatives: identical equations hold for the tilded fermions. The world-sheet action (64) is invariant against the supersymmetry transformation rules descending from the rheonomic parametrizations (58) and (59)\(\Gamma\) namely:

\begin{align}
\delta \phi^i &= \frac{i}{2} \partial_{\pm} X^i \epsilon^\pm \equiv \bar{\epsilon} \Gamma_{jk} \phi^j \psi^k \\
\delta \tilde{\phi}^i &= \frac{i}{2} \partial_{\pm} X^i \bar{\epsilon}^\pm \equiv \epsilon \Gamma_{jk} \phi^j \tilde{\psi}^k \\
\delta \psi^* &= + \frac{i}{2} \partial_{\pm} X^i \epsilon^\mp + \bar{\epsilon} \Gamma_{jk} \phi^j \psi^k \\
\delta \tilde{\psi}^* &= + \frac{i}{2} \partial_{\pm} X^i \bar{\epsilon}^\pm + \epsilon \Gamma_{jk} \phi^j \tilde{\psi}^k
\end{align}

(66)
Comparing with the transformation rules defined by eqs (54) we see that in the variation of the fermionic fields the term proportional to the derivative of the superpotential has been replaced with a fermion bilinear containing the Levi-Civita connection of the target manifold. Indeed one set of rules can be obtained from the other by means of the replacement:

$$
\eta^{ij} \partial_j W^* \leftrightarrow \Gamma^{*}_{jk} \bar{\psi}^i \psi^k \\eta^{ij} \partial_j W \leftrightarrow \Gamma^{*}_{jk} \bar{\psi}^j \psi^k
$$

(67)

This fact emphasizes that in the $\sigma$-model the form of the interaction and hence all the quantum properties of the theory are dictated by the Kähler structure namely by the real non-holomorphic Kähler potential $K(X, X^*)$ while in the Landau-Ginzburg case the structure of the interaction and the resulting quantum properties are governed by the holomorphic superpotential $\mathcal{W}(X)$. In spite of these differences both type of models can yield at the infrared critical point an N=2 superconformal theory and can be related to the same Calabi-Yau manifold. In the case of the $\sigma$-model the relation is most direct: it suffices to take as target manifold $\mathcal{M}$ the very Calabi-Yau n-fold one is interested in and to choose for the Kähler metric $g_{ij}$ one representative in one of the available Kähler classes:

$$
K = i g_{ij} dX^i \wedge dX^j \in \left[ K \right] \in H^{(1,1)} (\mathcal{M})
$$

(68)

If $c_1(\mathcal{M}) = 0\Gamma$ within each Kähler class we can readjust the choice of the representative metric $g_{ij}$ so that at each perturbative order the beta-function is made equal to zero. In this way we obtain conformal invariance and we associate an N=2 superconformal theory with any N=2 $\sigma$-model on a Calabi-Yau n-fold $\mathcal{M}$. The N=2 gauge model discussed in the previous sections interpolates between the $\sigma$-model and the Landau-Ginzburg theory with $\Gamma$ as superpotential the very function $\mathcal{W}(X)$ whose vanishing defines $\mathcal{M}$ as a hypersurface in a (weighted) projective space.

As a matter of comparison a very important issue are the left-moving and right-moving R-symmetries of the $\sigma$-model. Indeed also in this case the rheonomic parametrizations of the rheonomic and world-sheet actions are invariant under a global $U(1)_L \otimes U(1)_R$ group. The action of this group on the $\sigma$-model fields however is different from that on the Landau-Ginzburg fields namely we have:

$$
\zeta^\pm \leftrightarrow \exp[\pm i \alpha_L] \zeta^\pm \quad \tilde{\zeta}^\pm \leftrightarrow \exp[\pm i \alpha_R] \tilde{\zeta}^\pm
$$

$$
X^i \leftrightarrow X^i \quad \tilde{X}^i \leftrightarrow \exp[\pm i \alpha_R] \tilde{X}^i
$$

$$
\psi^i \leftrightarrow \exp[i \alpha_L] \psi^i \quad \tilde{\psi}^i \leftrightarrow \exp[i \alpha_R] \tilde{\psi}^i
$$

$$
\psi^i \leftrightarrow \exp[\mp i \alpha_L] \psi^i \quad \tilde{\psi}^i \leftrightarrow \exp[\mp i \alpha_R] \tilde{\psi}^i
$$

(69)

where $\alpha_L$ and $\alpha_R$ are the two constant phase parameters. The crucial difference of eqs (69) with respect to eqs (55) resides in the R-invariance of the scalar fields $X^i$ that applies to
the $\sigma$-model case but not the Landau-Ginzburg case. As a consequence, in the $\sigma$-model case the fermions have fixed integer R-symmetry charges while in the Landau-Ginzburg case they acquire fractional R-charges depending on the homogeneity degree of the corresponding scalar field and of the superpotential.

9 Extrema of the N=2 scalar potential, phases of the theory and reconstruction of the effective N=2 $\sigma$-model

Now we focus on the effective low-energy theory emerging from the $N = 2$ gauge plus matter systems described in the above sections. Our considerations remain at a classical level. We are mostly interested in the case where the effective theory is an $N = 2 \sigma$-model. We show how the $N = 2 \sigma$-model Lagrangian is technically retrieved in a manner that is intimately related with the momentum map construction. Indeed this latter is just the geometrical counterpart of the physical concept of low-energy effective Lagrangian. To be simple we perform our computations in the case where the target space of the low-energy $\sigma$-model is the manifold $\mathbb{CP}^N$.

First of all we need to recall the structure of the classical vacua for a system described by the Lagrangian (29) referring to the linear superpotential case: $U = (\frac{1}{\xi} \leftrightarrow \frac{\delta}{\delta \phi}) M$; this structure was studied in Witten's paper [2]. We set the fermions to zero and we have to extremize the scalar potential (34). Since $U$ is given by a sum of moduli squared, this amounts to equate each term in (34) separately to zero. A particularly interesting situation arises when the Landau-Ginzburg potential has the form

$$ W = X^0 \mathcal{W}(X^i) $$

(70)

Here $\mathcal{W}(X^i)$ is a quasihomogeneous function of degree $d$ of the fields $X^i$ that are assigned the weights $q^i$; i.e., their charges with respect to the abelian gauge group. In the case all the charges $q^i$ are equal (say all equal to $\Gamma$ for simplicity) $\mathcal{W}(X^i)$ is homogeneous. $X^0$ is a scalar field of charge $\leftrightarrow d$. $\mathcal{W}(X^i)$ must moreover be transverse: $\partial_i \mathcal{W} = 0 \forall i$ iff $X^i = 0 \forall i$.

In this case we have:

$$ U = \frac{1}{2}(r + d|X^0|^2 \leftrightarrow \sum_i q^i |X^i|^2)^2 + 8|\mathcal{W}(X^i)|^2 + 8|X^0|^2 |\partial_i \mathcal{W}|^2 $$

$$ + 8 |M|^2 \left( d^2 |X^0|^2 + \sum_i (q^i)^2 |X^i|^2 \right), $$

(71)

and two possibilities emerge.

- $r > 0$. In this case some of the $X^i$ must be different from zero. Due to the transversality of $\mathcal{W}$ it follows that $X^0 = 0$. The space of classical vacua is characterized not only by having $X^0 = 0$ and $M = 0 \forall$ but also by the condition $\sum_i q^i |X^i|^2 = r$. When $q^i = 1 \forall i$ this condition together with the $U(1)$ gauge invariance is equivalent to the statement
that the $X^i$ represent coordinates on $\mathbb{CP}^N$. In general $\Gamma$ the $X^i$'s are coordinates on the weighted projective space $\mathbb{WP}^N_{q_1,\ldots,q_N}$. The last requirement $\Gamma \mathcal{W}(X^i) = 0 \forall i$ defines the space of classical vacua as a transverse hypersurface embedded in $\mathbb{CP}^N$ or $\Gamma$ in general $\Gamma$ in $\mathbb{WP}^N_{q_1,\ldots,q_N}$. The low energy theory around these vacua is expected to correspond to the $N = 2$ $\sigma$-model on such a hypersurface. Indeed $\Gamma$ studying the quadratic fluctuations one sees that the gauge field $A$ acquires a mass due to a Higgs phenomenon; the gauge scalar $M$ becomes massive together with those modes of the matter fields that are not tangent to the hypersurface. The only massless degrees of freedom $\Gamma$ i.e. those described by the low energy theory $\Gamma$ are the excitations tangent to the hypersurface. The fermionic partners behave consistently. We are in the “$\sigma$-model phase”.

• $r < 0$. In this case $X^0$ must be different from zero. Then it is necessary that $\partial_i \mathcal{W} = 0 \forall i$; this implies by transversality that all the $X^i$ vanish. The space of classical vacua is just a point. Indeed utilizing the gauge invariance we can reduce $X^0$ to be real so that it is fixed to have the constant value $X^0 = \sqrt{-r}$. $M$ vanishes together with the $X^i$. The low energy theory can now be recognized to be a theory of massless fields $\Gamma$ the $X^i$'s governed by a Landau Ginzburg potential which is just $\mathcal{W}(x^i)$. We are in the “Landau-Ginzburg phase”.

Now we turn our attention to the $\mathbb{CP}^N$-model $\Gamma$ which corresponds to the particular case in which all the charges are equal to 1 and $W = 0$. As it is easy to see from the above discussion $\Gamma$ in this case the only possible vacuum phase is the $\sigma$-model phase $\Gamma$ i.e one must have $r > 0$. We start by writing the complete rheonomic lagrangian of the system consisting of $N + 1$ chiral multiplets with no selfinteraction $(X^A, \bar{\psi}^A, \bar{\psi}^A), A = 1, \ldots, N$ coupled to an abelian gauge multiplet $\Gamma$ each with charge one. Differently to what we did in the previous sections $\Gamma$ in this section we make the dependence on the gauge coupling constant $g$ explicit. To reinstall $g$ appropriately $\Gamma$ after reinserting it into the covariant derivatives $\Gamma \nabla X^A = dx^A + igAX^A \Gamma$ we redefine the fields of the gauge multiplet as follows:

$$A \leftrightarrow \frac{1}{g} A ; \quad M \leftrightarrow \frac{1}{g} M ; \quad \lambda \leftrightarrow \frac{1}{g} \lambda$$

(72)

so that at the end no modification occurs in the matter lagrangian $\Gamma$ while the gauge kinetic lagrangian is multiplied by $\frac{1}{g^2}$. Altogether we have:

$$\mathcal{L} = \frac{F}{g^2} \left[ F + \frac{i}{2}(\lambda^+ \zeta^- + \lambda^- \zeta^+) \epsilon^- \leftrightarrow \frac{i}{2}(\lambda^+ \hat{\zeta}^- + \lambda^- \hat{\zeta}^+) \epsilon^+ \leftrightarrow M \hat{\zeta}^- \hat{\zeta}^+ \leftrightarrow M^* \zeta^- \zeta^+ \right]$$

$$\leftrightarrow \frac{1}{2g^2} F^2 \epsilon^+ \epsilon^- \leftrightarrow \frac{i}{2g^2}(\lambda^+ d\lambda^- + \lambda^- d\lambda^+) \epsilon^- + \frac{i}{2g^2}(\lambda^+ d\lambda^+ + \lambda^- d\lambda^-) \epsilon^+$$

$$\leftrightarrow \frac{4}{g^2} \left[ dM^* \leftrightarrow - \frac{1}{4}(\lambda^+ \zeta^- \leftrightarrow \lambda^- \zeta^+) \right] (M_+ \epsilon^+ \leftrightarrow M_- \epsilon^-)$$
\[\begin{align*}
&\Rightarrow \frac{4}{g^2}[dM + \frac{1}{4}(\lambda^--\zeta^+ \Leftrightarrow \lambda^+\zeta^-)](M_+^*e^+ \Leftrightarrow M_-^*e^-) \\
&\Rightarrow \frac{4}{g^2}(M_+^*M_- + M_-^*M_+) e^+e^- \Rightarrow \frac{1}{g^2}dM(\bar{\lambda}^-\zeta^+ + \lambda^+\zeta^-) + \frac{1}{g}dM(\bar{\lambda}^+\zeta^- + \lambda^-\zeta^+) \\
&\Rightarrow \frac{1}{4g^2}(\lambda^+\lambda^-\zeta^-\zeta^+ + \lambda^-\lambda^+\zeta^+\zeta^-) + \frac{2}{g^2}\mathcal{P}^2 e^+e^- \Rightarrow 2r\mathcal{P} e^+e^- + \frac{\theta}{2\pi}F \\
&+ \frac{r}{2g^2}\left[(\lambda^+\zeta^- \Leftrightarrow \lambda^-\zeta^+) e^- + (\lambda^+\zeta^- \Leftrightarrow \lambda^-\zeta^+) e^+\right] + i\frac{r}{g\pi}\left(M\zeta^-\zeta^+ + M^*\zeta^+\zeta^-\right) \\
&(\nabla X^A \Leftrightarrow \psi^A \zeta^- \Leftrightarrow \tilde{\psi}^A \zeta^-)(\Pi_A^*e^+ \Leftrightarrow \Pi_A^*e^-) \\
&(\nabla X^{A*} + \psi^A \zeta^+ + \tilde{\psi}^A \zeta^+)(\Pi_A^*e^+ \Leftrightarrow \Pi_A^*e^-) \\
&+ (\Pi_A^*\Pi_A^* \Leftrightarrow \Pi_A^*\Pi_A^*) e^+e^- + 2i(\psi^A \nabla \psi^A^* + \psi^*\nabla \psi^A) e^+ \\
&\Rightarrow 2i(\tilde{\psi}^A \nabla \tilde{\psi}^A^* + \tilde{\psi}^A \nabla \tilde{\psi}^A) e^- \Leftrightarrow \psi^A \psi^A^* \zeta^-\zeta^+ + \tilde{\psi}^A \tilde{\psi}^A^* \zeta^-\zeta^+ \\
&\Rightarrow 2i(\tilde{\psi}^A \tilde{\psi}^A^* \zeta^-\zeta^+ \Leftrightarrow \psi^A \psi^A^* \zeta^-\zeta^+ + \tilde{\psi}^A \tilde{\psi}^A^* \zeta^-\zeta^+) \\
&\Rightarrow \nabla X^A(\psi^A \zeta^- \Leftrightarrow \tilde{\psi}^A \zeta^-) + 4M X^A \psi^A \zeta^-\zeta^+ e^- \Leftrightarrow 4M X^A \psi^A \zeta^-\zeta^+ e^- \\
&+ 4M X^A \tilde{\psi}^A \tilde{\psi}^A^* \zeta^-\zeta^+ e^- \Rightarrow 4M X^A \tilde{\psi}^A \tilde{\psi}^A^* \zeta^-\zeta^+ e^- \\
&+ \left\{8iM^* \tilde{\psi}^A \psi^A + 8iM \tilde{\psi}^A \psi^A + 2i\lambda^+ \psi^A X^A + 2i\lambda^- \psi^A X^A^* \right\} e^+e^- \\
&\Rightarrow 2i\tilde{\lambda}^+ \tilde{\psi}^A X^A \Leftrightarrow 2i\tilde{\lambda}^- \tilde{\psi}^A X^A + 2\mathcal{P} X^A X^A \Leftrightarrow 8M^*M X^A X^A \Rightarrow (73)
\end{align*}\]

The procedure that we follow to extract the effective lagrangian is the following. We let the gauge coupling constant go to infinity and we are left with a gauge invariant lagrangian describing matter coupled to gauge fields that have no kinetic terms. Varying the action in these fields the resulting equations of motion express the gauge fields in terms of the matter fields. Substituting back their expressions in the lagrangian we end up with a matter coupled to gauge fields that have no kinetic terms. Varying the action in these fields, the gauge fields, we obtain the low-energy effective action around the classical vacua of the complete manifold the quotient of the manifold spanned by the matter fields with respect to the action of the gauge group \([9]\). This procedure is nothing else from the functional integral viewpoint but the gaussian integration over the gauge multiplet in the limit \(g \Leftrightarrow \infty\). As already pointed out in the introduction to consider a gauge coupled lagrangian without gauge kinetic terms is not a mere trick to implement the quotient procedure in a Lagrangian formalism. It rather amounts to deriving the low-energy effective action around the classical vacuum of the complete gauge plus matter system. Indeed we have seen that around these vacua the oscillations of the gauge fields are massive and thus decouple from the low-energy point of view. So we integrate over them: furthermore all masses are proportional to \(\frac{1}{g}\) and the integration makes sense for energy-scales \(E \ll \frac{1}{g}\) namely in the limit \(g \Leftrightarrow \infty\).

Here we show in detail how the above-sketched procedure works at the level of the rheonomic approach. In this way we retrieve the rheonomic lagrangian and the rheonomic parametrizations of the \(N = 2\) \(\sigma\)-model as described in section (8) the target space being \(\mathbb{CP}^N\) equipped with the standard Fubini-Study metric. The whole procedure amounts geometrically to realize \(\mathbb{CP}^N\)
Let us consider the lagrangian \((73)\Gamma\) in the limit \(g \leftrightarrow \infty\) and let us perform the variations in the gauge fields.

The variations in \(\lambda^-, \tilde{\lambda}^+, \lambda^-, \lambda^+\) give the following “fermionic constraints”:

\[
X^A \psi^A = X^A \bar{\psi}^A = X^A \bar{\psi}^A = 0
\]  

(74)

Here the summation on the capital index \(A\) is understood. In the following we use simplified notations such as \(X\psi^\ast\) for \(X^A \psi^A \Gamma\) and the like everywhere it is possible without generating confusion.

The fermionic constraints (74) are explained by the bosonic constraint \(X^*X = r\Gamma\) for which the auxiliary field \(\mathcal{P}\Gamma\) in the limit \(g \leftrightarrow \infty\) becomes a Lagrange multiplier. Indeed taking the exterior derivative of this bosonic constraint we obtain \(0 = d(X^*X) = X^*dX + XdX^*\) and substituting the rheonomic parametrizations (24) in the gravitino sectors this implies

\[
X^*(\psi \bar{\zeta}^- + \bar{\psi} \zeta^-) \Leftrightarrow X(\psi^* \zeta^+ + \bar{\psi}^* \bar{\zeta}^+) = 0
\]  

(75)

from which (74) follows.

