Quantum Affine Algebra and Universal $R$-Matrix with Spectral Parameter

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Abstract:

Using the previous obtained universal $R$-matrix for the quantized nontwisted affine Lie algebras $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$, we determine the explicitly spectral-dependent universal $R$-matrix for the corresponding quantum Lie algebras $U_q(A_1)$ and $U_q(A_2)$. As their applications, we reproduce the well-known results in the fundamental representations and we also derive for the first time the extremely explicit formula of the spectral-dependent $R$-matrix for the adjoint representation of $U_q(A_2)$, the simplest non-trivial case when the tensor product of the representations is not multiplicity-free.

PACS numbers: 03.65.Fd; 02.20.+b
1. Introduction: In this Letter, we present two main results: (i) the universal $R$-matrix for quantum Lie algebras $U_q(A_1)$ and $U_q(A_2)$ with explicit spectral parameter dependence; (ii) an extremely explicit formula for the spectral-dependent $R$-matrix in the adjoint representation of $U_q(A_2)$, the simplest nontrivial case when the tensor product of the representations is not multiplicity-free.

Recently quantum groups have attracted much attention [1] [2][3]. Finding spectral parameter dependent solution to quantum Yang-Baxter equation (QYBE) (that is spectral-dependent $R$-matrix) has been a central issue (see, for example, [2][4][5]) in many aspects. The usual approach to this effect is Jones' "Yang-Baxterization" procedure \(^1\) [4]. Namely, given some representation of braid group, it is possible to obtain the spectral-dependent solution to QYBE by Yang-Baxterizing the former. This approach is powerful for the case of "abelian Yang-Baxterization" where the tensor product of representations is multiplicity-free. In fact, as far as we know, all the previous research in the literature has limited to this simple case. When the tensor product of representations is not multiplicity-free, Jones conjectures that "non-abelian Yang-Baxterization" occurs. This suggests that one cannot any more use the simple ansatz that spectral-dependent $R$-matrix takes the form of spectral-dependent scalar function times spectral-independent projection operator and thus makes it difficult to solve Jimbo-type equations [2].

In this Letter we will present a new way of obtaining spectral-dependent $R$-matrix for quantum Lie algebras. Our idea reverses, in some sense, the above process. Namely, we start from universal $R$-matrix of $U_q(G^{(1)})$ and then apply it to finite-dimensional loop representations $V(z)$ of $U_q(G^{(1)})$, which is isomorphic to $V \otimes C(z, z^{-1})$ of $U_q(G)$. In this way, we obtain the spectral-dependent solution to QYBE for the latter. One of the advantages lying our approach is that the multiplicity-free and non-multiplicity-free cases can be treated in a unified fashion. As a matter of fact, we are able to get a spectral-dependent universal $R$-matrix for $U_q(A_1)$ and $U_q(A_2)$. Applying to some concrete representations, we are able to reproduce the well-known results in the fundamental representation and to obtain for the first time the extremely explicit

\(^1\)Jones uses the terminology "Baxterization" in his paper. We find it's more fitting to use "Yang-Baxterization", as most people do in literature.
formula for $R$-matrix of $U_q(A_2)$ for the adjoint representation.

2. Universal $R$-Matrix for $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$: For self-contained, this section is devoted to a brief and quick review of the construction of the universal $R$-matrix for $U_q(A_1^{(1)})$ [6] and for $U_q(A_2^{(1)})$ [6][7]. Throughout the paper, we use the notations: $(ad_q x_\alpha) x_{\beta} = [x_\alpha, x_\beta] = x_\alpha x_\beta - q^{[\alpha,\beta]} x_\beta x_\alpha$, $\theta(q^h) = q^{\pm h}$, $\theta(E_i) = F_i$, $\theta(F_i) = E_i$, $\theta(q) = q^{-1}$, $(n)_q = (1 - q^n) / (1 - q)$, $[n]_q = (q^n - q^{-n}) / (q - q^{-1})$, $q_\alpha = q^{[\alpha,\alpha]}$, $\exp_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]_q!}$, $(n)_q! = (n)_q(n - 1)_q \ldots (1)_q$.