The variation of the action with respect to \(M^*\) in the gravitino sectors implies again the fermionic constraints (74). In the \(e^+e^-\) sector we get the following equation of motion:

\[
M = \frac{i \bar{\psi}^* \psi}{X^*X}
\]  

(76)

The terms in the lagrangian (73) containing the connection \(A\) are hidden in the covariant derivatives. Explicitly they are:

\[
\Leftrightarrow \mathcal{A}X^A(\Pi^*_+ e^+ \Leftrightarrow \Pi^*_+ e^-) + i \mathcal{A}X^A(\Pi^*_+ e^+ \Leftrightarrow \Pi^*_+ e^-) + 2i \psi^A(\Leftrightarrow i)\mathcal{A} \psi^A e^+ \\
+ 2i \psi^A i \mathcal{A} \psi^A e^+ \Leftrightarrow 2i \bar{\psi}^A(\Leftrightarrow i)\mathcal{A} \bar{\psi}^A e^- \Leftrightarrow 2i \bar{\psi}^A i \mathcal{A} \bar{\psi}^A e^- \\
+ i \mathcal{A} X^A(\psi^A \bar{\zeta}^+ \Leftrightarrow \bar{\psi}^A \zeta^+) + i \mathcal{A} X^A(\psi^A \zeta^- \Leftrightarrow \bar{\psi}^A \bar{\zeta}^-) + \frac{\theta}{2\pi} d\mathcal{A}
\]  

(77)

In the gravitino sector we again retrieve the constraints (74). In the \(e^+\) and \(e^-\) sector we respectively obtain:

\[
i X^A \Pi^*_+ \Leftrightarrow i X^A \Pi^*_+ \Leftrightarrow 4 \psi^A \psi^A = 0 \\
is X^A \Pi^*_+ + i X^A \Pi^*_+ + 4 \bar{\psi}^A \bar{\psi}^A = 0
\]  

(78)

At this point we take into account the variations with respect to the first order fields \(\Pi\) that give \(\Pi^*_+ = \nabla_+ X^A = \nabla_+ X^A + i \mathcal{A}_+ X^A \Gamma\) and so on. Substituting into eq.s (78) and solving for...
\[ A_+ \quad \text{and} \quad A_- \ \text{we get:} \]

\[
A_+ = \frac{\left\langle (X \partial_+ X^* \leftrightarrow X^* \partial_+ X) + 4 \psi \tilde{\psi}^* \right\rangle}{2X^2} \]

\[
A_- = \frac{\left\langle (X \partial_- X^* \leftrightarrow X^* \partial_- X) + 4 \tilde{\psi} \psi^* \right\rangle}{2X^2} \quad (79)
\]

Substituting back the expression (76) for \( M \) into the lagrangian (73) in the \( g \leftrightarrow \infty \) limit we have

\[
\mathcal{L} = \left\langle dX^A + iX^A (A_+ e^+ + A_- e^-) \right\rangle \psi^A \zeta^- \equiv \tilde{\psi}^A \tilde{\zeta}^- \left( \Pi^A_+ e^+ \equiv \Pi^A_- e^- \right)
\]

\[
\equiv \left[ dX^A \equiv iX^A (A_+ e^+ + A_- e^-) + \psi^A \zeta^+ + \tilde{\psi}^A \tilde{\zeta}^+ \left( \Pi^A_+ e^+ \equiv \Pi^A_- e^- \right) \right] \]

\[
\equiv \left( \Pi^A_+ \Pi^A_+ + \Pi^A_- \Pi^A_- \right) e^+ e^- + 2i \left( \psi^A d\psi^A^* + \psi^A^* d\psi^A \right) \equiv 2i \left( A_+ e^+ \tilde{\psi}^A \psi^A \right) e^+ e^- \]

\[
\equiv 2i \left( \tilde{\psi}^A d\tilde{\psi}^A^* + \tilde{\psi}^A^* d\tilde{\psi}^A \right) \equiv 2i \left( A_+ e^+ \tilde{\psi}^A \psi^A \right) e^+ e^-
\]

\[
\equiv \psi^A \tilde{\psi}^A \zeta^- \tilde{\zeta}^+ \equiv \psi^A \tilde{\psi}^A \zeta^- \tilde{\zeta}^+ \equiv \psi^A \tilde{\psi}^A \zeta^- \tilde{\zeta}^+ \equiv \psi^A \tilde{\psi}^A \zeta^- \tilde{\zeta}^+ \equiv \psi^A \tilde{\psi}^A \zeta^- \tilde{\zeta}^+ \equiv \psi^A \tilde{\psi}^A \zeta^- \tilde{\zeta}^+
\]

\[
+ dX^A (\psi^A \zeta^+ \equiv \tilde{\psi}^A \tilde{\zeta}^+ \equiv dX^A (\psi^A \zeta^- \equiv \tilde{\psi}^A \tilde{\zeta}^-)
\]

\[
\approx 8 \frac{\tilde{\psi}^A \psi^B \tilde{\psi}^B \tilde{\psi}^B \psi^A \zeta^-}{X^2X} e^+ e^- + 2 \mathcal{P} (v \leftrightarrow X^* X) e^+ e^-
\quad (80)
\]

where \( A_+ \) and \( A_- \) are to be identified with their expressions (79). To obtain this expression we have also used the “fermionic constraints” (74). The \( U(1) \) gauge invariance of the above lagrangian can be extended to a \( C^* \)-invariance where \( C^* \equiv C \{ 0 \} \) is the complexification of the \( U(1) \) gauge group \( \Gamma \) by introducing an extra scalar field \( v \) transforming appropriately.

Consider the \( C^* \) gauge transformation given by

\[
X^A \quad \equiv \quad e^{i \Phi} X^A \quad ; \quad \psi^A \quad \equiv \quad e^{i \Phi} \psi^A
\]

\[
X^{A*} \quad \equiv \quad e^{-i \Phi^*} X^{A*} \quad ; \quad \ldots \quad ; \quad (\Phi \in C) \quad (81)
\]

which is just the complexification of the \( U(1) \) transformation \( \Gamma \) the latter corresponding to the case \( \Phi \in \mathbb{R} \Gamma \) supplemented with

\[
v \quad \leftrightarrow \quad v + \frac{i}{2} \left( \Phi \leftrightarrow \Phi^* \right) \quad (82)
\]

One realizes that under the transformations \( (81) \Gamma (82) \) the combinations \( e^{-v} X^A \) (and similar ones) undergo just a \( U(1) \) transformation:

\[
\begin{align*}
\quad e^{-v} X^A & \iff e^{i \Re \Phi} e^{-v} X^A \\
\quad e^{-v} X^{A*} & \iff e^{-i \Re \Phi} e^{-v} X^{A*}
\end{align*}
\quad (83)
\]

By substituting

\[
X^A, \psi^A, \psi^A, \psi^A, \Pi^A_+ \ldots \iff e^{-v} X^A, e^{-v} \psi^A, e^{-v} \psi^A, \ldots
\quad (84)
\]

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into the lagrangian \((80)\) we obtain an expression which is invariant with respect to the \(C^*\)-transformations \((81)\). In particular the last term of \((85)\) becomes

\[
\epsilon^{-\nu} = \frac{\rho}{\sqrt{X^*X}} \quad (85)
\]

If at this point we perform the so far delayed variation with respect to the auxiliary field \(\mathcal{P}\) the resulting equation of motion identifies the extra scalar field \(v\) in terms of the matter fields. Introducing \(\rho^2 \equiv r\) the result is that

\[
\epsilon^{-\nu} = \frac{\rho}{\sqrt{X^*X}} \quad (86)
\]

What is the geometrical meaning of the above “tricks” (introduction of the extra field \(v\)) consideration of the complexified gauge group)? The answer relies on the properties of the Kähler quotient construction; extensively discussed in \([9]\)\'[11]. Let us recall few concepts keeping always in touch with the example we are dealing with. We use the notions and notations introduced in section I.

Let \(Y(s) = Y^s k_s(s)\) be a Killing vector on \(S\) (in our case \(C^{N+1}\)) belonging to \(G\) (in our case \(R\)). \(Y\) is the algebra of the gauge group. In our case \(Y\) has a single component: \(Y = i\Phi(X^A \frac{\partial}{\partial X^A} \Leftrightarrow X^{\star A} \frac{\partial}{\partial X^{\star A}})\) (\(\Phi \in R\)). The \(X^A\)'s are the coordinates on \(S\). Consider the vector field \(\mathbf{I}Y \in G^c\) (the complexified algebra) \(I\) being the complex structure acting on \(T_S\). In our case \(\mathbf{I}Y = \Phi(X^A \frac{\partial}{\partial X^A} + X^{\star A} \frac{\partial}{\partial X^{\star A}})\). This vector field is orthogonal to the hypersurface \(D^{-1}(\zeta)\) for any level \(\zeta\); that is it generates transformations that change the level of the surface. In our case the surface \(D^{-1}(\rho^2) \in C^{N+1}\) is defined by the equation \(X^A X^{*A} = \rho^2\). The infinitesimal transformation generated by \(\mathbf{I}Y\) is \(X^A \rightarrow (1 + \Phi)X^A \Gamma X^{*A} \rightarrow (1 + \Phi)X^{*A}\) so that the transformed \(X^A\)'s satisfy \(X^A X^{*A} = (1 + 2\Phi)\rho^2\). As recalled in section \(\Pi\) the Kähler quotient consists in starting from \(\mathcal{S}\) restricting to \(\mathcal{N} = D^{-1}(\zeta)\) and taking the quotient \(\mathcal{M} = \mathcal{N}/G\). The above remarks about the action of the complexified gauge group suggest that this is equivalent (at least if we skip the problems due to the non-compactness of \(G^c\)) to simply taking the quotient \(S/G^c\) the so-called “algebro-geometric” quotient \([9]\)\'[12].

The Kähler quotient allows in principle to determine the expression of the Kähler form on \(\mathcal{M}\) in terms of the original one on \(S\). Schematically, let \(j\) be the inclusion map of \(\mathcal{N}\) into \(\mathcal{S}\) the projection from \(\mathcal{N}\) to the quotient \(\mathcal{M} = \mathcal{N}/G\) the Kähler form on \(S\) and \(\omega\) the Kähler form on \(\mathcal{M}\). It can be shown \([9]\) that

\[
\begin{align*}
S & \leftrightarrow \mathcal{N} = D^{-1}(\zeta) \leftrightarrow \mathcal{M} = \mathcal{N}/G \\
\Omega & \leftrightarrow j^*\Omega = p^*\omega \leftrightarrow \omega
\end{align*}
\]

(87)

In the algebro-geometric setting the holomorphic map that associates to a point \(s \in S\) (for us \(\{X^A\} \in C^{N+1}\)) its image \(m \in \mathcal{M}\) is obtained as follows:
i) Bringing \( s \) to \( \mathcal{N} \) by means of the finite action infinitesimally generated by a vector field of the form \( \mathbf{V} = I \mathbf{Y} = V^a k_a \)

\[
\pi : \quad s \in \mathcal{S} \iff e^{-V}s \in D^{-1}(\zeta) \quad (88)
\]

ii) Projecting \( e^{-V} \) to its image in the quotient \( \mathcal{M} = \mathcal{N}/G \).

Thus we can consider the pullback of the Kähler form \( \omega \) through the map \( p \cdot \pi \):

\[
\mathcal{S} \xrightarrow{\pi} \mathcal{N} = D^{-1}(\zeta) \xrightarrow{\pi} \mathcal{N}/G \quad \pi^* p^* \omega \iff p^* \omega \iff \omega \quad (89)
\]

Looking at (87) we see that \( \pi^* p^* \omega = \pi^* j^* \Omega \) so that at the end of the day in order to recover the pullback of \( \omega \) to \( \mathcal{S} \) it is sufficient:

i) to restrict \( \Omega \) to \( \mathcal{N} \)

ii) to pull back this restriction to \( \mathcal{M} \) with respect to the map \( \pi = e^{-V} \).

We see from (88) that the components of the vector field \( \mathbf{V} \) must be determined by requiring

\[
D(e^{-V}s) = \zeta \quad (90)
\]

But this is precisely effected in the lagrangian context by the term having as Lagrange multiplier the auxiliary field \( \mathcal{P} \) see eq. (86) through the equation of motion of \( \mathcal{P} \) once we have introduced the extra field \( v \) (which is now interpreted as the unique component of the vector field \( \mathbf{V} \)) to make the lagrangian invariant under the complexified gauge group \( C^* \). The lagrangian formalism of \( N = 2 \) supersymmetry perfectly matches the key points of the momentum map construction. This allows us to determine the form of the map \( \pi : \) it corresponds to the transformations (84). The steps that we are going to discuss in treating the lagrangian just consist in implementing the Kähler quotient as in (89). Thus it is clear why at the end we obtain the \( \sigma \)-model on the target space \( \mathcal{M} \) (in our case \( \mathbb{C}P^N \)) endowed with the Kähler metric corresponding to the Kähler form \( \omega \). In our example such metric is the Fubini-Study metric. Indeed one can show in full generality [9] that the Kähler potential \( \hat{K} \) for the manifold \( \mathcal{M} \) such that \( \omega = 2i\partial \bar{\partial} \hat{K} \) is given by

\[
\hat{K} = K|_{\mathcal{N}} + V^a \zeta_a \quad (91)
\]

Here \( K \) is the Kähler potential on \( \mathcal{S} \); \( K|_{\mathcal{N}} \) is the restriction of \( K \) to \( \mathcal{N} \) that is it is computed after acting on the point \( s \in \mathcal{S} \) with the transformation \( e^{-V} \) determined by eq. (90); \( V^a \) are the components of the vector field \( \mathbf{V} \) along the \( a \)-th generator of the gauge group \( \Gamma \) and \( \zeta_a \) those of the level \( \zeta \) of the momentum map. In our case we have the single component \( v \) given by eq. (86) and we named \( \rho^2 \) the single component of the level. The original Kähler potential on \( \mathcal{S} = \mathbb{C}^{N+1} \) is \( K = \frac{1}{2} X^*X \) so that when restricted to \( D^{-1}(\rho^2) \) it takes an irrelevant constant value \( \frac{\rho^2}{2} \). Thus we deduce from (91) that the Kähler potential for \( \mathcal{M} = \mathbb{C}P^N \) that we obtain is \( \hat{K} = \frac{1}{2} \rho^2 \log (X^*X) \). Fixing a particular gauge to perform the quotient with respect to \( C^* \) (see later) this potential can be rewritten as \( \hat{K} = \frac{1}{2} \rho^2 \log(1 + x^*x) \) namely the Fubini-Study potential.

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Let us now proceed with our manipulations of the lagrangian. It is a trivial algebraic matter to rewrite the lagrangian (80) after the substitutions (84) with $e^{-\nu}$ given by eq. (86). For convenience we divide the resulting expressions into three parts to be separately handled.

First we have what we can call the “bosonic kinetic terms”:

$$
L_1 = \left[ \rho^2 \sum_A \sum_B \left[ \left( \delta_{AB} \implies X^A X^B \right) \right] dX^B \implies \frac{X^A X^B}{2 X^† X} dX^B \right]
$$

We would like to recognize in the above expressions the bosonic kinetic terms of an $N = 2$ $\sigma$-model. By looking at the $\sigma$-model rheonomic lagrangian (63) we are inspired to perform a series of manipulations.

Collecting some suitable terms we can rewrite

$$
X^A (X \partial_+ X^* e^+ + X \partial_- X^* e^-) \implies X^A X dX^* \\
X^A (X^* \partial_+ X e^+ + X^* \partial_- X e^-) \implies X^A X^* dX
$$

due to the fact that the further terms in the rheonomic parametrizations of $dX, dX^* \Gamma$ proportional to the gravitinos give here a vanishing contribution in force of the constraints (74).

We introduce the following provisional notation:

$$
G_{AB} = \frac{\rho^2}{X^† X} \left( \delta_{AB} \implies \frac{X^A X^B}{X^† X} \right).
$$

Noting that because of the constraints (74) $\Gamma$

$$
G_{AB} \psi^A = \frac{\rho^2}{X^† X} \psi^A
$$

we can write

$$
L_1 = \left[ G_{AB} \right] \left[ dX^A \implies \psi^A \zeta^- \implies \bar{\psi}^A \bar{\zeta}^- \right]
$$

$$
+ 2i \rho^2 \frac{X^B}{X^† X} (\psi \psi^* e^+) \implies X^A (\psi^A \psi^* e^-) \implies c.c.
$$

$$
\implies \frac{\rho^2}{X^† X} \left( \Pi^+_+, \Pi^*_+ - \Pi^+ \Pi^+ \right) e^+ e^-\left( \Pi^+_+, \Pi^*_+ - \Pi^+ \Pi^+ \right)
$$

In order to eliminate the terms containing the first order fields $\Pi$′s multiplied by fermionic expressions we redefine the $\Pi$′s:

$$
\Pi^A \rightarrow \Pi^A + 2i X^A \bar{\psi}^* \frac{\psi^*}{X^† X} \quad \Pi^A \rightarrow \Pi^A \equiv 2i X^A \bar{\psi}^* \frac{\psi^*}{X^† X}
$$

$$
\Pi^+_+ \rightarrow \Pi^+_+ + 2i X^A \bar{\psi}^* \frac{\psi^*}{X^† X} \quad \Pi^+_+ \rightarrow \Pi^+_+ \equiv 2i X^A \bar{\psi}^* \frac{\psi^*}{X^† X}
$$

(97)
Then we perform a second redefinition of the $\Pi$’s:

$$
\Pi^A_\pm \to \left( \delta_{AB} \pm \frac{X^A X^B}{X^* X} \right) \Pi^B_\pm \\
\Pi^{*\pm} \to \left( \delta_{AB} \pm \frac{X^A X^B}{X^* X} \right) \Pi^{*B}_{\pm}
$$

in such a way that the quadratic term in the first order fields takes the form

$$
\Longleftrightarrow G_{AB^*}(\Pi^A_+ \Pi^B_{-} + \Pi^A_- \Pi^B_+)^{e^+ e^-}
$$

(99)

After the redefinitions (97) and (98) we can rewrite the part $\mathcal{L}_1$ of the Lagrangian in the following way; we take into account besides the constraints (74) the fact that

$$
G_{AB^*} a^A X^{B^*} \propto \left( \delta_{AB} \Longleftrightarrow \frac{X^A X^B}{X^* X} \right) a^A X^{B^*} = 0
$$

(100)

and we obtain:

$$
\mathcal{L}_1 = \Longleftrightarrow G_{AB^*}(dX^A \Longleftrightarrow \bar{\psi}^A \zeta^- \Longleftrightarrow \bar{\psi}^A \bar{\zeta})(\Pi^B_e \equiv \Pi^B_+ \equiv \Pi^B_-)
$$

$$
\Longleftrightarrow G_{AB^*}(dX^{B^*} + \bar{\psi}^{B^*} \bar{\zeta} + \bar{\psi}^{B^*} \bar{\zeta}^+)(\Pi^B_e \equiv \Pi^B_+ \equiv \Pi^B_-)
$$

$$
\Longleftrightarrow G_{AB^*}(\Pi^A_+ \Pi^B_- + \Pi^A_- \Pi^B_+)^{e^+ e^-} + \frac{8 \rho^2}{(X^* X)^2} \bar{\psi}^* \bar{\psi}^* \bar{\psi}^* \bar{\psi}^* e^+ e^-
$$

(101)

Next we consider the fermionic kinetic terms in eq. (80). Performing the substitutions (84) with $v$ given by eq. (86) and using the fact that for instance

$$
\frac{\rho^2}{X^* X} \psi^A d\psi^A = G_{AB^*} \psi^A d\psi^{B^*}
$$

(102)

these terms are

$$
\mathcal{L}_2 = 2i \left\{ G_{AB^*} (\psi^A d\psi^{B^*} + \psi^{B^*} d\psi^A) \Longleftrightarrow \frac{\rho^2}{(X^* X)^2} \psi^A \bar{\psi}^A (X \partial_- X^* \Leftrightarrow X^* \partial_- X)^{e^-} \right\} e^+
$$

$$
\Longleftrightarrow 2i \left\{ G_{AB^*} (\bar{\psi}^A d\bar{\psi}^{B^*} + \bar{\psi}^{B^*} d\bar{\psi}^A) + \frac{\rho^2}{(X^* X)^2} \bar{\psi}^A \bar{\psi}^A (X \partial_+ X^* \Leftrightarrow X^* \partial_+ X)^{e^-} \right\} e^-
$$

$$
\Longleftrightarrow 16 \frac{\rho^2}{(X^* X)^2} \bar{\psi}^* \bar{\psi}^* \bar{\psi}^* \bar{\psi}^* e^+ e^-
$$

(103)

Let us introduce another provisional notation:

$$
\gamma^A_{BC} = \frac{1}{X^* X} (\delta^A_B X^C + \delta^A_C X^B)
$$

(104)

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It is not difficult to check that the expression (103) can be rewritten as follows:

\[
\mathcal{L}_2 = 2i \left\{ G_{AB} \psi^A (d\psi^B* \Leftrightarrow \gamma^B_{B'C'} \psi^{C*} \, dX^{D*}) + G_{AB} \psi^B (d\psi^A \Leftrightarrow \gamma^A_{CD} \psi^C \, dX^D) \right\} e^+ \\
\Rightarrow 2i \left\{ G_{AB} \tilde{\psi}^A (d\tilde{\psi}^B* \Leftrightarrow \gamma^B_{B'C'} \tilde{\psi}^{C*} \, dX^{D*}) + G_{AB} \tilde{\psi}^B (d\tilde{\psi}^A \Leftrightarrow \gamma^A_{CD} \tilde{\psi}^C \, dX^D) \right\} e^- \\
\Rightarrow 16 \frac{\beta^2}{(X^*X)^2}(\tilde{\psi} \psi^*)(\tilde{\psi} \psi^*)
\] (105)

The remaining terms in the lagrangian (80) become after the substitutions (84)

\[
\mathcal{L}_3 = \left\{ \frac{8\beta^2}{(X^*X)^2} \psi^A \psi^B \tilde{\psi}^A \tilde{\psi}^B \equiv G_{AB} \psi^A \psi^B \right\} \left\{ \psi^A \psi^B \tilde{\psi}^A \tilde{\psi}^B \right\} + \psi^B \psi^A \tilde{\psi}^B \tilde{\psi}^A \equiv G_{AB} \psi^A \psi^B \tilde{\psi}^A \tilde{\psi}^B + G_{AB} \psi^B \psi^A \tilde{\psi}^B \tilde{\psi}^A
\]

\[+ \psi^A \psi^B \tilde{\psi}^A \tilde{\psi}^B + G_{AB} \psi^A \psi^B \tilde{\psi}^A \tilde{\psi}^B + G_{AB} \psi^B \psi^A \tilde{\psi}^B \tilde{\psi}^A
\]

(106)

We have succeeded so far in making the lagrangian (80) invariant under the C*-transformations (81) and to write it in a nicer form consisting of the sum of the three parts \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \) as given in eqs. (101), (102), (103) respectively:

\[
\mathcal{L} = \left\{ G_{AB} \psi^A \psi^B \tilde{\psi}^A \tilde{\psi}^B \equiv G_{AB} \psi^A \psi^B \right\} \left\{ \psi^A \psi^B \tilde{\psi}^A \tilde{\psi}^B \right\} + \psi^B \psi^A \tilde{\psi}^B \tilde{\psi}^A \equiv G_{AB} \psi^A \psi^B \tilde{\psi}^A \tilde{\psi}^B + G_{AB} \psi^B \psi^A \tilde{\psi}^B \tilde{\psi}^A
\]

\[+ 2i \left\{ G_{AB} \psi^A (d\psi^B* \Leftrightarrow \gamma^B_{B'C'} \psi^{C*} \, dX^{D*}) + G_{AB} \psi^B (d\psi^A \Leftrightarrow \gamma^A_{CD} \psi^C \, dX^D) \right\} e^+ \\
\Rightarrow 2i \left\{ G_{AB} \tilde{\psi}^A (d\tilde{\psi}^B* \Leftrightarrow \gamma^B_{B'C'} \tilde{\psi}^{C*} \, dX^{D*}) + G_{AB} \tilde{\psi}^B (d\tilde{\psi}^A \Leftrightarrow \gamma^A_{CD} \tilde{\psi}^C \, dX^D) \right\} e^- \\
\Rightarrow 16 \frac{\beta^2}{(X^*X)^2}(\tilde{\psi} \psi^*)(\tilde{\psi} \psi^*) + \psi^B \psi^A \tilde{\psi}^B \tilde{\psi}^A \equiv G_{AB} \psi^A \psi^B \tilde{\psi}^A \tilde{\psi}^B + G_{AB} \psi^B \psi^A \tilde{\psi}^B \tilde{\psi}^A
\]

(107)

We can now utilize the gauge invariance to fix for instance (in the coordinate patch where \( X^0 \neq 0 \)) \( X^0 = 1 \) fixing completely the gauge. In practice we perform the transformation

\[
X^A \rightarrow \epsilon^{-q} X^A = \frac{1}{X^0} X^A
\]

(108)

that is we go from the homogeneous coordinates \((X^0, X^i)\) to the inhomogeneous coordinates \((1, x^i = X^i/X^0)\) on CP^N.

Having chosen our gauge we rewrite the lagrangian (107) in terms of the fields \( x^i \) (and of their fermionic partners \( \psi^i, \tilde{\psi}^i \)). Note that now \( dx^0 = 0 \) implies (because of the rheonomic
parametrizations) \( \psi^0 = 0 \) and \( \tilde{\psi}^0 = 0 \). The expression \( X^*X \equiv X^A X^A \) becomes \( 1 + x^i x^i \). Of the expressions \( G_{AB} \) and \( \gamma_{BC}^A \) only the components not involving the index zero survive. We introduce the following notations:

\[
\begin{align*}
G_{AB} & \equiv \frac{\rho^2}{X^A X^B} \left( \delta_{AB} \Leftrightarrow X^A X^B \right), \\
\gamma_{BC}^A & \equiv \frac{1}{X^A (\delta_B^A X^C + \delta_C^A X^B)}
\end{align*}
\]

We see that \( g_{ij} \) is just the standard Fubini-Study metric on \( \mathbb{CP}^N \) which is a Kähler metric of Kähler potential \( K = \rho^2 \log(1 + x^i x^i) \); \( \Gamma^i_{jk} \) is just the purely holomorphic part of its associated Levi-Civita connection. Moreover the Riemann tensor for the Fubini-Study metric is given by:

\[
R_{ijkl} = \frac{\rho^2}{(1 + x^i x^i)} \left( \frac{1}{\rho^2} \left( \delta_i^k \delta_j^l + \delta_j^k \delta_i^l \right) \frac{\rho^2}{x^i x^i} \left( \delta_i^j x^l + \delta_j^l x^i \right) \right) + \frac{2 \rho^2}{(1 + x^i x^i)^2} \left( \delta_i^j x^l x^k x^l \right)
\]

and we see that using once more the fermionic constraints (74) the four-fermion terms in (107) can be rewritten as follows:

\[
\frac{\rho^2}{(1 + x^i x^i)} \left( \psi^i \psi^{i*} \tilde{\psi}_j \tilde{\psi}^{j*} + \psi^i \psi^{i*} \tilde{\psi}_j \tilde{\psi}^{j*} \tilde{\psi}_k \right) = R_{ijkl} \psi^i \psi^{i*} \tilde{\psi}_j \tilde{\psi}^{j*} \tilde{\psi}_k \tilde{\psi}^{k*}
\]

Thus at the end of the above manipulations corresponding to the procedure of obtaining \( \mathbb{CP}^N \) as the Kähler quotient of \( \mathbb{C}^{N+1} \) we have reduced our initial rheonomic lagrangian (73)\( \Gamma \) in the limit \( g \to \infty \) to a form which is that of the \( N = 2 \) \( \sigma \)-model as given in eq. (63). The target space is \( \mathbb{CP}^N \) equipped with the Kählerian Fubini-Study metric.

10 The \( N = 4 \) Abelian gauge multiplet,

Having exhausted our rheonomic reconstruction of the \( N=2 \) models we now turn our attention to the \( N=4 \) case. We start with the gauge multiplet. The \( N=4 \) vector multiplet\( \Gamma \) in addition to the gauge bosons namely the 1-form, \( A_1 \) contains four spin 1/2 gauginos whose eight components are denoted by \( \lambda^+ \Lambda^\dagger \Gamma \lambda^+ \Lambda^\dagger \lambda^+ \Lambda^\dagger \Gamma \) two complex physical scalars \( M \neq M^* \Gamma N \neq N^* \Gamma \) and three auxiliary fields arranged into a real scalar \( \mathcal{P} = \mathcal{P}^* \) and a complex scalar \( \mathcal{Q} \neq \mathcal{Q}^* \). The rheonomic parametrization of the abelian field-strength \( F = dA \) and of the exterior derivatives of the scalars\( \Gamma \) gauginos and auxiliary fields is given below. It is uniquely determined from the Bianchi identities:

\[
F = \mathcal{F} e^+ e^- \Leftrightarrow \frac{i}{2} (\lambda^+ \zeta^- + \bar{\lambda}^- \zeta^+ + \bar{\mu}^+ \chi^- + \bar{\mu}^- \chi^+) e^- + \frac{i}{2} (\lambda^+ \zeta^+ + \lambda^- \zeta^-) e^+
\]
\[ dM = \partial_+ M e^+ + \partial_- M e^- \Rightarrow \frac{1}{4}(\lambda^- \zeta^+ \Leftrightarrow \lambda^+ \zeta^- + \mu^+ \chi^- \Leftrightarrow \mu^- \bar{\chi}^+) \]

\[ dN = \partial_+ N e^+ + \partial_- N e^- \Rightarrow \frac{1}{4}(\mu^+ \zeta^- + \bar{\mu}^+ \xi^- \Leftrightarrow \lambda^- \chi^+ \Leftrightarrow \lambda^+ \chi^-) \]

\[ d\tilde{\lambda}^+ = \partial_+ \lambda^+ e^+ + \partial_- \lambda^+ e^- + \left(\frac{\mathcal{F}}{2} + i\mathcal{P}\right) \zeta^+ \Leftrightarrow 2i \partial_- M \tilde{\zeta}^+ + \mathcal{Q} \chi^- + 2i \partial_- N^* \tilde{\chi}^+ \]

\[ d\lambda^+ = \partial_+ \lambda^+ e^+ + \partial_- \lambda^+ e^- + \left(\frac{\mathcal{F}}{2} + i\mathcal{P}\right) \zeta^+ \Leftrightarrow 2i \partial_- M^* \xi^+ + \mathcal{Q} \tilde{\chi}^- + 2i \partial_- N^* \xi^- \]

\[ d\bar{\mu}^+ = \partial_+ \bar{\mu}^+ e^+ + \partial_- \bar{\mu}^+ e^- + \left(\frac{\mathcal{F}}{2} + i\mathcal{P}\right) \chi^+ + 2i \partial_- M^* \tilde{\chi}^+ \Leftrightarrow \mathcal{Q} \xi^- + 2i \partial_- N^* \xi^- \]

\[ d\mu^+ = \partial_+ \mu^+ e^+ + \partial_- \mu^+ e^- + \left(\frac{\mathcal{F}}{2} + i\mathcal{P}\right) \chi^+ + 2i \partial_- M \xi^+ + \mathcal{Q} \tilde{\chi}^- + 2i \partial_- N \xi^- \]

\[ d\mathcal{P} = \partial_+ \mathcal{P} e^+ + \partial_- \mathcal{P} e^- \Leftrightarrow \frac{1}{4}(\partial_+ \lambda^+ \zeta^- \Leftrightarrow \partial_+ \lambda^- \zeta^+ \Leftrightarrow \partial_- \lambda^+ \tilde{\zeta}^- \Leftrightarrow \partial_- \lambda^- \tilde{\zeta}^+) \]

\[ d\mathcal{Q} = \partial_+ \mathcal{Q} e^+ + \partial_- \mathcal{Q} e^- + \frac{i}{2}(\partial_+ \mu^+ \zeta^+ \Leftrightarrow \partial_- \bar{\mu}^+ \tilde{\zeta}^+ \Leftrightarrow \partial_+ \lambda^+ \chi^- \Leftrightarrow \partial_- \lambda^+ \chi^-) \quad (112) \]

The rheonomic parametrizations of the complex conjugate fields \(d\tilde{\lambda}^- \Gamma d\lambda^- \Gamma d\mu^- \Gamma d\bar{\mu}^- \Gamma dM^* \Gamma dQ^*\) are immediately obtained by applying the rules of complex conjugation.

Using these results by means of lengthy but straightforward algebra we can derive the rheonomic action of the N=4 abelian gauge multiplet. The result is given below.

\[ \mathcal{L}_{\text{gauge}}^{(r,\text{rheo})}(N = 4) = \mathcal{F} \left[ F + \frac{i}{2}(\lambda^+ \zeta^- + \bar{\lambda}^+ \xi^- + \mu^+ \chi^- + \bar{\mu}^+ \tilde{\chi}^+) e^- \Rightarrow \frac{i}{2}(\lambda^+ \zeta^+ + \lambda^- \tilde{\zeta}^- + \mu^+ \chi^- + \bar{\mu}^+ \tilde{\chi}^+) e^+ \right] \]

\[ + M(\zeta^- \tilde{\zeta}^+ + \chi^+ \tilde{\chi}^-) + M^*(\zeta^+ \tilde{\zeta}^- + \chi^- \tilde{\chi}^+) \Leftrightarrow N(\zeta^+ \tilde{\zeta}^- + \chi^- \tilde{\chi}^+) \]

\[ + N^*(\zeta^- \tilde{\zeta}^+ + \chi^+ \tilde{\chi}^-) \]
By means of the usual manipulations from eq. (113) we immediately retrieve the N=4 globally supersymmetric world-sheet action of the abelian vector multiplet. It is the following:

\[
\begin{align*}
\mathcal{L}^{(\text{wz})}_{\text{gauge}}(N = 4) &= \frac{1}{2} \mathcal{F}^2 - i(\lambda^+ \partial_+ \lambda^- + \mu^+ \partial_+ \mu^- + \lambda^+ \partial_- \lambda^- + \mu^+ \partial_- \mu^-) \\
&\quad + 4(\partial_+ M^+ \partial_- M + \partial_- M^+ \partial_+ M + \partial_+ N^* \partial_- N + \partial_- N^* \partial_+ N) \\
&\quad + \frac{\theta}{2\pi} \mathcal{F} + 2\mathcal{P}^2 + 2\mathcal{Q}^* \mathcal{Q} \Leftrightarrow 2r\mathcal{P} \Leftrightarrow (s \mathcal{Q}^* + s^* \mathcal{Q})
\end{align*}
\]

(114)

In the above action we note the announced N=4 generalization of the Fayet-Iliopoulos term. In addition to the \(\theta\)-term proportional to the first Chern-class of the gauge field \(\mathcal{F}\) and to the \(r\)-term linear in the real auxiliary field \(\mathcal{P}\) we have term linear in the complex auxiliary field \(\mathcal{Q}\) involving a new complex parameter \(s\). Differently from the N=2 case the only allowed self interaction of the N=4 vector multiplet is given by the analogue of a linear superpotential term namely the above N=4 generalization of the Fayet-Iliopoulos term existing only for abelian gauge fields. As we are going to see shortly the parameters \(r\) and \(s\) correspond in the Lagrangian realization of the HyperKähler quotients to the levels of the triholomorphic momentum map. To discuss this point that is one of our main goals we have to revert to the discussion of the quaternionic hypermultiplets. These are the N=4 analogues of the N=2 chiral multiplets.

11 \(N = 4\) Quaternionic hypermultiplets with an abelian gauge symmetry

As in four-dimensions the N=2 analogue of the N=1 Wess-Zumino multiplets is given by the hypermultiplets that display a quaternionic structure in the same way in two dimensions the N=4 analogues of the complex N=2 chiral multiplets are the quaternionic hypermultiplets that parametrize a HyperKähler manifold. If this manifold is curved we have an N=4 \(\sigma\)-model similarly to the N=2 \(\sigma\)-model that is constructed on a Kähler manifold. Alternatively if the HyperKähler variety is flat we are dealing with the N=4 analogue of the N=2 Landau-Ginzburg model. Here however the more stringent constraints of N=4 supersymmetry rule out the
insertion of any self-interaction driven by a holomorphic superpotential. On the other hand, what we can still do just as in the N=2 case is to couple the flat hypermultiplets to abelian or non-abelian gauge multiplets. In full analogy with the N=2 case this construction will generate an N=4 \( \sigma \)-model as the effective low-energy action of the gauge \( \oplus \) matter system. The target manifold will be the HyperKähler quotient of the flat quaternionic manifold with respect to the triholomorphic action of the gauge group. Hence in the present section we consider quaternionic hypermultiplets minimally coupled to abelian gauge multiplets. For simplicity we focus on the case of one gauge-multiplet. All formulae can be straightforwardly generalized to the case of many abelian multiplets at the end.

Consider a set of bosonic complex fields \( u^i, v^j \) that can be organized in a set of quaternions

\[
Y^i = \begin{pmatrix} u^i & iv^j \end{pmatrix}
\]

(115)

On these matter fields the abelian gauge group acts in a triholomorphic fashion. According to the discussion of section I (see eq.(6) the triholomorphic character of this action corresponds to the following definition of the covariant derivatives:

\[
\nabla u^i = du^i + iA^j_q u^j
\]

\[
\nabla v^i = dv^i \Leftrightarrow iA^j_q v^j
\]

where \( A^j_q \) is a hermitean matrix. Correspondingly the Bianchi identities take the form:

\[
\nabla^2 u^i = +iF^j_q u^j
\]

\[
\nabla^2 v^i = \Leftrightarrow iF^j_q v^j
\]

(116)

We solve these Bianchi identities parametrizing the covariant derivatives \( \nabla^2 u^i \) and \( \nabla^2 v^i \) in terms of four spin 1/2 fermions whose eight components are given by \( \psi^i_u, \bar{\psi}^i_u, \psi^i_v, \bar{\psi}^i_v \) together with their complex conjugates \( \psi^i_u^*, \bar{\psi}^i_u^*, \psi^i_v^*, \bar{\psi}^i_v^* \). In the background of the abelian gauge multiplet (112) we obtain:

\[
\nabla u^i = \nabla_u u^i e^+ + \nabla_u v^j e^- + \psi^i u^i \zeta^- + \psi^i v^j \zeta^+ + \bar{\psi}^i u^i \bar{\zeta}^- + \bar{\psi}^i v^j \bar{\zeta}^+
\]

\[
\nabla v^i = \nabla_u v^i e^+ + \nabla_u v^j e^- + \psi^i u^i \zeta^- + \psi^i v^j \zeta^+ + \bar{\psi}^i u^i \bar{\zeta}^- + \bar{\psi}^i v^j \bar{\zeta}^+
\]

\[
\nabla \psi^i_u = \nabla_u \psi^i_u e^+ + \nabla_u \psi^j_v e^- \Leftrightarrow \frac{i}{2} \nabla_u u^i \zeta^+ \Leftrightarrow \frac{i}{2} \nabla_u v^j \zeta^-
\]

\[
+ iA^j_q (Mu^i \zeta^+ + Nu^j \zeta^- \Leftrightarrow Nu^i \bar{\zeta}^+ + Mu^j \bar{\zeta}^-)
\]

\[
\nabla \psi^i_v = \nabla_u \psi^i_v e^+ + \nabla_u \psi^j_v e^- \Leftrightarrow \frac{i}{2} \nabla_u v^i \zeta^+ \Leftrightarrow \frac{i}{2} \nabla_u u^j \zeta^-
\]

\[
+ iA^j_q (Mu^i \zeta^+ + Nu^j \zeta^- \Leftrightarrow Nu^i \bar{\zeta}^+ + Mu^j \bar{\zeta}^-)
\]

\[
\nabla \bar{\psi}^i_u = \nabla_u \bar{\psi}^i_u e^+ + \nabla_u \bar{\psi}^j_v e^- \Leftrightarrow \frac{i}{2} \nabla_u u^i \bar{\zeta}^+ \Leftrightarrow \frac{i}{2} \nabla_u v^j \bar{\zeta}^-
\]

(37)
\[ \nabla \bar{\psi}_i^j = \nabla_+ \bar{\psi}_i^j e^+ + \nabla_- \bar{\psi}_i^j e^- \iff \frac{i}{2} \nabla \bar{\psi}_i^j e^+ + \frac{i}{2} \nabla \bar{\psi}_i^j e^- \iff i q_i^j (M^* w_i^j \bar{\zeta}^+ + N^* v_i^j \bar{\zeta}^- \iff N^* w_i^j \chi^+ + M^* v_i^j \chi^-) \]

(118)

Note that the field content of the N=4 hypermultiplet is the same as the field content of two N=2 chiral multiplets. For each complex coordinate \( u \) or \( v \) we have two complex spin 1/2 Weyl fermions \( \psi_i^u \Gamma_i \psi_i^u \) or \( \psi_i^v \Gamma_i \psi_i^v \). The additional supersymmetries associated with the gravitinos \( \bar{\chi}^\pm \) and \( \bar{\chi}^\pm \) simply mix the fields of one N=2 chiral multiplet \( u \) with the other \( v \). Note also that contrarily to the N=2 case the rheonomic solution (118) does not involve any auxiliary field \( \Gamma \) namely in the N=4 case there is no room for an arbitrary interaction driven by a Landau-Ginzburg superpotential \( U(u, v) \).