We start with rank 2 case. Fix the ordering in the positive root system $\Delta_+^2$ of $A_1^{(1)}$: $\alpha$, $\alpha + \delta$, $\cdots$, $\alpha + n\delta$, $\delta$, $2\delta$, $\cdots$, $m\delta$, $\cdots$, $(\delta - \alpha) + \ell\delta$, $\cdots$, $\delta - \alpha$, where $\alpha$, $\delta - \alpha$ are simple roots and $\delta$ is the minimal positive imaginary root. One has [6] the universal $R$-matrix for $U_q(A_1^{(1)})$:

$$R = \left( \prod_{n \geq 0} \exp_q\left( (q - q^{-1})(E_{\alpha+n\delta} \otimes F_{\alpha+n\delta}) \right) \right) \cdot \exp\left( \sum_{n > 0} \frac{n}{[n]_q!} (q_\alpha - q^{-1}_\alpha)(E_{\alpha \delta} \otimes F_{n\delta}) \right) \cdot \left( \prod_{n \geq 0} \exp_q\left( (q - q^{-1})(E_{(\delta - \alpha)+n\delta} \otimes F_{(\delta - \alpha)+n\delta}) \right) \right) \cdot q^{(h_\alpha \otimes h_{-\alpha} \otimes d_+ d_+ d_+ c}) (1)$$

where $c = h_\alpha + h_{-\alpha}$ and the Cartan-Weyl generators, $E_\gamma$, $F_\gamma = \theta(E_\gamma)$, $\gamma \in \Delta_+$, are given by:

$$E_\delta = [(\alpha, \alpha)]_q^{-1}[E_\alpha, E_{\delta - \alpha}]_q$$

$E_{\alpha+n\delta} = (-1)^n \left( ad E_\delta \right)^n E_\alpha$, $E_{(\delta - \alpha)+n\delta} = \left( ad E_\delta \right)^n E_{\delta - \alpha}$, $\tilde{E}_{n\delta} = [(\alpha, \alpha)]_q^{-1}[E_{\alpha+(n-1)\delta}, E_{\delta - \alpha}]_q$ and

$$\tilde{E}_{n\delta} = \sum_{p_1 + 2p_2 + \cdots + np_n = n} \frac{\binom{q^{[\alpha,\alpha]} - q^{-[\alpha,\alpha]}}{p_1! \cdots p_n!}}{(E_\delta)^{p_1}(E_{2\delta})^{p_2} \cdots (E_{n\delta})^{p_n}} (2)$$

The order in the product (1) coincides with the above chosen normal order.

We now consider rank 3 case. Fix the order in positive root system $\Delta_+^3$ of $A_2^{(1)}$: $\alpha$, $\alpha + \delta$, $\cdots$, $\alpha + m_1\delta$, $\cdots$, $\alpha + \beta$, $\alpha + \beta + \delta$, $\cdots$, $\alpha + \beta + m_2\delta$, $\cdots$, $\beta$, $\beta + \delta$, $\cdots$, $\beta + m_3\delta$, $\cdots$, $\delta$, $2\delta$, $\cdots$, $k\delta$, $\cdots$, $(\delta - \beta) + l_1\delta$, $\cdots$, $\delta - \beta$, $\cdots$, $(\delta - \alpha) + l_2\delta$, $\cdots$, $\delta - \alpha$, $\cdots$, $(\delta - \alpha - \beta) + l_3\delta$, $\cdots$, $\delta - \alpha - \beta$, where $m_i, k, l_i \geq 0$, $i = 1, 2, 3$. Then one can show [6][7] (see, in particular, [7]) that the universal $R$-matrix for $U_q(A_2^{(1)})$ takes the explicit form