From the Bianchi identities one gets the following fermionic equations of motion:

\[
\frac{i}{2} \nabla_+ \bar{\psi}_i^j + i q_j^i \left( \frac{1}{4} \lambda^+ u^j + \frac{1}{4} \mu^- v^j + M^* \bar{\psi}_i^u \iff N^* \bar{\psi}_i^v \right) = 0
\]

(119)

Applying the supersymmetry transformation of parameter \( e^+ \) to the first two of eqs (119) we obtain the bosonic equations of motion namely:

\[
\frac{1}{8} (\nabla_+ \nabla_- + \nabla_- \nabla_+) u^j = \frac{i}{4} q_j^i (\lambda^- \bar{\psi}_i^j \iff \lambda^- \bar{\psi}_i^j \iff \mu^- \bar{\psi}_i^j)
\]

\[
\frac{1}{8} (\nabla_+ \nabla_- + \nabla_- \nabla_+) v^i = \frac{i}{4} q_j^i (\lambda^- \bar{\psi}_i^j \iff \lambda^- \bar{\psi}_i^j \iff \mu^- \bar{\psi}_i^j)
\]

(120)

The rheonomic action that yields the rheonomic parametirizations (118) and the field equations (119) and (120) as variational equations is given below:

\[
\mathcal{L}_{\text{quaternion}}^{\text{rheonomic}} = (\nabla u_i^j \iff \psi_i^u \zeta^- \iff \psi_i^u \chi^- \iff \psi_i^v \zeta^- \iff \psi_i^v \chi^-) (U_i^+ e^+ \iff U_i^- e^-) + (\nabla w_i^j + \psi_i^u \zeta^+ + \psi_i^v \chi^+ \iff \psi_i^u \zeta^+ + \psi_i^v \chi^+) (U_i^+ e^+ \iff U_i^- e^-) + (U_i^+ U_i^+ + U_i^- U_i^-) e^+ e^-
\]

\[
(\nabla v_i^j \iff \psi_i^u \zeta^+ \iff \psi_i^v \chi^+ \iff \psi_i^v \zeta^+ \iff \psi_i^v \chi^+) (V_i^+ e^+ \iff V_i^- e^-) + (\nabla v_i^j + \psi_i^u \zeta^+ \iff \psi_i^u \chi^+ \iff \psi_i^v \zeta^+ \iff \psi_i^v \chi^+) (V_i^+ e^+ \iff V_i^- e^-)
\]

38
\[ + (V^*_i V^*_j + V^*_i V^*_j) e^+ e^- \\
\Leftrightarrow 4i(\tilde{\psi}^*_i \nabla \tilde{\psi}^*_i + \tilde{\psi}^*_i \nabla \tilde{\psi}^*_i) e^+ + 4i(\tilde{\psi}^*_i \nabla \tilde{\psi}^*_i + \tilde{\psi}^*_i \nabla \tilde{\psi}^*_i) e^- \\
+ (\psi^*_i \psi^*_j + \psi^*_i \psi^*_j)(\zeta^+ \zeta^- + \chi^+ \chi^-) \\
\Leftrightarrow \tilde{\psi}^*_i \tilde{\psi}^*_i + \tilde{\psi}^*_i \tilde{\psi}^*_i)(\zeta^+ \zeta^- + \chi^+ \chi^-) \\
+ \left[ (\psi^*_i \tilde{\psi}^*_i + \psi^*_i \tilde{\psi}^*_i)(\zeta^+ \zeta^- + \chi^+ \chi^-) + \text{c.c.} \right] \\
+ \left[ (\psi^*_i \tilde{\psi}^*_i + \psi^*_i \tilde{\psi}^*_i)(\zeta^+ \zeta^- + \chi^+ \chi^-) + \text{c.c.} \right] \\
+ \left[ \nabla u^*(\tilde{\psi}^*_i \zeta^- \Leftrightarrow \tilde{\psi}^*_i \zeta^- + \psi^*_i \chi^- \Leftrightarrow \psi^*_i \chi^- - \text{c.c.} \right] \\
+ \left[ \nabla v^*(\tilde{\psi}^*_i \zeta^- \Leftrightarrow \tilde{\psi}^*_i \zeta^- + \psi^*_i \chi^- \Leftrightarrow \psi^*_i \chi^- - \text{c.c.} \right] \\
+ 4 \left[ \psi^*_i q^*_j (M^* u^* \zeta^- + N v^i \zeta^+ \Leftrightarrow N u^* \chi^- + M^* v^i \chi^+ + \text{c.c.} \right] \\
+ 4 \left[ q^*_i d^*_j (M^* v^i \zeta^- + N u^* \chi^- + M^* u^* \chi^- + \text{c.c.} \right] \\
+ 4 \left[ \psi^*_i q^*_j (M u^* \zeta^- + N v^i \chi^- + M v^i \chi^- + \text{c.c.} \right] \\
+ 4 \left[ \psi^*_i q^*_j (M^* v^i \zeta^- + N u^* \chi^- + M^* u^* \chi^- + \text{c.c.} \right] \\
+ \left\{ 2i \left[ \psi^*_i q^*_j (\lambda^- u^* + \mu^+ v^i) \Leftrightarrow \text{c.c.} \right] \Leftrightarrow 2i \left[ \psi^*_i q^*_j (\lambda^- v^i + \mu^+ u^*) \Leftrightarrow \text{c.c.} \right] \\
\Leftrightarrow 2i \left[ \psi^*_i q^*_j (\lambda^- u^* + \mu^+ v^i) \Leftrightarrow \text{c.c.} \right] + 2i \left[ \psi^*_i q^*_j (\lambda^- v^i + \mu^+ u^*) \Leftrightarrow \text{c.c.} \right] \\
+ 8 \left[ M^* (\psi^*_i q^*_j \tilde{q}^*_j \Leftrightarrow \psi^*_i q^*_j \tilde{q}^*_j) \Leftrightarrow \text{c.c.} \right] \Leftrightarrow 8i \left[ M (\psi^*_i q^*_j \tilde{q}^*_j + \psi^*_i q^*_j \tilde{q}^*_j) \Leftrightarrow \text{c.c.} \right] \\
+ 8(|M|^2 + |N|^2) \left[ u^* (q^2)^i_j u^i + v^i (q^2)^i_j v^i \right] \Leftrightarrow 2P(u^* q^*_j \tilde{q}^*_j v^i \Leftrightarrow v^i q^*_j \tilde{q}^*_j v^i) \\
+ 2i (Q u^* q^*_j \tilde{q}^*_j v^i \Leftrightarrow \text{c.c.}) \right\} e^+ e^- \tag{121} \]

In the above formula, the fields implementing the first order formalism for the scalar kinetic
terms have been denoted by $U_{ij}^2, V_{ij}^2$. Eliminating these fields through their own equations and
deleting the terms proportional to the gravitinos, we obtain the world-sheet supersymmetric
Lagrangian of the N=4 quaternionic hypermultiplets coupled to the gauge multiplet. We write
it in a basis where the $U(1)$ generator has been diagonalised: $q^*_j \equiv q^i \delta^i_j$:

\[
\mathcal{L}_{\text{quat}}^{(us)} = \left\{ \nabla_+ u^* \nabla_- u^* + \nabla_- u^* \nabla_+ u^* + \nabla_+ v^i \nabla_- v^i + \nabla_- v^i \nabla_+ v^i \right\} \\
+ 4i(\psi^*_i \nabla_+ \psi^*_i + \psi^*_i \nabla_- \psi^*_i + \tilde{\psi}^*_i \nabla_+ \tilde{\psi}^*_i + \tilde{\psi}^*_i \nabla_- \tilde{\psi}^*_i) \\
+ 2i \sum_i q^i \left\{ \left[ \psi^*_i (\lambda^- u^* + \mu^+ v^i) \Leftrightarrow \text{c.c.} \right] \Leftrightarrow \left[ \psi^*_i (\lambda^- v^i + \mu^+ u^*) \Leftrightarrow \text{c.c.} \right] \\
n \left[ \tilde{\psi}^*_i (\lambda^- u^* + \mu^+ v^i) \Leftrightarrow \text{c.c.} \right] + \left[ \tilde{\psi}^*_i (\lambda^- v^i + \mu^+ u^*) \Leftrightarrow \text{c.c.} \right] \right\} 
\]
\[ + 8i \left[ M^* \sum_i q_i^\dagger (\psi_i^\dagger \bar{\psi}_i^\dagger \Leftrightarrow \psi_i^\dagger \bar{\psi}_i^\dagger) \Rightarrow \text{c.c.} \right] + 8i \left[ N \sum_i q_i^\dagger (\bar{\psi}_i \psi_i^\dagger \Leftrightarrow \bar{\psi}_i \psi_i^\dagger) \Rightarrow \text{c.c.} \right] \\
+ 8(|M|^2 + |N|^2) \sum_i (q_i^\dagger)^2 (|u_i|^2 \Leftrightarrow |v_i|^2) \equiv 2 \mathcal{P} \sum_i q_i^\dagger (|u_i|^2 \Leftrightarrow |v_i|^2) \\
+ 2i \left( \mathcal{Q} \sum_i q_i^\dagger u_i^* v_i \Leftrightarrow \text{c.c.} \right) \] (122)

The most interesting feature of the action (122) is the role of the auxiliary fields. Recalling our discussion of the HyperKähler quotient in section I and comparing with formulae (7) we see that the auxiliary field \( \mathcal{P} \) multiplies the real component \( \mathcal{D}^3(u_i, v_i) = \sum_i q_i^\dagger (|u_i|^2 \Leftrightarrow |v_i|^2) \) while \( \mathcal{Q} \) multiplies the holomorphic component \( \mathcal{D}^+(u_i, \bar{v}_i) = \Leftrightarrow 2i \left( \sum_i q_i^\dagger u_i^* \bar{v}_i \right) \) of the momentum map for the triholomorphic action of the gauge group. This fact is the basis for the Lagrangian realization of the HyperKähler quotients. Indeed the vacuum of the combined \( \text{gauge} \oplus \text{matter} \) system breaks the abelian gauge invariance giving a mass to all the fields in the gauge multiplet and to all the quaternionic scalars that do not lie on the momentum-map surface of level \( \mathcal{D}^3 = r, \mathcal{D}^+ = s \). Integrating on the massive modes one obtains an \( \text{N}=4 \sigma \)-model with target manifold the HyperKähler quotient. This mechanism will be evident from the study of the scalar potential of the combined system.

12 The scalar potential in the \( \text{N}=4 \) hypermultiplet-gauge system

As in the \( \text{N}=2 \) case the correct way of putting together the gauge and the matter lagrangian fixed by positivity of the energy is the following:

\[ \mathcal{L} = \mathcal{L}_{\text{gauge}} \Leftrightarrow \mathcal{L}_{\text{quaternion}}. \] (123)

As a result the bosonic scalar potential is:

\[ \Leftrightarrow U = 2 \mathcal{P}^2 + 2 |\mathcal{Q}|^2 \Leftrightarrow 2r \mathcal{P} \Leftrightarrow (s \mathcal{Q}^* + s^* \mathcal{Q}) \Leftrightarrow 8(|M|^2 + |N|^2) \sum_i (q_i^\dagger)^2 \left( |u_i|^2 \Leftrightarrow |v_i|^2 \right) \]
\[ + 2 \mathcal{P} \sum_i q_i^\dagger \left( |u_i|^2 \Leftrightarrow |v_i|^2 \right) \Leftrightarrow 2i \left( \mathcal{Q} \sum_i q_i^\dagger u_i^* v_i \Leftrightarrow \text{c.c.} \right) \] (124)

Varying the lagrangian in \( \mathcal{P} \) and \( \mathcal{Q} \) we obtain the algebraic equations:

\[ \mathcal{P} = \frac{1}{2} \left[ r \Leftrightarrow \sum_i q_i^\dagger \left( |u_i|^2 \Leftrightarrow |v_i|^2 \right) \right] = \frac{1}{2} \left[ r \Leftrightarrow \mathcal{D}^3(u, v) \right] \]
\[ \mathcal{Q} = \frac{1}{2} \left[ s \Leftrightarrow 2i \sum_i q_i^\dagger u_i^* v_i \right] = \frac{1}{2} \left[ s \Leftrightarrow \mathcal{D}^+(u, v) \right] \] (125)
and substituting back eq.s (125) in eq.(124) we get the final form of the N=4 bosonic potential:

\[
U = \frac{1}{2} (r \equiv D^3)^2 + \frac{1}{2} |s \equiv D^+|^2 + 8(|M|^2 + |N|^2) \sum (q_i)^2(|u_i|^2 + |v_i|^2) \tag{126}
\]

As we see the parameters \( r, s \) of the Fayet-Iliopoulos term are identified with the levels of the triholomorphic momentum-map as we announced. In the next section we discuss the structure of the N=4 scalar potential extrema.

### 13 Phase structure of the \( N = 4 \) theory and reconstruction of the associated low-energy theory

We address now the questions related with the structure of the classical vacuum of the \( N = 4 \) theory discussed above and with the low energy theory around this vacuum.

To minimize the potential (126) which is given by a sum of squares we must separately equate each addend to zero. If we compare the \( N = 4 \) bosonic potential with the \( N = 2 \) one given in eq. (34) we note that the absence of an \( N = 4 \) analogue of the Landau-Ginzburg potential reduces the possibilities. There is only an \( N = 4 \) \( \sigma \)-model phase. Beside \( M = 0, N = 0 \) we must impose \( D^3(u, v) = r \) and \( D^+ = s \). Taking into account the gauge invariance of the Lagrangian this means that the classical vacua are characterized by having \( M = N = 0 \) and the matter fields \( u, v \) lying on the HyperKähler quotient

\[
\mathcal{M} = D^{-1}_3(r) \cap D^{-1}_+ (s)/U(1) \tag{127}
\]

defined by the quaternionic space \( \mathbb{H}^n \) spanned by the fields \( u^i, v^i \) with respect to the triholomorphic the action of the \( U(1) \) gauge group (see section (1)). Considering the fluctuations around this vacuum we can see that the fields of the gauge multiplet are massive together with the modes of the matter fields not tangent to \( \mathcal{M} \). The low-energy theory will turns out to be the \( N = 4 \) \( \sigma \)-model on \( \mathcal{M} \).

Here neither we write the explicit derivation of the general form for an \( N = 4 \) \( \sigma \)-model nor we give the \( N = 4 \) analogue of the reconstruction of the low-energy \( N = 2 \) \( \sigma \)-model discussed in section (9). We just recall the basic fact that a \( \sigma \)-model is \( N = 4 \) supersymmetric only under the condition that the target space be a hyperKähler manifold. The reason of this omission is not just to save space; the key point of what happens in the \( N = 4 \) case can be fully understood also in an \( N = 2 \) language.

Indeed \( N = 4 \) theories are nothing else but particular \( N = 2 \) theories whose structure allows the existence of additional supersymmetries.

Which kind of \( N = 2 \) theory is the \( N = 4 \) gauge plus matter system described in sections (10I,XIII.2)? The answer is easily given. If we suppress the additional gravitinos \( \chi^{\pm} \) and \( \tilde{\chi}^{\pm} \) the \( N=4 \) rheonomic parametrizations (112)/(118) and the \( N=4 \) action (121)/(113) of \( n \) quaternionic
multiplets coupled to a gauge multiplet become those of an N=2 theory (see eqs (16Γ24Γ27Γ18)
containing one gauge multiplet \((A, \lambda^+, \lambda^-, \lambda^+, \lambda^-, M, P)\) and \(2n + 1\) chiral multiplets namely

\[
(X^A, \psi^A, \psi^{A*}, \tilde{\psi}^A, \tilde{\psi}^{A*}) = \begin{pmatrix}
\psi^i_a, \psi^{i*}_a, \tilde{\psi}^i_a, \tilde{\psi}^{i*}_a \\
\psi^i_v, \psi^{i*}_v, \tilde{\psi}^i_v, \tilde{\psi}^{i*}_v \\
X^0, \psi^{0*}, \tilde{\psi}^0, \tilde{\psi}^{0*}
\end{pmatrix}
\]

(128)

where the index \(A\) runs on \(2n + 1\) values \(i\) the index \(i\) takes the values \(i = 1, \ldots, n\) and where we have defined:

\[
\begin{align*}
X^0 &= 2N \\
\psi^0 &= \frac{1}{2} \mu^+ \\n\tilde{\psi}^0 &= \frac{1}{2} \tilde{\mu}^+
\end{align*}
\]

(129)

The match between the N=4 theory and the general form of the N=2 model is complete if we write the generator of the U(1) transformations on the \(X^A\) chiral multiplets as the following \((2n + 1) \times (2n + 1)\) matrix:

\[
q^A_B = \begin{pmatrix}
q^i_j \delta^i_j & 0 & 0 \\
0 & q^i_j \delta^i_j & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

(130)

and we choose as superpotential the following cubic function:

\[
W(X^A) = \frac{1}{4} X^0 \left( s^* \Leftrightarrow D^-(u, v) \right) = \frac{1}{4} X^0 \left( s^* + 2i \sum_i q^i u^i v^i \right)
\]

(131)

where \(D^-(u, v) \Leftrightarrow 2i \sum q^i u^i v^i\) is the holomorphic part of the momentum map for the triholomorphic action of the gauge group on \(H^n \cong \mathbb{C}^n\). The superpotential (131) is quasi-homogeneous of degree

\[
d_W = 1
\]

if we assign the following weights to the various chiral fields:

\[
\omega_0 = 1 ; \ \omega_{u^i} = \omega_{v^i} = 0
\]

(133)

See the discussion at the end of section (7) for the meaning of these assignments.

In particular it is easy to check that the form (31) of the \(\bar{N} = 2\) bosonic potential reduces the Landau-Ginzburg potential being given by eq. (131) exactly to the potential of eq. (126):

\[
U = \frac{1}{2} \left( r \Leftrightarrow D^2 \right)^2 + \sum_A |\partial_4 W|^2 + 8|M|^2 \sum_i (q^i)^2 \left( |u^i|^2 + |v^i|^2 \right)
\]

\[
= \frac{1}{2} \left( r \Leftrightarrow D^2 \right)^2 + \frac{1}{2} |s \Leftrightarrow D^*|^2 + (8|M|^2 + 2|X^0|^2) \sum_i (q^i)^2 \left( |u^i|^2 + |v^i|^2 \right)
\]

(134)
From this $N = 2$ point of view why do we not see two different phases in the structure of the classical vacuum? To answer this question let us compare the above potential with that of eq. (71) i.e. with the simplest of the examples considered in Witten’s paper [2]. The crucial difference resides in the expression of the real component $D^3(u, v)$ of the momentum map $\Gamma$ see eq. (125). It is indeed clear that by setting

$$r \leftrightarrow \sum_i q^i \left(|u^i|^2 \leftrightarrow |v^i|^2\right) = 0 \quad (135)$$

the exchange of $r > 0$ with $r < 0$ just corresponds to the exchange of the $u$’s with the $v$’s. Since in all the other expressions the $u$’s and the $v$’s play symmetric roles the two phases $r > 0$ and $r < 0$ are actually the same thing. This is far from being accidental. The reason why the charge of $v^i$ is opposite to the one of $u^i$ is the triholomorphicity of the action of the gauge group $\Gamma$ as already noted in section I. The triholomorphicity is essential in order to have an $N = 4$ theory; thus the indistinguishability of the two phases is intrinsic to any $N = 4$ theory of the type we are considering in this paper.

venuto in mente sarebbe interessante considerare It would be interesting to investigate in detail what happens at $r = 0\Gamma$ or better in general for the values of the momentum map parameters ($r$ and $s$ here) where the hyperKähler quotient degenerates $\Gamma[11]\Gamma[4]$. This might be particularly relevant in the case of the ALE spaces $\Gamma[5]\Gamma[11]\Gamma$four dimensional spaces with $c_1 = 0\Gamma$ obtained via a hyperKähler construction. Note that the supersymmetric $\sigma$-model on such spaces $\Gamma$ because of the vanishing of the first Chern class $\Gamma$ gives rise $\Gamma$ at the quantum level $\Gamma$ to a superconformal theory. In these cases $\Gamma$ for certain values of the momentum map parameters the hyperKähler quotient degenerates into an orbifold. If for these particular values of the parameters there is no real singularity in the complete theory (the gauge plus matter system) then we have an explicit unification of the "singular" case where the effective theory is the superconformal theory of an orbifold space with the case where the effective theory is a $\sigma$-model on a ALE space.

To complete the definition of the vacuum $\Gamma$ we must set $M = 0, X^0 = 0$ and require $D^+(u, v) = s$.

We have found that considering an $N = 2$ theory with a Landau-Ginzburg potential (131) does not introduce the possibility of a Landau-Ginzburg phase for the vacuum. We can understand this fact because such a potential has a “geometrical” origin and at the level of the $N = 4$ theory it is related to the gauge sector; it does not come from a self-interaction of the $N = 4$ matter fields (quaternionic multiplets). This self-interaction cannot exist as we noted above.

To reconstruct the low-energy theory $\Gamma$ we must follow the procedure outlined in section 9. The only difference is that there we considered the $\mathbf{CP}^N$ case in which there is no Landau-Ginzburg potential. On the other hand here we must take into account also the constraint $D^+ = s$ which comes from the potential (131).
To be definite we consider in an extremely sketchy way the case that corresponding to the obvious $N = 4$ generalization of $\mathbb{CP}^N$ namely in the above formulæ we take all the charges $q^i = 1$. The spaces obtained by means of the hyperKähler quotient procedure of $H^\alpha$ with respect to this $U(1)$ action have real dimension $4(n \Rightarrow 1)$; the Kähler metric they inherit from the quotient construction are called Calabi metrics [14].

First of all if we restore the gauge coupling constant (extending the redefinitions (72) to the other fields of the $N = 4$ gauge multiplet) before reducing the theory in its $N = 2$ components at the end also the kinetic terms for $X^0$ and its fermionic partners acquire a factor $\frac{1}{g^2}$. They disappear together with the kinetic terms for the remaining $N = 2$ gauge multiplet when we take the limit $g \to \infty$ which should correspond to integrate over the massive fluctuations. This matches the fact that also the fluctuations of $X^0$ and of its partners around the vacuum are massive.

In analogy with section 9 we consider the variations of the action with respect to the non-propagating fields. The variations in $X^0, \psi^0, \bar{\psi}^0$ are on the same footing as those in $\mathcal{M}, \bar{\lambda}^+, \lambda^+$. In particular we get fermionic constraints that by supersymmetry correspond to the two momentum map equations

$$D^3 = r \quad \leftrightarrow \quad \sum_i (|u^i|^2 \leftrightarrow |v^i|^2) = r \quad \quad \quad (136)$$
$$D^+ = s \quad \leftrightarrow \quad 2i \sum_i u^{i*} \bar{v}^i = s \quad \quad \quad (137)$$

The fermionic constraints are crucial in the technical reconstruction of the correct form of the rheonomic lagrangian of the $N = 2$ $\sigma$-model on a space $TCP^N$ endowed with a Calabi metric the Calabi space. We omit all the details confining ourselves to pointing out the essential differences with the $N=2$ case.

Note that the holomorphic constraint $D^+ = s$ is not implemented in the $N = 2$ lagrangian we are starting from eqs (2927118) through a Lagrange multiplier. This would be the case (by means of the auxiliary field $Q$) we had chosen to utilize the $N = 4$ formalism see eq.(122) and this is the case for the real constraint $D^3 = r$ through the auxiliary field $P$. This fact causes no problem as it is perfectly consistent with what happens from the geometrical point of view taking the hyperKähler quotient. Indeed the hyperKähler quotient procedure is schematically represented by

$$\mathcal{S} \quad \leftrightarrow \quad D_+^{-1}(s) \quad \leftrightarrow \quad \mathcal{N} \equiv D_3^{-1}(r) \cap D_+^{-1}(s) \quad \leftrightarrow \quad \mathcal{M} \equiv \mathcal{N} / G \quad \quad \quad (138)$$

where we have gone back to the general case and we have extended in an obvious way the notation of eq. (87); $j^+$ and $j^3$ are inclusion maps and $p$ the projection on the quotient.

We remarked in section (9) that the surface $D_3^{-1}(r)$ is not invariant under the action of the complexified gauge group $G^c$. It is easy to verify instead that the holomorphic surface $D_+^{-1}(s)$ is invariant under the action of $G^c$. Just as in the Kähler quotient procedure of section
(9) we can therefore replace the restriction to $\mathcal{D}_s^{-1}(r)$ and the $G$ quotient with a $G^c$ quotient $\Gamma$ without modifying the need of taking the restriction to $\mathcal{D}_+^{-1}(s)$. The hyperKähler quotient can be realized as follows:

$$S \xrightarrow{i^+} \mathcal{D}_+^{-1}(s) \xrightarrow{\Phi^c} \mathcal{M} \equiv \mathcal{D}_+^{-1}(s)/G^c \quad (139)$$

We see that in any case we have to implement the constraint $\mathcal{D}^s = s$. This does not affect the procedure of extending the action of the gauge group to its complexification $\Gamma$ which in our case is given by:

$$u^i \leftrightarrow e^{i\Phi} u^i \quad ; \quad v^i \leftrightarrow e^{-i\Phi} v^i$$

$$u'^* \leftrightarrow e^{-i\Phi^*} \quad ; \quad v'^* \leftrightarrow e^{i\Phi^*} v'^*$$

$$v \leftrightarrow v + \frac{i}{2}(\Phi \leftrightarrow \Phi^*) \quad (140)$$

and of obtaining the invariance of the lagrangian under this action $\Gamma$ by means of the substitutions

$$u^i \leftrightarrow e^{-u} u^i \quad ; \quad v^i \leftrightarrow e^{v} v^i \quad (141)$$

and similarly for the other fields $\Gamma$ as it happened in eq. (84).

The variation in the auxiliary field $\mathcal{P}\Gamma$ that acts as a Lagrange multiplier for the real momentum map constraints $\Gamma$ gives $\Gamma$ after the substitutions (141) $\Gamma$ the equation $\mathcal{D}^3(e^{-v} u, e^v v) = r \Gamma$ that is

$$r \leftrightarrow e^{-2v} \sum_i |u^i|^2 + e^{2v} \sum_i |v^i|^2 = 0 \quad (142)$$

This equation is solved for $v$ as follows (we introduce the notation $\rho^2 \equiv r$):

$$\epsilon^{2v} = \frac{\rho^2 + \sqrt{\rho^4 + 4 \sum_i |u^i|^2 \sum_i |v^i|^2}}{2 \sum_i |v^i|^2} \quad (143)$$

We have still to implement the holomorphic constraint $\mathcal{D}_s = s$; we have also at our disposal the $C^*$ gauge invariance of our lagrangian. We can utilize this invariance choosing a gauge which can simplify the implementation of the constraint [9]. One can for instance $\Gamma$ as it is clear from the form (140) of the $C^*$-transformations $\Gamma$ choose the gauge where $u^v = v^v$. In this gauge the constraint

$$\mathcal{D}^v = \leftrightarrow 2i \sum_i u^i v^i = s^* \quad (144)$$

is solved by setting

$$u^i = \sqrt{\frac{is^*}{2(1 + \sum_j \hat{u}^j \hat{v}^j)}} (\hat{u}^l, 1)$$

$$v^i = \sqrt{\frac{is^*}{2(1 + \sum_j \hat{v}^j \hat{v}^j)}} (\hat{v}^l, 1) \quad (145)$$
where the capital indices $I, J, \ldots$ run from 1 to $n \leftrightarrow 1$. The final result of the appropriate manipulations that should be made on the lagrangian following what was done in section 9 will be the reconstruction of the rheonomic action (63) for the $N = 2 \sigma$-model having as target space the hyperKähler quotient $\mathbb{H}^n/U(1)$ endowed with that the Kähler metric which is naturally provided by the hyperKähler quotient construction exactly in the same way as it happened in the Kähler quotient case of section 9. The Kähler quotient is again obtained through eq. (91). In expressing the result it is convenient to assign a name to the expressions $\sum_i |u_i|^2$ and $\sum_i |v_i|^2$ which must be reexpressed in terms of the true coordinates on the target space the $\hat{u}$'s and the $\hat{v}$'s. Therefore we set

$$
\beta \equiv \sum_i |u_i|^2 = \frac{is^a}{2|1 + \sum J \hat{u}_I \hat{v}_J|^2} (1 + \sum_I |\hat{u}_I|^2)
$$

$$
\gamma \equiv \sum_i |v_i|^2 = \frac{is^a}{2|1 + \sum J \hat{v}_I \hat{u}_J|^2} (1 + \sum_I |\hat{v}_I|^2)
$$

(146)

We note that differently from the $\mathbb{CP}^N$ case the part of the Kähler potential on the target space that comes from the restriction of the potential for the flat metric on the manifold $\mathbb{H}^n$ to the momentum-map surface $D_3^{-1}(r) \cap D_n^{-1}(s)$ is not an irrelevant constant. Indeed it is given (see section 1) by:

$$
K|_N = \frac{1}{2}(e^{-2v} \sum_i |u_i|^2 + e^{2v} \sum_i |v_i|^2) = \frac{1}{2} \rho^4 + 4 \beta \gamma
$$

(147)

The final expression of the Kähler potential for the Calabi metric is:

$$
\hat{K} = \frac{1}{2} \sqrt{\rho^4 + 4 \beta \gamma} + \frac{\rho^2}{2} \log \frac{\rho^2 + \sqrt{\rho^4 + 4 \beta \gamma}}{2 \gamma}
$$

(148)

In the case $n = 2\Gamma$ the target space has 4 real dimensions and the Calabi metric is nothing else that the Eguchi-Hanson metric i.e. the simplest Asymptotically Locally Euclidean gravitational instanton [11].

### 14 Extension to the case where the quaternionic hypermultiplets have several abelian gauge symmetries

The extension of the above results to the case of several $U(1)$ multiplets is fairly simple. This case is relevant to implement the Kronheimer construction of the multi Eguchi-Hanson spaces belonging to $A_k$-series [11] l[4]. Let the gauge group be $U(1)^n$ and let the corresponding gauge fields be the 1-forms $A^a$ ($a = 1, \ldots, n$); let the triholomorphic action of these groups on the hypermultiplets $Y^i$ be generated by the matrices $(F^a)_{ij}$ then the covariant derivatives of the
quaternionic scalars $u^i, v^i$ will be:

\[
\nabla u^i = du^i + iA^a(F^a)_{ij}u^j \\
\nabla v^i = dv^i \iff iA^a(F^a)_{ij}v^j
\]

(149)

Since the group is abelian and the generators $F^a$ are commuting, the gauge part of the action should simply be given by $n$ replicas of the $U(1)$ lagrangian; thus the world-sheet lagrangian is given by

\[
\mathcal{L}^{(w)}_{\text{gauge}} = \frac{1}{2}F^aF^a \iff i(\lambda^a_+\partial_+\lambda^- - \tilde{\mu}^a_+\partial_+\mu^- - \lambda^a_-\partial_-\lambda^a - \mu^a_+\partial_-\mu^-) + \\
+ 4(\partial_+M^a_+\partial_-M_a + \partial_-M^a_+\partial_+M_a + \partial_+N^a_+\partial_-N_a + \partial_-N^a_+\partial_+N_a) + \\
+ \frac{\theta^a}{2\pi}F^a + 2P^a\mathcal{P}^a + 2(Q^a)^*Q^a \iff r^a\mathcal{P}^a \iff \left(s^a(Q^a)^* + (s^a)^*Q^a\right)
\]

(150)

where the summation on the index $a$ enumerating the $U(1)$ generators is understood as usual. Similar formulae hold for the rheonomic action. For the matter part of the Lagrangian, note that the covariant derivatives (12) are nearly identical to the ones utilized in the case of one multiplet (116) We just have to take into account the substitution $\dot{q}^i_j \iff (F^a)_{ij}$ and the summation over the index $a$. The modification of the rheonomic parametrizations and of the action are almost trivially substantial because of the abelian nature of the group that we consider. Let us therefore quote here only the spacetime lagrangian:

\[
\mathcal{L}^{(w)}_{\text{quaternion}} \iff \left(\nabla_+ u^{i*}\nabla_-u^i + \nabla_+ u^i\nabla_-u^{i*} + \nabla_+ v^{i*}\nabla_-v^i + \nabla_+ v^i\nabla_-v^{i*}\right) \\
+ 4i(\dot{\psi}^i\nabla_-\psi^{i*} + \dot{\psi}^{i*}\nabla_-\psi^i + \dot{\bar{\psi}}^i\nabla_+\bar{\psi}^{i*} + \dot{\bar{\psi}}^{i*}\nabla_+\bar{\psi}^i) \\
+ 2i\sum_a\sum_i(f^a)_{ij}\left\{(\dot{\psi}^i\bar{\lambda}^-_a\psi^{i*} + \dot{\bar{\psi}}^i\lambda^+_a\psi^i) \iff \text{c.c.}\right\} \iff \left(\dot{\psi}^i(\lambda^-_a\psi^{i*} \iff \bar{\mu}^+_a\psi^i) \iff \text{c.c.}\right) \\
\iff \left[\dot{\psi}^i(\lambda^-_a u^{i*} \iff \bar{\mu}^+_a u^i) \iff \text{c.c.}\right] + \left[\dot{\bar{\psi}}^i(\lambda^+_a v^{i*} \iff \mu^-_a v^i) \iff \text{c.c.}\right] \\
+ 8i\sum_a\sum_i(F^a)_{ij}(\dot{\psi}^i\psi^{j*)} \iff \psi^i\dot{\bar{\psi}}^{j*)} \iff \text{c.c.}\right) \iff \text{c.c.} \\
+ 8i\sum_a\sum_i(F^a)_{ij}(\dot{\bar{\psi}}^i\psi^{j*)} \iff \bar{\psi}^i\psi^{j*)} \iff \text{c.c.}\right) \iff \text{c.c.} \\
+ 8\sum_a(|M_a|^2 + |N_a|^2)(f^a)^{ij}(u^iu^j + v^iv^j) \\
\iff 2\sum_a\mathcal{P}^a(F^a)^{ij}(u^iu^j \iff v^{i*}v^j) \\
+ 2i\sum_a(Q^a(F^a)^{ij}u^iv^j \iff \text{c.c.})
\]

(151)

As expected, the auxiliary fields $\mathcal{P}^a, Q^a$ multiply the $a^{th}$ component of the momentum-map respectively the real part $\mathcal{D}^3$ and the anti-holomorphic part $\mathcal{D}^-$. 47
The complete bosonic potential takes therefore the following direct sum form:

$$U = \sum_a \left[ \frac{1}{2} (r_a \leftrightarrow D^2_a) + \frac{1}{2} |s_a \leftrightarrow D^4_a|^2 + 8(|M_a|^2 + |N_a|^2) \sum_{i,j} (F^a F^a)_{ij} (u^i u^j + v^i v^j) \right]$$

(152)

15 Quaternionic notation for the \( N = 4 \) gauge-matter system and identification of the \( N = 4 \) \( R \)-symmetries

The above construction of the \( N = 4 \) gauge \( \oplus \) matter system can be recast in a more compact quaternionic notation that allows a simple identification of a \( U(2)_L \) and a \( U(2)_R \) global \( R \)-symmetry group \( \Gamma \) respectively acting on the left-moving and right-moving degrees of freedom. As it is well known [17] in the \( N = 2 \) case the \( R \)-symmetries play an important role in the identification of the superconformal theories emerging at the critical point and in the calculation of the so called elliptic genus which a very interesting type of genus one path integral with twisted boundary conditions that has a topological meaning as the index of one of the supersymmetry charges. Therefore the identification of the \( N = 4 \) \( R \)-symmetries is particularly valuable as their \( SU(2) \) subgroups will turn into the \( SU(2)_L \) and \( SU(2)_R \) currents of the \( N = 4 \) superalgebras for the left-moving and right-moving sectors respectively. Let us then introduce the quaternionic formalism. Setting \( \omega = 0 \Gamma \) we can write the super-world-sheet structure equations as follows:

$$\begin{align*}
  de^+ &= \frac{i}{4} \text{Tr}(Z^\dagger Z) \\
  de^- &= \frac{i}{4} \text{Tr}(\bar{Z}^\dagger \bar{Z})
\end{align*}$$

(153)

where

$$Z = \begin{pmatrix}
  \zeta^- & i\chi^+
  \\
  i\chi^- & \zeta^+
\end{pmatrix} \quad \bar{Z} = \begin{pmatrix}
  \bar{\zeta}^- & \bar{i}\bar{\chi}^+
  \\
  \bar{i}\bar{\chi}^- & \bar{\zeta}^+
\end{pmatrix}$$

(154)

To describe the abelian gauge multiplet we group the gauginos into quaternions \( \Gamma \) according to:

$$\Lambda = \begin{pmatrix}
  \lambda^- & i\mu^+
  \\
  i\mu^- & \lambda^+
\end{pmatrix} \quad \bar{\Lambda} = \begin{pmatrix}
  \bar{\lambda}^- & i\bar{\mu}^+
  \\
  i\bar{\mu}^- & \bar{\lambda}^+
\end{pmatrix}$$

(155)

and the gauge scalars \( \Gamma \) according to:

$$\Sigma = \begin{pmatrix}
  M & iN \\
  \bar{N}^* & M^*
\end{pmatrix}$$

(156)

It is also useful although we do not use such a notation in the Lagrangian \( \Gamma \) to group the field strength \( F \) and the auxiliary fields \( P, Q \) into another quaternion:

$$f = \left( \frac{\xi}{\tau} + i\mathcal{P} \leftrightarrow \mathcal{Q} \quad \mathcal{Q}^* \leftrightarrow i\mathcal{P} \right) \quad \bar{f} = \left( \frac{\bar{\xi}}{\bar{\tau}} \leftrightarrow i\bar{\mathcal{P}} \quad \bar{\mathcal{Q}}^* \leftrightarrow i\bar{\mathcal{P}} \right)$$

(157)
Then the rheonomic parametrizations (112) can be written as follows:

\[ F = \text{Tr}e^+e^- \Leftrightarrow \frac{i}{2} \text{Tr}(\bar{\Lambda}^1 Z) e^- + \frac{i}{2} \text{Tr}(\Lambda^1 \bar{Z}) e^+ + \text{Tr}(Z\sigma_3 \bar{Z}^\dagger \Sigma) \]

\[ d\Sigma = \partial_+ \Sigma e^+ + \partial_- \Sigma e^- \Leftrightarrow \frac{1}{4} \Lambda \sigma_3 Z^\dagger + \frac{1}{4} \bar{Z} \sigma_3 \bar{\Lambda} \]

\[ d\Lambda = \partial_+ \Lambda e^+ + \partial_- \Lambda e^- + \bar{Z}f + 2i\partial_+ \Sigma Z\sigma_3 \]

\[ df = \partial_+ f e^+ + \partial_- f e^- + \frac{i}{2} \partial_+ \bar{\Lambda} Z \Leftrightarrow \frac{i}{2} \bar{Z}^\dagger \partial_- \Lambda \]  

These parametrizations (158) are invariant under the following *left-moving* and *right-moving* \( R \)-symmetries\( \) where \( U_L, U_R \in U(2) \) are arbitrary unitary \( 2 \times 2 \) matrices:

\[ Z \Leftrightarrow U_L Z \quad ; \quad \bar{\Lambda} \Leftrightarrow U_L \bar{\Lambda} \]

\[ \bar{Z} \Leftrightarrow U_R \bar{Z} \quad ; \quad \Lambda \Leftrightarrow U_R \Lambda \quad ; \quad \Sigma \Leftrightarrow U_R \Sigma U_L^{-1} \]  

(159)

The rheonomic action (113) can also be rewritten in this notation as it follows:

\[ \mathcal{L}_{\text{gauge}}^{(\psi^i)} = \frac{1}{2} \mathcal{F}^2 + \mathcal{F} \left[ F + \frac{i}{2} \text{Tr}(\bar{\Lambda}^1 Z) e^- \Leftrightarrow \frac{i}{2} \text{Tr}(\Lambda^1 \bar{Z}) e^+ \Leftrightarrow \text{Tr}(\Sigma \bar{Z}^\dagger \sigma_3 Z) \right] \]

\[ \Leftrightarrow \frac{i}{4} \text{Tr}(\bar{\Lambda}^1 d\bar{\Lambda}) e^- + \frac{i}{4} \text{Tr}(\Lambda^1 d\Lambda) e^+ \]

\[ \Leftrightarrow 4 \text{Tr} \left\{ \left[ d\Sigma^\dagger + \frac{1}{4} Z\sigma_3 \Lambda^1 \Leftrightarrow \frac{1}{4} \bar{\Lambda} \sigma_3 \bar{Z}^\dagger \right] (S_i e^+ \Leftrightarrow S_i e^-) + (S_i^1 S_i^1 + S_i^1 S_i^i)e^+e^- \right\} \]

\[ + \text{Tr}(d\Sigma \Lambda \sigma_3 \bar{Z}^\dagger + d\Sigma^\dagger \Lambda \sigma_3 Z^\dagger) \Leftrightarrow \frac{1}{4} \text{Tr}(\Lambda^1 \bar{Z} \sigma_3 \bar{\Lambda}^1) \sigma_3 \]

\[ + \frac{1}{2} \text{Tr} \left\{ \begin{pmatrix} r & s^* \\ \bar{s} & r \end{pmatrix} \left[ \Lambda^1 \bar{Z} e^+ + \bar{\Lambda}^1 Z e^- \right] \right\} + 2i \text{Tr} \left\{ \begin{pmatrix} r & s^* \\ \bar{s} & r \end{pmatrix} \bar{Z}^\dagger \Sigma Z \right\} \]

\[ \left\{ + \frac{\theta}{2 \pi} F + \left[ 2\mathcal{P}^2 + 2\mathcal{Q}^* \mathcal{Q} \Leftrightarrow 2r \mathcal{P} \Leftrightarrow (s \mathcal{Q}^* + s^* \mathcal{Q}) \right] \right\} e^+e^- \]  

(160)

Written in this form the superspace Lagrangian is invariant by inspection against the \( R \)-symmetries (159).