$$R = \left( \prod_{n \geq 0} \exp_q\left( (q - q^{-1})(E_{\alpha+n\delta} \otimes F_{\alpha+n\delta}) \right) \right) \cdot \left( \prod_{n \geq 0} \exp_q\left( (q - q^{-1})(E_{\alpha+\beta+n\delta} \otimes F_{\alpha+\beta+n\delta}) \right) \right)$$
\[ \prod_{n \geq 0} \exp_{q^n} \left( (q - q^{-1})(E_{\beta + n \delta} \otimes F_{\beta + n \delta}) \right) \]

\[ \cdot \exp \left( \sum_{n > 0, i, j = 1}^{2} C^n_{ij}(q)(q - q^{-1})(E_{n \delta}^{(i)} \otimes F_{n \delta}^{(j)}) \right) \]

\[ \prod_{n \geq 0} \exp_{q^n} \left( (q - q^{-1})(E_{\beta - \alpha + n \delta} \otimes F_{\beta - \alpha + n \delta}) \right) \]

\[ \prod_{n \geq 0} \exp_{q^n} \left( (q - q^{-1})(E_{\beta - \alpha} \otimes F_{\beta - \alpha}) \right) \]

\[ \prod_{n \geq 0} \exp_{q^n} \left( (q - q^{-1})(E_{\beta - \alpha} \otimes F_{\beta - \alpha}) \right) \]

\[ \sum_{n, i, j = 1}^{2} \sigma_{g_{n i}}^{-1} h_{\alpha \otimes h_{\beta} \otimes c \otimes d \otimes c} \]

(3)

where \( c = h_0 + h_\psi \) with \( \psi = \alpha + \beta \) being the highest root of \( A_2^{(1)} \) and \( (a_{ij}^{(m)}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \).

\[ (C^n_{ij}(q)) = (C^n_{ji}(q)) = \frac{1}{[n]_q} \frac{[2]^2}{q^{2n} + 1 + q^{2n}} \begin{pmatrix} q^n + q^{-n} & (-1)^n \\ (-1)^n & q^n + q^{-n} \end{pmatrix} \]

(4)

and the Cartan-Weyl generators, \( E_{\gamma}, \ F_{\gamma} = \theta(E_{\gamma}), \ \gamma \in \Delta_+, \) are given by \( (\alpha_i = \alpha, \beta, \alpha + \beta \) below) : \( E_{\alpha + \beta} = [E_\alpha, E_\beta]_q \), \( E_{\beta - \alpha} = [E_\beta, E_{\beta - \alpha}]_q \), \( E_{\beta - \alpha} = [E_\beta, E_{\beta - \alpha}]_q \), \( \tilde{E}_h^{(i)} = [\alpha_i, \alpha_i]_q^{-1}[E_{\alpha_i}, E_{\beta - \alpha}]_q \), \( E_{\alpha + n \delta} = (-1)^n(a \tilde{E}_h^{(i)})^n E_{\alpha_i} \), \( E_{\beta - \alpha + n \delta} = (-1)^n(a \tilde{E}_h^{(i)})^n E_{\beta - \alpha} \), \( \tilde{E}_h^{(i)} = [\alpha_i, \alpha_i]_q^{-1}[E_{\alpha_i + (n-1) \delta}, E_{\beta - \alpha}]_q \) and

\[ \tilde{E}_h^{(i)} = \sum_{p_1 + 2p_2 + \ldots + np_n = n} \frac{(q^{(\alpha, \alpha)} - q^{-\alpha_i, \alpha_i})^{p_1 + \ldots + p_n}}{p_1! \ldots p_n!} (E_h^{(i)})^{p_1} (E_{2h}^{(i)})^{p_2} \ldots (E_{nh}^{(i)})^{p_n} \]

(5)

The order in the product of (3) coincides with the above defined order.

3. Universal R-matrix with Spectral Parameter: We come to our main concerns in this Letter. We only list our results and details will be published elsewhere [8].