The hypermultiplets are rewritten in quaternionic notation as follows:

\[ Y^i = \begin{pmatrix} u^i & iv^* \\ iv^i & u^i \end{pmatrix} \quad ; \quad \Psi^i = \begin{pmatrix} \psi^i & \psi^{i*} \\ \bar{\psi}^i & \bar{\psi}^{i*} \end{pmatrix} \quad ; \quad \bar{\Psi}^i = \begin{pmatrix} \bar{\psi}^i & i\psi^{i*} \\ \bar{\psi}^* & \psi^{i*} \end{pmatrix} \]  

(161)

The Bianchi Identities take the form:

\[ \nabla^2 Y^i = i F \bar{q}_j^i \sigma_3 Y^j \]  

(162)

and the rheonomic parametrizations (118) become:

\[ \nabla Y^i = \nabla_+ Y^i e^+ + \nabla_- Y^i e^- + \sigma_3 \Psi^i Z + \bar{\Psi}^i \bar{Z} \sigma_3 \]
\[ \nabla \Psi^i = \nabla_+ \Psi^i e^+ + \nabla_- \Psi^i e^- \iff \frac{i}{2} \sigma_3 \nabla_+ Y^i Z^j + i Y^i q_j \sigma_3 \bar{Z}^i \Sigma^j \]
\[ \nabla \bar{\Psi}^i = \nabla_+ \bar{\Psi}^i e^+ + \nabla_- \bar{\Psi}^i e^- \iff \frac{i}{2} \nabla_- Y^i \sigma_3 \bar{Z}^i + i Y^i q_j \sigma_3 Z^j \Sigma^i \]

These parametrizations are invariant under the left- and right-moving R-symmetries provided the transformations (159) are adjoined to the following ones:

\[ \Psi^i \iff \Psi^i U_L^{-1} \quad ; \quad \bar{\Psi}^i \iff \bar{\Psi}^i U_R^{-1} \]

The rheonomic action (121) is rewritten as follow in quaternionic notation:

\[ L_{quatern}^{(rheo)} = \text{Tr} \left\{ (\nabla Y^i + \sigma_3 \Psi^i Z + \sigma_3 \bar{\Psi}^i \bar{Z})(Y^i e^+ \iff Y^i e^-) + Y^i Y^i e^+ e^- \right. \\
\iff 4 i(\Psi^i) \nabla \Psi^i e^+ \iff \bar{\Psi}^i \nabla \bar{\Psi}^i e^- + \bar{\Psi}^i \sigma_3 \Psi^i Z \sigma_3 \bar{Z}^i \\
+ \frac{1}{2}(\Psi^i \sigma_3 \Psi^i \text{Tr} Z^i Z \iff \frac{1}{2} \bar{\Psi}^i \bar{\Psi}^i \bar{Z}^i \bar{Z}) \iff \nabla Y^i (\sigma_3 \Psi^i Z \iff \bar{\Psi}^i \bar{Z} \sigma_3) \\
\iff 4 \Psi^i \bar{\Psi}^i \bar{Z}^i \sigma_3 Y^i e^+ \iff 4 \bar{\Psi}^i \Sigma Z q_j Y^i \sigma_3 e^- \\
\iff \frac{1}{2} \left( \begin{array}{c} D^3 \\
\iff D^- \iff D^3 \end{array} \right) \left[ \Lambda^i \bar{Z} e^+ + \bar{\Lambda}^i Z e^- \right] + 2 i \left( \begin{array}{c} D^3 \\
\iff D^- \iff D^3 \end{array} \right) \bar{Z}^i \Sigma Z \]
\[ + \left( 2 i \text{Tr} [q_j Y^i \Psi \Lambda + q_j \sigma_3 Y^i \sigma_3 \Lambda \bar{\Psi}^i] \iff 8 i \text{Tr} [\Psi^i \Sigma q_j \bar{\Psi}^i] \right) \\
+ 8 \text{Tr} (\Sigma^i \Sigma Y^i)(q^2)^i \bar{Y} \\
\iff 2 PD^3 + i [Q D^+ \iff Q^* D^-] e^+ e^- \] (165)

Written in this form also the hypermultiplet action is invariant by inspection with respect to the R-symmetries (159ff 64).

16 The A and B Topological Twists of the N=2 and N=4 Theories

We now discuss the two possible topological twists (A and B models in Witten’s nomenclature) of the above reviewed N=2 Landau-Ginzburg theories with local gauge symmetries. The topological twists of the corresponding N=4 theories are covered by this discussion since as we have shown they can be regarded as special instances of N=2 theories with a specific field-content and a special choice of the superpotential.

We focus on the formal aspects of the topological twist procedure relying on the clarification of the involved steps recently obtained in the D=4 case [7] and making preparatory remarks for our planned extension of the whole procedure to the case of locally N=2 supersymmetric theories (matter-coupled 2D topological gravity).

As discussed at length in [7] the topological twist extracts from any N=2 supersymmetric theory a topological field-theory that is already gauge-fixed namely where the BRST-algebra
already contains the antighosts whose Slavnov variation is proportional to the gauge-fixings. The appropriate instanton conditions that play the role of gauge-fixings for the topological symmetry are thus automatically selected when the topological field-theory is obtained via the topological twist. This latter consists of the following steps:

i) First one BRST quantizes the ordinary N=2 theory. (This step is relevant when the ordinary N=2 theory is locally supersymmetric (supergravity) and/or it contains gauge-fields as in our specific case. For rigid N=2 theories containing only matter multiplets as the N=2 σ-model or the rigid N=2 Landau-Ginzburg models [this step is empty].)

ii) Then one redefines the spins of all the fields taking as new Lorentz group the diagonal of the old Lorentz group with the internal automorphism group of N=2 supersymmetry.

iii) After this redefinition, one recognizes that at least one component of the N=2 multiplet of supercharges, say \( Q_0 \) has spin zero, is nilpotent and anticommutes with the old BRST-charge \( Q^{\text{BRST}}_{\text{old}} Q_0 + Q_0 Q^{\text{BRST}}_{\text{old}} = 0 \). Then one defines the new BRST-charge as \( Q^{\text{new}} = Q_0 + Q^{\text{BRST}}_{\text{old}} \).

iv) Next one redefines the ghost-number \( gh^{\text{new}} = J gh^{\text{old}} + F \) where \( F \) is some appropriate fermion number in such a way that the operator \( (\mathbb{A})_{\alpha}^{\beta} \) anticommutes with the new BRST-charge, in the same way as the operator \( (\mathbb{A})^{\alpha}_{\beta} \) did anticommute with the old BRST-charge. In this way all the fields of the BRST-quantized N=2 theory acquire a new well-defined ghost-number.

v) Reading the ghost-numbers one separates the physical fields from the ghosts and the antighosts, the BRST-variation of these latter yielding the gauge fixing instanton equations. The gauge-free BRST algebra (that involving no-antighosts) \([16]\) should, at this point, be recognizable as that associated with a well defined topological symmetry: for instance the continuous deformations of the vielbein (topological gravity), the continuous deformations of the gauge-connection (topological Yang-Mills theory), the continuous deformations of the embedding functions (topological σ-model) and so on.

STEP 1

The first step is straightforward. The case of interest to us just involves an ordinary gauge symmetry. Hence we just make the shift:

\[
\mathcal{A} \leftrightarrow \hat{\mathcal{A}} = \mathcal{A} + c^{(\text{gauge})}
\]

where \( c^{(\text{gauge})} \) are ordinary Yang-Mills ghosts. Imposing the BRST-rheonomic conditions:

\[
\hat{F} \overset{\text{def}}{=} \hat{d} \hat{\mathcal{A}} + \hat{\mathcal{A}} \wedge \hat{\mathcal{A}}
\]

\[
\hat{F} = (d + s) \left( \mathcal{A} + c^{(\text{gauge})} \right) + \left( \mathcal{A} + c^{(\text{gauge})} \right) \wedge \left( \mathcal{A} + c^{(\text{gauge})} \right)
\]

\[
\mathcal{F} e^+ e^- \leftrightarrow \frac{i}{2} (\lambda^+ \zeta^- + \lambda^- \zeta^+) e^- + \frac{i}{2} (\lambda^+ \tilde{\zeta}^+ + \lambda^- \tilde{\zeta}^+) e^+ + M \zeta^- \tilde{\zeta}^+ \leftrightarrow M^* \zeta^+ \tilde{\zeta}^-
\]

we obtain the ordinary BRST algebra of an N=2 supersymmetric gauge theory. We do not dwell on this trivial point.
STEP 2

The second step is the delicate one. In two dimensions the Lorentz group is $O(1, 1)$ which becomes $O(2)$ after Wick-rotation. Let us name $J_S$ the Lorentz generator: the eigenvalues $s^i$ of this operator are the spins of the various fields $\varphi^i$. The number $s^i$ appears in the Lorentz covariant derivative of the field $\varphi^i$:

$$\nabla \varphi^i = d\varphi^i \Leftrightarrow s^i \omega \varphi^i$$  \hspace{1cm} (168)

The automorphism group of the supersymmetry algebra that can be used to redefine the Lorentz group is the R-symmetry group $U(1)_L \otimes U(1)_R$. Denoting by $J_L, \Gamma J_R$ the two R-symmetry generators\(\Gamma\)we redefine the Lorentz generator according to the formula:

$$J'_S = J_S + \frac{1}{2} [J_R \pm J_L]$$  \hspace{1cm} (169)

Correspondingly the new spin quantum number is given by:

$$s' = s + \frac{1}{2} [q_R \pm q_L]$$  \hspace{1cm} (170)

The choice of sign in eq.s (169)$\Gamma$(170) corresponds to the existence of two different topological twists for the same $N=2$ theory. Following Witten [8] they will be named the A-twist leading to the A-model (upper choice of the sign) and the B-twist leading to the B-model (lower choice of the sign). It might seem arbitrary to restrict the possible linear combinations of the operators $J_S, \Gamma J_L$ and $J_R$ to those in eq.s (169)$\Gamma$(170)$\Gamma$but$\Gamma$these are the only possible ones if we take into account the following requirements. In the gravitational sector the spin redefinition must transform $N=2$ supergravity into topological gravity$\Gamma$ hence the spins of the vielbein $\epsilon^\pm$ must remain the same before and after the twists: this fixes the coefficient of $J_S$ to be equal to one as in eq. (169). Furthermore of the four gravitino 1-forms $\zeta^+, \zeta^-, \tilde{\zeta}^+, \tilde{\zeta}^-$ two must acquire spin $s = 1$ and $s = \Leftrightarrow \Gamma$ respectively$\Gamma$and the other two must have spin zero. This is so because two of the gravitinos have to become the topological ghosts corresponding to continuous deformations of the vielbein (so they must have the same spins as the vielbein) while the other two must be the gauge fields of those supersymmetry charges that$\Gamma$ acquiring spin zero$\Gamma$can be used to redefine the BRST-charge. These constraints have two solutions: indeed they fix the coefficients of $J_L$ and $J_R$ to the values displayed in eq.s (169)$\Gamma$(170)$\Gamma$the choice of sign distinguishing the two solutions.

STEP 3

Naming $Q_{BRST}$ the BRST-charge of the original gauge theory and $Q^\pm \Gamma \tilde{Q}^\pm$ the supersymmetry charges generating the transformations of parameters $\varepsilon^\pm \Gamma \tilde{\varepsilon}^\pm$ whose corresponding gauge fields are the gravitinos $\zeta^\pm \Gamma \tilde{\zeta}^\pm$, we realize that in the A-twist the spinless supercharges are $Q^-$ and $\tilde{Q}^+$ while in the B-twist they are $Q^+$ and $\tilde{Q}^-$. In both cases the two spinless supercharges
anticommute among themselves and with the BRST-charge so that we can define the new 
BRST-charge of the topological theory according to the formula:

$$Q'_{BRST} = Q_{BRST} \mp \bar{Q} + \hat{Q}^+$$  \hfill (171)

The choice of signs in eq. (171) is just a matter of convention: once more the upper choice of 
sign corresponds to the A-twist while the lower corresponds to the B-twist. The physical states 
of the topological theory are the cohomology classes of the operators (171).

**STEP 4**

What matters in the definition of the ghost number are the differences of ghost numbers for 
the fields related by a BRST-transformation. Indeed ghost number is one of the two gradings 
in a double elliptic complex. Hence to all the fields we must assign an integer grading which 
has to be increased of one unit by the application of the BRST-charge (or Slavnov operator). 
In other words $Q'_{BRST}$ must have ghost-number $gh = 1$. These requirements are satisfied iff 
for the redefinition of the ghost number $gh' = gh + F$ we use the generator $F$ of some $U(1)$ 
symmetry of the original N=2 theory with respect to which all the fields have integer charges 
and in particular the new BRST-generator (175) has charge $q_{BRST} = 1$: furthermore the two 
gravitinos that acquire the same spin as the vielbein and become the ghosts of topological 
gravity must have $gh' = 1$. In this case the action being invariant under the chosen symmetry 
has ghost number $gh' = 0$. This is the situation that can be realized in the A-twist. In the 
B-twist the situation is more complicated since there is no symmetry of the original theory that 
satisfies all the requirements: yet ghost-numbers can be consistently assigned to all the fields 
in such a way that $(\Leftrightarrow) gh'$ does anticommute with the new BRST-charge. The action however 
has no fixed ghost number rather it is the sum of terms having different values of $gh'$. However 
modulo BRST-exact terms the ghost-number is conserved since modulo these terms the action 
has a fixed ghost number $gh_{action} = \leftrightarrow 2$.

We examine first the situation for the A-twist. In this case naming $\# gh$ the ghost number 
of the original gauge theory we fulfill all the desired properties if we define the ghost number 
of the topological theory according to the formula:

$$\# gh' = \# gh + q_L \Leftrightarrow q_R$$  \hfill (172)

In this case the $U(1)$ symmetry utilized to redefine the ghost-number is generated by 
$F = J_L \Leftrightarrow J_R$ and it is a subgroup of the R-symmetry group $U(1)_L \otimes U(1)_R$.

In the B-twist case the new ghost numbers are defined as follows. For the fields belonging 
to the N=2 gauge multiplet we set:

$$\# gh' = \# gh \Leftrightarrow q_L \Leftrightarrow q_R$$  \hfill (173)

while for the fields in the chiral matter multiplet we put:

$$\# gh \left[ X^i \right] = 0$$
The spin and charge assignments before and after the twist in the N=2 abelian gauge theory are summarized in table I. In this table the fermions \( \psi^i \) and \( \tilde{\psi}^i \) appear with non diag. on the ghost-number column of the B-twist because their \( gh \)-number is undefined: indeed as it appears from eq. (174) only their sum and difference have a well-defined ghost-number. The same will happen for the corresponding fields of the N=4 theory.

As a preparation to STEP 5 namely the identification of the topological BRST-algebras and theories generated by the twists we consider the explicit form of the BRST-transformations of all the fields. In view of a very simple and powerful fixed point theorem due to Witten [8] we also recall that the topological theory besides being BRST-invariant with respect to the supercharge (171) has also a supergroup (0|2) of fermionic symmetries commuting with the BRST-transformations and generated by the two spinless supercharges utilized to redefine the BRST-charge. Hence while writing the topological BRST-transformations we write also the (0|2)-transformations. As Witten pointed out the topological functional integral is concentrated on those configurations that are a fixed point of the (0|2)-transformations: these are the true instantons of our theory and can be read from the formulae we are going to list.

In the A-twisted case the BRST-charge is given by:

\[
Q_{BRS}^{(A)} = Q_{BRS}^{(auge)} \leftrightarrow Q^- + \tilde{Q}^+
\]

Correspondingly we rename the supersymmetry parameters as follows:

\[
\begin{align*}
\varphi^- & = \alpha \\
\varphi^+ & = \alpha' \\
\alpha^{(A)} & = \varphi^+ = \varphi^- = \alpha_g
\end{align*}
\]

where \( \alpha_g \) is the nilpotent BRST-parameter associated with the original gauge symmetry and \( \alpha^{(A)} \) is the BRST-parameter of the A-twisted model. The parameters \( \alpha \) and \( \alpha' \) correspond to the two fermionic nilpotent transformations commuting with the BRST transformations and generating the (0|2) supergroup of exact symmetries of the topological action. Using the above conventions the form of the BRST-transformations and of the (0|2) symmetries in the A-twisted version of the N=2 gauge coupled Landau-Ginzburg model is given by the following formulae:

\[
\delta \mathcal{A}_+ = \alpha^{(A)} \left( \frac{i}{2} \lambda^- + \partial_+ e^{auge} \right) = \frac{i}{2} \alpha \lambda^- + \alpha_g \partial_+ e^{auge}
\]
\[ \delta A_\pm = \alpha^{(A)} \left( \frac{i}{2} \tilde{\lambda}^\pm + \partial_\pm e^{gaug\text{e}} \right) = \frac{i}{2} \alpha' \tilde{\lambda}^\pm + \alpha_g \partial_\pm e^{gaug\text{e}} \]

\[ \delta M = 0 \]

\[ \delta M^* = \frac{1}{4} \alpha^{(A)} \left( \lambda^+ + \tilde{\lambda}^- \right) = \frac{1}{4} \left( \alpha \lambda^+ + \alpha' \tilde{\lambda}^- \right) \]

\[ \delta \lambda^+ = \alpha^{(A)} \left( \frac{\mathcal{F}}{2} \Leftrightarrow i \mathcal{P} \right) = \alpha' \left( \frac{\mathcal{F}}{2} \Leftrightarrow i \mathcal{P} \right) \]

\[ \delta \lambda^- = \alpha^{(A)} \left( \frac{\mathcal{F}}{2} \Leftrightarrow i \mathcal{P} \right) = \alpha \left( \frac{\mathcal{F}}{2} \Leftrightarrow i \mathcal{P} \right) \]

\[ \delta \mathcal{P} = \frac{1}{4} \left( \alpha \partial_+ \lambda^+ + \partial_- \lambda^- \right) = \frac{1}{4} \left( \alpha \partial_+ \lambda^+ + \alpha' \partial_- \lambda^- \right) \]

\[ \delta X^i = \alpha^{(A)} \left( \psi^i + i e^{gaug\text{e}} q^i_j X^j \right) = \left( \alpha \psi^i + i \alpha_g e^{gaug\text{e}} q^i_j X^j \right) \]

\[ \delta X^{i*} = \alpha^{(A)} \left( \tilde{\psi}^{i*} \Leftrightarrow i e^{gaug\text{e}} \tilde{q}^i_j X^{j*} \right) = \left( \alpha' \tilde{\psi}^{i*} \Leftrightarrow i \alpha_g e^{gaug\text{e}} \tilde{q}^i_j X^{j*} \right) \]

\[ \delta \psi^i = \alpha^{(A)} \left( i M q^i_j X^j + i e^{gaug\text{e}} q^i_j \tilde{\psi}^j \right) \]

\[ = \left( i \alpha' M q^i_j X^j + i \alpha_g e^{gaug\text{e}} q^i_j \tilde{\psi}^j \right) \]

\[ \delta \tilde{\psi}^i = \alpha^{(A)} \left( \frac{i}{2} \nabla_- X^i + \eta^{i*} \partial_j W^* + i e^{gaug\text{e}} q^i_j \tilde{\psi}^j \right) \]

\[ = \left( \frac{i}{2} \nabla_- X^i + \alpha \eta^{i*} \partial_j W^* + i \alpha_g e^{gaug\text{e}} q^i_j \tilde{\psi}^j \right) \]

\[ \delta \tilde{\psi}^{i*} = \alpha^{(A)} \left( \frac{i}{2} \nabla_+ X^i + \eta^{i*} \partial_j W \Leftrightarrow i e^{gaug\text{e}} \tilde{q}^{i*}_j \tilde{\psi}^j \right) \]

\[ = \left( \frac{i}{2} \nabla_+ X^i + \alpha' \eta^{i*} \partial_j W \Leftrightarrow i \alpha_g e^{gaug\text{e}} \tilde{q}^{i*}_j \tilde{\psi}^j \right) \]

\[ \delta \tilde{\psi}^{i*} = \alpha^{(A)} \left( i M \tilde{q}^{i*}_j X^{j*} + i e^{gaug\text{e}} \tilde{q}^{i*}_j \tilde{\psi}^{j*} \right) \]

\[ = \left( i \alpha' M \tilde{q}^{i*}_j X^{j*} + i \alpha_g e^{gaug\text{e}} \tilde{q}^{i*}_j \tilde{\psi}^{j*} \right) \]

\[ (177) \]

On the other hand in the B-twisted version of the same N=2 theory the BRST-charge is given by:

\[ Q_{BRST}^{(A)} = Q^{(gaug\text{e})} + Q^+ + \tilde{Q}^+ \]

(178)

In view of eq. (178) and of our previous discussion of the ghost number in the B-twist case we rename the supersymmetry parameters as follows:

\[ \frac{1}{2} \left( \varepsilon^+ + \bar{\varepsilon}^+ \right) = \alpha \]

\[ \frac{1}{2} \left( \varepsilon^+ \Leftrightarrow \bar{\varepsilon}^+ \right) = \alpha' \]

\[ \alpha^{(B)} = \bar{\varepsilon}^+ = \varepsilon^+ = \alpha_g \]

(179)
$\alpha^{(B)}$ being the new BRST-parameter and $\alpha \Gamma \alpha'$ the parameters of the $(0|2)$ fermionic supergroup relevant to this case. With these notations the BRST-transformations and $(0|2)$ symmetries of the B-model are the following:

\[
\begin{align*}
\delta A_+ & = \alpha^{(B)} \left( \frac{i}{2} \lambda^- + \partial_+ e^{gauge} \right) = \left( \frac{i}{2} \lambda^- + \partial_+ e^{gauge} \right) \\
\delta A_- & = \alpha^{(B)} \left( \frac{i}{2} \lambda^- + \partial_- e^{gauge} \right) = \left( \frac{i}{2} \lambda^- + \partial_- e^{gauge} \right) \\
\delta M & = \alpha^{(B)} \frac{1}{4} \lambda^- = \frac{1}{4} \lambda^- \\
\delta M^* & = \alpha^{(B)} \frac{1}{4} \lambda^- = \frac{1}{4} \lambda^- \\
\delta \lambda^+ & = \alpha^{(B)} \left( \mathcal{F} \Leftrightarrow i \mathcal{P} \right) \Leftrightarrow 2i \partial_+ M^* \\
\delta \lambda^- & = 0 \\
\delta \bar{\lambda}^+ & = \alpha^{(B)} \left( \mathcal{F} + i \mathcal{P} \right) \Leftrightarrow 2i \partial_- M \\
\delta \bar{\lambda}^- & = 0 \\
\delta \mathcal{P} & = \alpha^{(B)} \frac{1}{4} \left[ \Rightarrow \partial_+ \lambda^- + \partial_- \lambda^- \right] = \frac{1}{4} \left[ \Rightarrow \left( \alpha + \alpha' \right) \partial_+ \lambda^- + \left( \alpha \Leftrightarrow \alpha' \right) \partial_- \lambda^- \right] \\
\delta X^i & = \alpha^{(B)} \left[ i e^{gauge} q_j^i X^j \right] = \left[ i \alpha_g e^{gauge} q_j^i X^j \right] \\
\delta X^{j*} & = \alpha^{(B)} \left[ \psi^{j*} + \bar{\psi}^{j*} + i e^{gauge} q_j^i X^j \right] \\
\delta \psi^i & = \alpha^{(B)} \left[ \Leftrightarrow \frac{1}{2} \nabla_+ X^i + i M q_j^i X^j + i e^{gauge} q_j^i \psi^j \right] \\
\delta \bar{\psi}^j & = \alpha^{(B)} \left[ \Leftrightarrow \frac{1}{2} \nabla_- X^j \Rightarrow i M^* q_j^i X^j + i e^{gauge} q_j^i \bar{\psi}^j \right] \\
\delta \bar{\psi}^{j*} & = \alpha^{(B)} \left[ \eta^{j*} \partial_j W \Leftrightarrow i e^{gauge} q_j^i \bar{\psi}^{j*} \right]
\end{align*}
\]
To discuss the topological twists of the N=4 matter coupled gauge theory it might seem necessary to write down the analogues of eq.s (179) and (180) as they follow from the rheonomic parametrizations of the N=4 theory (see eq.s (112) and (118)). Actually this is not necessary since the N=4 model is just a particular kind of N=2 theory so that the BRST-transformations relevant to the N=4 case can be obtained with a suitable specialization of eq.s (179) and (180) according to what shown in section XIII.

Using eq.s (132) and (133) in the general formulae (51) for the R-symmetries of the N=2 theory we obtain a result that coincides with the assignments of R-charges given in table II. In this table the charge assignments were deduced by restricting the non-abelian $U(2)_L \otimes U(2)_R$ R-symmetry group of the N=4 theory to its abelian subgroup $U(1)_L \otimes U(1)_R$ generated by the two third components of the $SU(2)_L \otimes SU(2)_R$ isospin generators. As a consequence the twisted spins and twisted ghost numbers displayed in table II can be alternatively deduced from the N=4 formulae (159) or from the N=2 formulae (51) upon use of the special values of $d$ and $\omega_A$ given in eq. (132) and (133).

In this way we have completly reduced the twisted N=4 models to special instances of the twisted N=2 models the crucial point being the identification of the superpotential (131). From now on we discuss the structure of the twisted models in the N=2 language. The next point is the analysis of STEP 5; we devote the next section to this.

17 Identification of the topological systems described by the A and B models

In this section we consider the interpretation of the topological field-theories described by the A and B models.

THE A-MODEL

We begin with the A model. To this effect we start by recalling the structure of a pure topological Yang-Mills theory [18]. In any space-time dimensions the field-content of this theory is given by table III where $A = A_\mu \, dx^\mu$ is the gauge-field $\psi = \psi_\mu \, dx^\mu$ the ghost of the topological symmetry $\Gamma c = c^{\text{gauge}}$ the ghost of the ordinary gauge symmetry and $\phi$ the ghost for the ghosts (indeed the ghost 1-form $\psi$ is by itself a gauge field). These fields enter the gauge-free
topological BRST-algebra that has the following form:

\[
\begin{align*}
    s A &= \Leftrightarrow (Dc + \psi) \\
    s F &= D\psi \Leftrightarrow [c, F] \\
    s c &= \phi \Leftrightarrow \frac{1}{2} [c, c] \\
    s \psi &= D\phi \Leftrightarrow [c, \phi] \\
    s \phi &= \Leftrightarrow [c, \phi]
\end{align*}
\]

(181)

The above algebra follows from the ghost-form Bianchi identities:

\[
\dot{d} \hat{F} + [\hat{A}, \hat{F}] = 0
\]

(182)

where

\[
\begin{align*}
    \hat{A} &= A + c \\
    \hat{F} &= \dot{d} \hat{A} + \frac{1}{2} J [\hat{A}, \hat{A}] \\
    \dot{d} &= d + s
\end{align*}
\]

(183)

by removing the BRST-rheonomic conditions

\[
\hat{F} = F_{ab} V^a \wedge V^b
\]

(184)

that characterize the BRST quantization of the ordinary gauge-theory. Indeed if we write the decomposition:

\[
\hat{F} = F_{(20)} + F_{(1,1)} + F_{(0,2)}
\]

(185)

and we remove the BRST-rheonomic conditions (184) that imply \(F_{(1,1)} = F_{(0,2)} = 0\) we see that eq.s (181) follow from eq. (182) upon use of the identifications:

\[
\begin{align*}
    \psi &= \Leftrightarrow F_{(1,1)} \\
    \phi &= \Leftrightarrow F_{(0,2)}
\end{align*}
\]

(186)

The other fields appearing in table III are either antighosts or auxiliary fields. Indeed the complete \(gauge-fixed\) topological BRST-algebra is obtained by adjoining to eq.s (181) the following ones:

\[
\begin{align*}
    s \bar{c} &= b \\
    s \bar{\psi} &= T \\
    s \bar{\phi} &= \bar{\eta}
\end{align*}
\]

(187)
where \( \bar{\Gamma} \bar{\psi} \bar{\Gamma} \bar{\phi} \) are the antighosts and \( T \) and \( \bar{\eta} \) are the auxiliary fields as displayed in table III. Actually rather than \( \bar{\psi} \) and \( T \) it is more convenient to use \( \Gamma \) as antighost and auxiliary field the functionals \( \bar{\chi} \) and \( B \) defined by the following equations

\[
\begin{align*}
\bar{\chi} &= dx^\mu \wedge dx^\nu \bar{\chi}_{\mu\nu} = D \bar{\psi} \\
B &= dx^\mu \wedge dx^\nu B_{\mu\nu} = \Longleftrightarrow DT \Longleftrightarrow [Dc, \bar{\psi}] \Longleftrightarrow [\psi, \bar{\psi}]
\end{align*}
\] (188)

in such a way that:

\[
s \bar{\chi} = B
\] (189)

is an identity.

In the quantum action \( \Gamma b \) is the Lagrangian multiplier for the gauge-fixing of the ordinary gauge transformations \( \Gamma \) while \( T_\mu \) (or rather its functional \( B_{\mu\nu} \)) is the Lagrangian multiplier associated with the gauge-fixing of the topological symmetry. Finally \( \bar{\eta} \) is utilized to gauge fix the gauge invariance of the topological ghost \( \psi_\mu \). Indeed the quantum action has the form:

\[
S_{\text{quantum}} = S_{\text{class}} + \int_{M_k} s \left( \Psi_{\text{topol}} + \Psi_{\text{gauge}} + \Psi_{\text{ghost}} \right)
\] (190)

where the gauge fermion is the sum of a gauge fermion fixing the topological symmetry \( (\Psi_{\text{topol}}) \) plus one fixing the ordinary gauge symmetry \( (\Psi_{\text{gauge}}) \) plus a last one fixing the gauge of the ghosts. In \( D=4 \) the classical action is the integral of the first Chern-class \( S_{\text{class}} = \int tr (F \wedge F) \) while in \( D=2 \) as classical action one takes the integral of the field strength in the direction of the center of the gauge Lie-algebra \( S_{\text{class}} = \frac{1}{2 \pi} \int F_{\text{center}} \). The topological gauge-fixing must break the invariance under continuous deformations of the connection still preserving ordinary gauge-invariance. In four-dimensions a convenient gauge condition that satisfies this requirement is provided by enforcing self-duality of the field strength (the instanton condition). Setting:

\[
G^{\pm}_{\mu\nu} = F_{\mu\nu} \pm \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}
\] (191)

the four-dimensional topological gauge-fixing can be chosen to be:

\[
G^+_{\mu\nu} = 0
\] (192)

In two-dimensions eq.(191) has no meaning and the topological gauge-fixing (192) can just be replaced by the condition of constant curvature \( (F_{\text{center}} = \text{const}) \) where \( \Gamma \) as already stated \( \Gamma \) \( F_{\text{center}} \) denotes the field strength of the gauge group restricted to the center of the Lie-algebra \( \Gamma \) namely the only components of the field-strength that \( \Gamma \) being gauge invariant \( \Gamma \) can be given a constant value. This makes sense under the assumption that \( \Gamma \) within the set of all gauge-connections characterized by the same first Chern-class \( \int F_{\text{center}} \) (value of the classical action) \( \Gamma \)
there is always at least one that has a constant field strength in the center-direction and a vanishing field-strength in the other directions (almost flat connections). Hence we set:

\[
\Psi_{\text{topol}} = \text{tr} \left\{ \bar{\chi}_\rho \left( F_{\mu\nu} + \text{const} \, \varepsilon_{\mu\nu} \right) \, g^{\rho\mu} \, g^{\sigma\nu} \right\}
\]

\[
\Psi_{\text{gauge}} = \text{tr} \left\{ \tilde{\varepsilon} \left( \partial_\mu A_\mu \, g^{\rho\mu} + b \right) \right\}
\]

\[
\Psi_{\text{ghost}} = \text{tr} \left\{ \tilde{\phi} \, \partial_\rho \psi_\mu \, g^{\rho\mu} \right\}
\]

(193)

fixing the ordinary gauge symmetry of the physical gauge-boson \( A_\mu \) and of the topological ghost \( \psi_\mu \) by means of the Lorentz gauge. As we see \( \tilde{\varepsilon} \) is the antighost of ordinary gauge-symmetry while \( \tilde{\phi} \) is the antighost for the gauge symmetry of the topological ghost \( \psi \).

In the case the topological Yang-Mills theory is coupled to some topological matter system the gauge-fixing of the topological gauge-symmetry can be achieved by imposing that the field-strength \( F = F_{\mu\nu}^{\text{center}} \) be equal to some appropriate function of the matter fields:

\[
F = 2i \, \mathcal{P}(X)
\]

(194)

In this case we can also suppress the auxiliary field \( B \) and replace the antighost part of the BRST-algebra with the equations:

\[
\begin{align*}
\mathcal{S} \, \bar{\varepsilon} &= b \\
\mathcal{S} \, \bar{x}^{\pm} &= \left( \frac{1}{2} \mathcal{F} \Leftrightarrow i \mathcal{P} \right) \\
\mathcal{S} \, \tilde{\phi} &= \tilde{\eta}
\end{align*}
\]

(195)

that substitute eq.s (187). Correspondingly the gauge-fermion \( \Psi_{\text{topol}} \) of eq. (193) can be replaced with:

\[
\Psi_{\text{topol}} = 2\bar{x}^{\pm} \left( \frac{1}{2} \mathcal{F} + i \mathcal{P} \right)
\]

(196)

It is now worth noting that for consistency with the BRST algebra (181) if we define the 2-form \( \Theta^{(2)} = 2i \, \mathcal{P}(X, X^*) \, e^+ \wedge e^- \) we must have \( s \, \Theta^{(2)} = d \psi_\text{center} \). Indeed by restriction to the center of the Lie-algebra we obtain an abelian topological gauge theory for which \( s \mathcal{F} = d \tilde{\psi} \).

Reconsidering the supersymmetry transformation rules of the gauge multiplet (16) and the rules of A-twisting we realize that the property required for the function \( \mathcal{P}(X, X^*) \) is satisfied by the auxiliary field \( \mathcal{P} \) of the gauge multiplet provided we identify \( \psi = \frac{i}{2} \left( \lambda^- \, e^- + \lambda^+ \, e^+ \right) \). This is correct since looking at table I and at eq.s (177) we recognize that a subset of the fields does indeed describe a topological Yang-Mills theory upon the identifications:

\[
\begin{align*}
\psi &= \frac{i}{2} \left( \lambda^- \, e^- + \lambda^+ \, e^+ \right) \\
\phi &= M \\
\tilde{\phi} &= M^* \\
\tilde{\eta} &= \frac{1}{2} \left( \lambda^+ + \lambda^* \right) \\
\bar{x}^{\pm} &= \frac{1}{2} \left( \lambda^* \Leftrightarrow \lambda^- \right)
\end{align*}
\]

(197)
We also see that the descent equations:
\[ \begin{align*}
    s\Theta^{(2)} &= d\Theta^{(1)} \\
    s\Theta^{(1)} &= d\Theta^{(0)} \\
    s\Theta^{(0)} &= 0
\end{align*} \] (198)
are solved by the position:
\[ \begin{align*}
    \Theta^{(2)} &= 2i\mathcal{P}e^+ \wedge e^- = 2i\mathcal{P}(X,X^*)e^+ \wedge e^- \\
    \Theta^{(1)} &= \psi = \frac{i}{2}(\hat{\lambda}^- e^- + \lambda^+ e^+) \\
    \Theta^{(0)} &= \phi = M
\end{align*} \] (199)
so that the quantum action of the topological gauge-theory (190) can be topologically deformed by:
\[ S_{\text{quantum}} \leftrightarrow S_{\text{quantum}} \leftrightarrow i\int \Theta^{(2)} \] (200)
Altogether we see that the classical action \( S_{\text{class}} = \frac{\alpha}{2\pi} \int F_{\text{curv}} \) plus the topological deformation \( \Leftrightarrow i\int \Theta^{(2)} \) constitute the Fayet-Iliopoulos term while the remaining terms in the action (22) are BRST-exact and come from the gauge-fixings:
\[ s\int \left[ \bar{\chi}_{+-} \left( \frac{\mathcal{F}}{\xi} \Leftrightarrow i\mathcal{P} \right) + \bar{\phi} \left( \partial_+ \psi_- + \partial_- \psi_+ \right) \right] \] (201)
On the other hand the matter multiplets with their fermions span a topological \( \sigma \)-model [19] coupled to the topological gauge-system. The topological symmetry in this case is the possibility of deforming the embedding functions \( X^i(z,\bar{z}) \) in an arbitrary way. Correspondingly in the absence of gauge couplings the \textit{gauge-free} topological BRST-algebra is very simple:
\[ \begin{align*}
    sX^i &= e^i \\
    sX^* &= e^* \\
    sc^i &= 0 \\
    sc^* &= 0
\end{align*} \] (202)
\( e^i \) and \( e^* \) being the ghost of the deformation-symmetry. In the presence of a coupling to a topological gauge-theory defined by the covariant derivative:
\[ \nabla X^i = dX^i \Leftrightarrow i\mathcal{A}q^i_jX^j \] (203)
the \textit{gauge-free} BRST-algebra of the matter system becomes:
\[ \begin{align*}
    sX^i &= e^i \Leftrightarrow i e^a q^j_i X^j \\
    sX^* &= e^* + i e^a q^j_i X^j \\
    sc^i &= i q^i_j \left( c^j e^a + X^j \phi \right) \\
    sc^* &= i q^* j \left( c^j e^a + X^j \phi \right)
\end{align*} \] (204)
the last two of eq.s (204) being uniquely fixed by the nilpotency $s^2 = 0$ of the Slavnov operator.

Comparing with eq.s (177) we see that indeed eq.s (196) are reproduced if we make the following identifications:

$$
\begin{align*}
\psi^i & = c^i \\
\bar{\psi}^i & = \bar{c}^i
\end{align*}
$$

The remaining two fermions are to be identified with the antighosts:

$$
\begin{align*}
\bar{c}^i & = \bar{\psi}^i \\
\bar{c}^i & = \psi^i
\end{align*}
$$

and their BRST-variation following from eq.s (177) yields the topological gauge-fixing of the matter sector:

$$
\begin{align*}
s \bar{c}^i & = i \eta^j_{\beta} \bar{c}^i c^j + J \eta^j_{\beta} \partial_j W^* + \frac{i}{2} \nabla_- X^i \\
s \bar{c}^{i*} & = \leftrightarrow i \eta^j_{\beta} \bar{c}^{i*} c^j + J \eta^j_{\beta} \partial_j W^* \leftrightarrow \frac{i}{2} \nabla_+ X^i
\end{align*}
$$

Following Witten [8] and [2] we easily recover the interpretation of the "instantons" encoded in the topological gauge-fixings dictated by eq.s (207) and (195). Indeed we just recall that the functional integral is concentrated on those configurations that are a fixed point of the (0|2) supergroup transformations. Looking at eq.s (177) we see that such configurations have all the ghosts and antighosts equal to zero while the bosonic fields satisfy the following conditions:

$$
\begin{align*}
\eta^j_{\beta} \partial_j W^*(X^*) & = 0 \\
\eta^j_{\beta} \partial_j W(X) & = 0 \\
\nabla_- X^i & = 0 \\
\nabla_+ X^{i*} & = 0 \\
\mathcal{F} & = 2i \mathcal{P} = \leftrightarrow [\mathcal{D}^X(X, X^*) \leftrightarrow r]
\end{align*}
$$

where $\mathcal{D}^X(X, X^*) = \sum_i q^i |X^i|^2$ is the momentum map function defined in section V. Hence the instantons are holomorphic maps from the world-sheet to a locus in $\mathbb{C}^n$ characterized by the equations $\eta^j_{\beta} \partial_j W(X) = 0$. In the case chosen by Witten and reviewed in section IX where $W(X) = P \mathcal{W}(S^i)\Gamma$ this locus is the hypersurface $\mathcal{W}(S^i) = 0 \Gamma(P = 0)$ in a weighted projective space $\mathbb{W}C_{\mu - 2} \Gamma$ the weights of the homogeneous coordinates $S^i$ being their charges. In other words the instantons are holomorphic solutions of the corresponding $N = 2 \sigma$-model. The value of the action on these instantons has been calculated by Witten and the results is easily retrieved in our notations. Indeed the Lagrangian (29) restricted to the bosonic fields of zero
Using eqs (208) and \([\nabla_-, \nabla_+] X^i = i \mathcal{F} q^i_j X^j\Gamma\) we obtain:

\[
\int \mathcal{L}_{\text{ghost}} = \left( \frac{\theta}{2\pi} + ir \right) \int \mathcal{F} = 2\pi i t N
\]

where \(N\) is the winding number and the parameter \(t\) was defined in eq. (21).