It can be shown that for any \( z \in C^\times \), there is a homomorphism of algebras \( ev_z : U_q(A_2^{(1)}) \rightarrow U_q(A_1) \) given by: \( ev_z(E_{\alpha}) = E_{\alpha}, \ ev_z(F_{\alpha}) = F_{\alpha}, \ ev_z(h_\alpha) = h_\alpha, \ ev_z(c) = 0, \ ev_z(E_{\beta}) = zF_{\alpha}, \ ev_z(F_{\beta}) = z^{-1}E_{\alpha}, \ ev_z(h_{\beta}) = -h_{\beta} \). Then from eq.(1) one obtains the spectral-dependent universal R-matrix, \( R(x, y) \equiv (ev_x \otimes ev_y)R, \) for \( U_q(A_1) : \)

\[ R(x, y) = \prod_{n \geq 0} \exp_{q^n} \left( (q - q^{-1}) \left( \frac{x}{y} \right)^n (q^{-nh_{\alpha}} E_{\alpha} \otimes F_{\alpha} q^{nh_{\alpha}}) \right) \]

\[ \cdot \exp \left( \sum_{n > 0} n[a]_{q^2}^{-1}(q - q^{-1}) \left( \frac{x}{y} \right)^n (E_{n \delta} \otimes F_{n \delta}) \right) \]

\[ \cdot \prod_{n \geq 0} \exp_{q^n} \left( (q - q^{-1}) \left( \frac{x}{y} \right)^{n+1} (F_{\alpha} q^{-nh_{\alpha}} \otimes q^{nh_{\alpha}} E_{\alpha}) \right) \cdot q^{h_{\alpha} \otimes h_{\alpha}} \]

(6)
where $E_{n\delta}^\alpha$ and $F_{n\delta}^\alpha$ are determined by the following equalities of formal power series:

\begin{equation}
(q_\alpha - q_\alpha^{-1}) \sum_{k=1}^\infty \tilde{E}_{k\delta}^\alpha u^k = \exp \left((q_\alpha - q_\alpha^{-1}) \sum_{l=1}^\infty E_{l\delta}^\alpha u^l\right) - 1
\end{equation}

\begin{equation}
(q_\alpha - q_\alpha^{-1}) \sum_{k=1}^\infty \tilde{F}_{k\delta}^\alpha u^{-k} = \exp \left(-(q_\alpha - q_\alpha^{-1}) \sum_{l=1}^\infty F_{l\delta}^\alpha u^{-l}\right) - 1
\end{equation}

in which $\tilde{E}_{n\delta}^\alpha = [2]_{q^{-1}}^{-1}(1-n^{-1}q^{|n-1|h_\alpha} (E_\alpha F_\alpha - q^{-2}F_\alpha E_\alpha))$, $\tilde{F}_{n\delta}^\alpha = [2]_{q^{-1}}^{-1}(1-n^{-1}q^{|n-1|h_\alpha} (F_\alpha E_\alpha - q^2E_\alpha F_\alpha))$.

We now apply (6) to $V_{1/2} \otimes V_{1/2}$, where $V_{1/2}$ is the fundamental representation of $U_q(A_1)$.

One can show that in this case

\begin{equation}
R_{1/2,1/2}(x,y) = f_q(x,y) \cdot \left(e_{11} + e_{44} + \frac{q^{-1}(y-x)}{y-q^{-2}x}(e_{22} + e_{33}) + \frac{1-q^{-2}}{y-q^{-2}x}(ye_{33} + xe_{22})\right)
\end{equation}

where (and below) $e_{ij}$ is the matrix satisfying $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and

\begin{equation}
f_q(x,y) = q^{1/2} \cdot \exp \left(\sum_{n>0} \frac{q^n - q^{-n}}{q^n + q^{-n}} (x/y)^n\right)
\end{equation}

We thus reproduce the well-known result [2], up to the scalar factor $f_q(x,y)$. In [6] KT obtained (8) directly from (1).