**THE B-MODEL.**

In order to identify the system described by the B-model we discuss the structure of a topological Landau-Ginzburg theory [3] coupled to an ordinary abelian gauge theory. To this effect we begin with the structure of a topological rigid Landau-Ginzburg theory. The rigid Landau-Ginzburg model was defined in section VII and it is described by the action (53). It has the R-symmetries (55) and it is \(N=2\) supersymmetric under the transformations following from the rheonomic parametrizations (54). The rigid topological Landau-Ginzburg model has the same action (53) but the spin of the fields is changed; namely it is that obtained by B-twisting: the scalar fields \(X^i\) and \(X^{*i}\) maintain spin-zero as in the ordinary model while the spin 1/2 fermions acquire either spin zero or spin ±1. Specifically \(\psi^i\Gamma\bar{\psi}^{*i}\) have both spin zero, while \(\bar{\psi}^i\) and \(\psi_i\) have spin \(s = 1\) and \(s = 0\) respectively. In view of this fact it is convenient to introduce the new variables:

\[
\begin{align*}
C^* &= \psi^* + \bar{\psi}^*
\end{align*}
\]
\[
\begin{align*}
\bar{C}^i &= \left( \bar{C}^i_+ e^+ + \bar{C}^i_- e^- \right) \\
\theta^* &= \psi^* \Leftrightarrow \bar{\psi}^*
\end{align*}
\]

and rewrite the action (53) in the form:

\[
\begin{align*}
\mathcal{L}_{\text{topolLG}} &= \left( \partial_+ X^i\bar{\partial} X^i + \partial_- X^{*i}\bar{\partial} X^{*i} \right) \\
&+ 2i \left( \bar{C}^i_+ \partial_- C^{*i} + \bar{C}^i_- \partial_+ C^{*i} \right) \\
&+ 2i \left( \bar{C}^i_+ \partial_- \theta^* \Leftrightarrow \bar{C}^i_- \partial_+ \theta^* \right) \\
&+ 8 \bar{C}^i_+ \bar{C}^i_- \partial_i \partial_j \mathcal{W} + 4 C^{*i} \theta^i \partial_i \partial_j \bar{\mathcal{W}} \\
&+ 8 \partial_i \mathcal{W} \partial_j \bar{\mathcal{W}}
\end{align*}
\]

If we denote by \([\Omega]_+ = \Omega_+ e^+ \Leftrightarrow \Omega_- e^\Gamma\) the Hodge-dual of the 1-form \(\Omega = \Omega_+ e^+ + \Omega_- e^-\) then the action (21.2) can be rewritten in the following more condensed form:

\[
S_{\text{topolLG}} = \int \mathcal{L}_{\text{topolLG}} e^+ \wedge e^-
\]
and it is closed under the following BRST-transformations:

\[
\begin{align*}
    s X^i &= 0 \\
    s X^i &= C^i \\
    s C^i &= 0 \\
    s \theta^* &= 2 \eta^* \partial_j \mathcal{W} \\
    s \bar{C}^i &= \frac{i}{2} d X^i
\end{align*}
\]

where we have assigned ghost number \( \#gh = 0 \) to the physical fields \( X^i \) and \( \#gh = 1 \) to the ghost \( C^i \), and \( \#gh = \frac{1}{2} \) to the antighost \( \bar{C}^i \). In this way the gauge-free BRST-algebra is given by the first three of eqs (214): it quantizes a symmetry which corresponds to a deformation of the complex structure of the target coordinates \( X^i, X^{i*} \).

The variation of the antighosts defines the gauge-fixings:

\[
\begin{align*}
    \partial_i \mathcal{W}(X) &= 0 \\
    d X^i &= 0
\end{align*}
\]

that select the "instantons" \( \Gamma \) the constant maps \( (d X^i = 0) \) from the world-sheet to the critical points \( (\partial_i \mathcal{W}(X_0) = 0) \) of the superpotential \( \mathcal{W} \). The action (213) is the sum of a BRST non-trivial part:

\[
\Omega_{(-2)} [\mathcal{W}] = \int \left[ 4 \bar{C}^i \wedge C^j \partial_i \partial_j \mathcal{W} + 2 i \bar{C}^i \wedge d \theta^* \right]
\]

that is closed \( (s \Omega_{(-2)} = 0) \) but not exact \( (\Omega_{(-2)} \neq s(something)) \) and has ghost-number \( \#gh = \frac{1}{2} \) plus two BRST exact terms:

\[
\begin{align*}
    K^{(K_{(0)})}_{(K_{(0)})} &= \int \left\{ d X^i \wedge [d X^{i*}]_* + 2 i \bar{C}^i \wedge [d C^{i*}]_* \right\} \\
    &= s \int \Psi^{(K_{(0)})} = s \int 2 i \bar{C}^i \wedge [d X^{i*}]_* \\
    K^{(W)}_{(0)} &= \int 4 \left\{ C^i \theta^* \partial_i \partial_j \bar{\mathcal{W}} + 2 \partial_i \mathcal{W} \partial_j \bar{\mathcal{W}} \right\} e^+ \wedge e^- \\
    &= s \int \Psi^{(W)} = s \int 4 \partial_j \bar{\mathcal{W}} \theta^* e^+ \wedge e^-
\end{align*}
\]

that have ghost-number \( \#gh = 0 \) and correspond to the BRST-variation of the gauge-fermions associated with the two gauge-fixings (215). As already pointed out the rigid topological Landau-Ginzburg model has been extensively studied in the literature. Here we are interested in the case
where the topological Landau-Ginzburg model is coupled to an ordinary abelian gauge theory. Under this circumstance the BRST-algebra (214) is replaced by:

\[
\begin{align*}
\bar{s} A' &= d c^a \\
\bar{s} F' &= 0 \\
\bar{s} e^a &= 0 \\
\bar{s} X^i &= i e^a q_j^i X^j \\
\bar{s} X^{i*} &= C^{i*} \leftrightarrow i e^a q_j^i X^{j*} \\
\bar{s} C^{i*} &= i e^a q_j^i C^{j*} \\
\bar{s} \theta^{i*} &= 2 \eta^{i*} \partial_j W \leftrightarrow i e^a q_j^i \theta^{j*} \\
\bar{s} \tilde{C}^i &= \leftrightarrow \frac{i}{2} \nabla X^i + i e^a q_j^i \tilde{C}^j 
\end{align*}
\]

(218)

where \( \nabla(...)^i = d(...)^i + i A' q_j^i (...)^j \) denotes the gauge covariant derivative and the superpotential \( W(X) \) of the rigid theory has been replaced by \( W(X) \) namely the superpotential of the gauged-coupled model. The action (213) is also replaced by a similar expression where the ordinary derivatives are converted into covariant derivatives.

The topological system emerging from the B-twist of the N=2 model discussed in the present article is precisely a Landau-Ginzburg model of this type: in particular there is no topological gauge theory rather an ordinary gauge theory plus a topological massive vector. The identification is better discussed at the level of the BRST-algebra comparing eq.s (218) with eq.s (180) after setting:

\[
\begin{align*}
A' &= [A_+ + 2i \, M] \epsilon^+ + J \, [A_- \leftrightarrow 2i \, M^*] \epsilon^- \\
B &= M \epsilon^+ + M^* \epsilon^- \\
\psi^{(mass)} &= \frac{1}{4} \left( \lambda^- \epsilon^+ + \bar{\lambda}^- \epsilon^+ \right) \\
\lambda^+ &= \lambda^+ + \bar{\lambda}^+ \\
\bar{\lambda}^{(mass)} &= \frac{i}{2} \left[ \lambda^+ \leftrightarrow \bar{\lambda}^+ \right] \\
C^{i*} &= \psi^{i*} + \bar{\psi}^{i*} \\
\theta^i &= \psi^{i*} \leftrightarrow \bar{\psi}^{i*} \\
\tilde{C}^i &= \psi^i \epsilon^+ + \bar{\psi}^i \epsilon^-
\end{align*}
\]

(219)

With these definitions the BRST-transformations of eq.s (180) become indeed identical with those of eq.s (218) plus the following ones:

\[
\begin{align*}
\bar{s} B &= \psi^{(mass)} \\
\bar{s} \psi^{(mass)} &= 0
\end{align*}
\]
\[ s \tilde{\chi}^{(mass)} = \mathcal{P}(X, X^*) + (\partial_+ B_- \leftrightarrow \partial_- B_+) \]
\[ s \tilde{\chi} = \mathcal{F}' \]  
(220)

The first two of eq.s (220) correspond to the gauge-free BRST-algebra of the topological massive vector \( \Gamma^{(mass)} \) being the 1-form ghost associated with the continuous deformation symmetry of the vector \( B \). The second two of eq.s (220) are BRST-transformations of antighosts and the left hand side defines the gauge-fixings of the massive vector and gauge vector \( \Gamma \) respectively in namely:

\[ \mathcal{P}(X, X^*) + (\partial_+ B_- \leftrightarrow \partial_- B_+) = 0 \]
\[ \mathcal{F}' = \partial_+ A_- \leftrightarrow \partial_- A_+ = 0 \]  
(221)

Actually looking at eq.s (180) we realize that the configurations corresponding to a fixed point of the \( (0|2) \) supergroup are characterized by all the fermions (= ghosts + antighosts) equal to zero and by:

\[ M = M^* = 0 \implies B = 0 \]
\[ \mathcal{F} = 0 \implies \mathcal{F}' = 0 \]
\[ \mathcal{P}(X, X^*) = 0 \]
\[ \eta^{i_1 j} \partial_j W(X) = 0 \]
\[ dX^i = 0 \]  
(222)

Hence in the B-twist the functional integral is concentrated on the constant maps from the world-sheet to the extrema of the classical scalar potential (33). As we have seen in the A-twist the functional integral was concentrated on the holomorphic maps to such extrema: furthermore in the A-twist the classical extrema were somewhat modified by the winding number effect since the equation \( \mathcal{P} = 0 \) was replaced by \( \mathcal{P} = \mathcal{F} \). In the B-twist no instantonic effects modifies the definition of classical extremum. The extrema of the scalar potential can be a point (Landau-Ginzburg phase) or a manifold (\( \sigma \)-model phase). The B-twist selects the constant maps in either case whereas the A-twist selects the holomorphic maps in either case. However in the Landau-Ginzburg phase the holomorphic maps to a point are the same thing as the constant maps so that in this phase the instantons of the A-model coincide with those of the B-model.

In the case of those \( N=2 \) theories that are actually \( N=4 \) theories there is only the \( \sigma \)-model phase as we have already pointed out and the above coincidence does not occur.

18 Topological Observables of the A and B models and HyperKähler quotients

Having identified the topological theories produced by the A and B twists let us consider their meaning in relation with the HyperKähler quotient construction.
We recall that in any 2D topological field-theory the key objects are the solutions of the
descent equations (198). Indeed they provide the means to deform the topological action
according to the generalization of eq. (200):
\[
S_{\text{quantum}} \leftrightarrow S_{\text{quantum}} + \sum_A t_A \int \Theta_A^{(2)}
\]
\[\Theta_A^{(2)} \text{ being a complete base of solutions to eq. (198)}\]
and to study the deformed correlation functions:
\[
c_{A_1, A_2, \ldots, A_N}(t) = \langle \Theta^{(0)}_{A_1}, \ldots, \Theta^{(0)}_{A_N} \rangle \exp \left[ \sum_A t_A \int \Theta_A^{(2)} \right]
\]
In the case of the A-twist we have seen that a solution of the descent eq. (198) is associated
with each abelian factor of the gauge group and it is given by eq.(199). A set of topological
deformations of the action are therefore proportional in the A-model to the r-parameters of
the N=2 Fayet-Iliopoulos terms. In the N=4 case where the effective \( \sigma \)-model target space
\( \mathcal{M}_{\text{target}} \) namely the locus of the scalar potential extrema
\[
M = 0 ; \quad \mathcal{D}^3 (XX^*) = r ; \quad \partial_i W(X) = 0 \leftrightarrow \begin{cases} N = 0 \\ \mathcal{D}^+(u, v) = s \end{cases}
\]
is equal to the HyperKähler quotient \( \mathcal{D}^{-1}(\zeta)/G \) of flat space with respect the triholomorphic
action of the gauge group \( G \); the topological observables of the A-model associated with the
r-parameters correspond to the Kähler structure deformations of of \( \mathcal{M}_{\text{target}} \). To see this it
suffices to recall the way the Kähler potential of the quotient manifold \( \mathcal{D}^{-1}(\zeta)/G \) is determined
(see eq. s (91) \( \Gamma \) (90). Let \( K = \frac{1}{2} \sum_i (u^i u^i + v^i v^i) \) be the Kähler potential of flat space
and \( \mathcal{D}^3(u, \bar{u}, v, \bar{v}) \) be the real non-holomorphic part of the momentum map. By definition
both \( K \) and \( \mathcal{D}^3 \) are invariant under the action of the isometry group \( G \) but not under the
action of its complexification \( G^c \). On the other hand the holomorphic part \( \mathcal{D}^+(u, v) \) of the
momentum-map is invariant not only under \( G \) but also under \( G^c \); furthermore one shows that
the quotient manifold \( \mathcal{D}^{-1}(\zeta)/G \) is the same thing as the quotient \( \mathcal{H} = \frac{D^{+(u, v)} = s}{G} \)
of the holomorphic hypersurface \( D^{+(u, v)} = s \) (\( s = \zeta^1 + i \zeta^2 \Gamma \) being the complex level parameters)
modded by the action of \( G^c \). Naming \( e^V \in G^c \) an element of this complexified group eq. (90)
specializes in our case to the following equation:
\[
\mathcal{D}^3 \left( e^{-V}\{u, v\} \right) = r
\]
and it is true equation on the hypersurface \( \mathcal{H} = \frac{D^{+(u, v)} = s}{G^c} \). Then in agreement with eq. (91)
the Kähler potential of the HyperKähler quotient manifold \( \mathcal{D}^{-1}(\zeta)/G \) is:
\[
\hat{K} = K\vert_{\mathcal{H}} \left( e^{-V}\{u, v\} \right) + r V
\]
Consequently a variation of the \( r \) parameters uniquely affects the Kähler potential \( \Gamma \) the quotient \( \mathcal{H} = \frac{D^1[u,v]}{G^\ast} \Gamma \) as an analytic manifold being insensitive to such a variation. Summarizing the A-model is a cohomological theory in the moduli space of Kähler structure deformations.

Before addressing the structure of observables in the B-model it is also worth discussing the general form of observables in any topological theory described by the BRST-algebra (204) and coupled to a topological gauge-theory (181). In a topological \( \sigma \)-model the observables \( \Gamma \) namely the solutions of the descent equations (198) are in correspondence with the cohomology classes of the target-manifold. If \( \omega^{(n)} = \omega_{i_1,\ldots,i_n}(X) \, dX^{i_1} \wedge \ldots \wedge dX^{i_n} \) is a closed \( n \)-form \( d\omega^{(n)} = 0 \Gamma \) we promote it to a ghost-form \( \hat{\omega}^{(n)} \) by substituting \( d \Leftrightarrow d + s \Gamma dX^i \Leftrightarrow dX^i + c^i \) then by expanding this ghost-form into addends of definite ghost number \( \hat{\omega}^{(n)} = \sum_{g=0}^{n} \hat{\omega}^{(n-g)} \) we solve the descent equations by setting \( \Theta^{(2)} = \hat{\omega}^{(2)} \Gamma \Theta^{(1)} = \hat{\omega}^{(1)} \Gamma \Theta^{(0)} = \hat{\omega}^{(0)} \). (Note that in this discussion for simplicity we do not distinguish holomorphic and antiholomorphic indices).

In a similar way in the topological model described by eqs (204) \( \Gamma (181) \Gamma \) the solutions of the descent equations are in correspondence with the antisymmetric constant tensors \( a_{i_1,\ldots,i_n} \) that are invariant under the action of the gauge-group \( \Gamma \) namely that satisfy the condition:

\[
a_{p,[i_2,\ldots,i_n]} q^p_{i_1} = 0 \quad (228)
\]

Indeed setting

\[
\hat{\nabla} = \hat{d} \Leftrightarrow i \, q \hat{A} \\
\begin{align*}
\hat{d} & \Leftrightarrow i \, q \hat{A} \\
& = (d + s) \Leftrightarrow i \, q (A + c^a) \\
& = \nabla_{(1,0)} + \nabla_{(0,1)} \\
& = (d \Leftrightarrow i \, A \, q) + (s \Leftrightarrow i \, c^a \, q)
\end{align*}
\]

we obtain \( \hat{\nabla}^2 = \Leftrightarrow \hat{F} \, q \) and to every invariant antisymmetric tensor we can associate the \( \hat{d} \)-closed ghost-form \( \hat{\omega} = a_{i_1,\ldots,i_n} \hat{\nabla} X^{i_1} \ldots \hat{\nabla} X^{i_n} \). Expanding it in definite ghost-number parts \( \Gamma \) the solution of the descent equations is obtained in the same way as in the \( \sigma \)-model case.

Let us now turn our attention to the B-model which describes a topological gauge-coupled Landau-Ginzburg theory. Here the topological observables are in correspondence with the symmetric invariant tensors \( \Gamma \) rather than with the antisymmetric ones. To see it we recall the solutions of the descent equations in the case of the topological rigid Landau-Ginzburg model: in this case the topological observables are in correspondence with the elements of the local polynomial ring of the superpotential \( \mathcal{R}_W = \frac{C[X]}{\partial W} \). Indeed let \( P(X) \in \mathcal{R}_W \) be some non trivial polynomial of the local ring \( \Gamma \) a solution of the descent equations (198) is obtained by setting:

\[
\begin{align*}
\Theta_P &= P(X) \\
\Theta_P^{(1)} &= \Leftrightarrow 2 i \partial_i P \bar{C}^i \\
\Theta_P^{(2)} &= \Leftrightarrow 2 \partial_i \partial_j P \bar{C}^i \land \bar{C}^j \Leftrightarrow 4 \left[ \partial_x P \partial_t W \eta^{kt} \right] \epsilon^+ \land \epsilon^-
\end{align*}
\]

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The reason why $P(X)$ has to be a non trivial element of the local ring is simple. If $P(X)$ were proportional to the vanishing relations (i.e. if $P(X) = \sum_i p_i(X) \frac{\partial W}{\partial X_i}$) then using the BRST transformations (214) one could see that $P(X) = s K$ and so $\Theta_P$ would be exact. (For the proof it suffices to set $K = p^i(X) \frac{1}{2} \theta^* q_{ij} \theta$.) In our case where the Landau-Ginzburg theory is gauged-coupled and the BRST-transformations are given by eqs (218) the solution of the descent equations has the same form as in eq.(230) provided the polynomial $P(X)$ has the form

$$P(X) = s_{i_1,...,i_n} X^{i_1} ... X^{i_n}$$

(231)

the symmetric tensor $s_{i_1,...,i_n}$ being gauge invariant:

$$s_{p,(i_2,...,i_n q^p_{i_1})} = 0$$

(232)

and such that $P(X)$ is a non-trivial element of the the ring $\mathcal{R}_W$. Consider now the case of $N=4$ theories where the superpotential is given by eq. (131) and consider the polynomial:

$$P_s(X^A) = \text{cost } n$$

(233)

which is gauge-invariant ($n$ is neutral under the gauge-group) and non-trivial with respect to the vanishing relations $\frac{\partial}{\partial X^A} W(X^A) \approx 0$. The corresponding two-form is easily calculated from eq.s (230). We obtain:

$$\Theta_{P_s}^{(2)} = 2 \text{cost} \left( s^* \leftrightarrow i D^- (\bar{u}, \bar{v}) \right) e^+ \wedge e^-$$

(234)

Hence a topological deformation of the action is given by:

$$S_{\text{quantum}} \leftrightarrow \delta s \int \Theta_{P_s}^{(2)}$$

(235)

For a convenient choice of the constant $\text{const}$ this deformation is precisely the variation of the action (123)(124)(125) under a shift $s \leftrightarrow s + \delta s$ of the $s$ parameters of the triholomorphic momentum-map namely of the $N=4$ Fayet-Iliopoulos term. These parameters define the complex structure of the HyperKähler quotient manifold.

Summarizing we have seen that the three parameters $r = \zeta^3$, $s = \zeta^1 + i \zeta^2$ of the $N=4$ Fayet-Iliopoulos term are on one hand identified with the momentum-map levels in the geometrical HyperKähler quotient construction and on the other hand are the coupling constants of two topological field-theories: the $A$-twist selects the parameters $r$ that play the role of moduli of the Kähler structure while the $B$-twist selects the $s$ parameters that play the role of moduli of the complex structure. It is an obvious programme to apply now the topological field-theory framework to the investigation of the moduli space of interesting HyperKähler quotient manifolds like the ALE spaces [11][4]. This is left to future publications.
## Tables

### TABLE I

**N=2 THEORY: SPIN and CHARGES before and after the TWISTS**

<table>
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<tr>
<th>Field</th>
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$T_{\text{AB}}$, $F_{\mu\nu}$, $B_{\mu\nu}$
References