We then turn to $U_q(A_2^{(1)})$. In this case, one may show that for any $z \in C^\times$, there is a homomorphism of algebras $ev_z$: $U_q(A_2^{(1)}) \rightarrow U_q(A_2)$ given by: $ev_z(E_\alpha) = E_\alpha$, $ev_z(F_\alpha) = F_\alpha$, $ev_z(h_\alpha) = h_\alpha$, $ev_z(F_\beta) = F_\beta$, $ev_z(h_\beta) = h_\beta$, $ev_z(E_{\delta - \alpha - \beta}) = zF_{\alpha + \beta}q^{(h_\alpha - h_\beta)/3}$, $ev_z(F_{\delta - \alpha - \beta}) = z^{-1}q^{(h_\alpha - h_\beta)/3}E_{\alpha + \beta}$, $ev_z(h_{\delta - \alpha - \beta}) = -h_{\alpha + \beta}$, $ev_z(c) = 0$.

Thus, from eq.(3) one is able to deduce the explicitly spectral-dependent universal $R$-matrix for $U_q(A_2)$, $R(x,y) \equiv (ev_x \otimes ev_y)R$, given by

\begin{align*}
R(x,y) &= \prod_{n \geq 0} \exp_{q^2} \left((q^{-1})(\frac{x}{y})^n \left(q^{-nh_\alpha}E_\alpha q^{-n(h_\alpha + 2\delta_\alpha)/3} \otimes q^{n(h_\alpha + 2\delta_\alpha)/3}F_\alpha q^{nh_\alpha}\right)\right) \\
&\cdot \prod_{n \geq 0} \exp_{q^2} \left((q^{-1})(\frac{x}{y})^n \left(q^{-nh_\beta + \sigma}E_{\alpha + \beta}q^{n(h_\beta + h_\alpha)/3} \otimes q^{n(h_\beta + h_\alpha)/3}F_{\alpha + \beta}q^{nh_\beta + \sigma}\right)\right) \\
&\cdot \prod_{n \geq 0} \exp \left(\sum_{n>0 \atop i,j=1} \frac{2}{\delta_{ij}}(q-q^{-1})(E_{ni}^{(i)} \otimes F_{ni}^{(j)})\right) \\
&\cdot \prod_{n \geq 0} \exp_{q^2} \left((q^{-1})(\frac{x}{y})^{n+1} \left(E_{(i-\beta)^+ n\delta}^{(i-\beta)} \otimes F_{(i-\beta)^+ n\delta}^{(i)}\right)\right) \\
&\cdot \prod_{n \geq 0} \exp_{q^2} \left((q^{-1})(\frac{x}{y})^{n+1} \left(q^{-n+1}(h_\alpha + 2h_\sigma)/3F_\alpha q^{-nh_\alpha} \otimes q^{nh_\alpha}E_\alpha q^{n+1}(h_\alpha + 2h_\sigma)/3\right)\right)
\end{align*}
\[
\prod_{n \geq 0} \exp_{q_{n+\beta}} \left( (q - q^{-1}) \left( \frac{x}{y} \right)^{n+1} \left( q^{(n+1)(h_{\beta} - h_{\alpha})/3} F_{\alpha + \beta} q^{-n h_{\alpha} + \beta} \right) \right) \cdot q \sum_{\alpha, \beta=1}^{n} (\sigma_{\alpha \beta}^{-1}) u_{\alpha} \otimes u_{\beta} \right)
\]

where

\[
E_{\beta + n \delta}^{(i)} = (-1)^n [2]^{-n} q^n \left\{ (ad'_{q^{-1}} \mathcal{E})^n E_{\beta} \right\} q^{n(h_{\beta} - h_{\alpha})/3}
\]

\[
F_{\beta + n \delta}^{(i)} = [2]^{-n} q^n (h_{\alpha} - h_{\beta})/3 \left( (ad'_{q^{-1}} \mathcal{F})^n F_{\beta} \right)
\]

\[
E_{\beta + n \delta}^{(i)} = [2]^{-n} q^n \left\{ (ad'_{q} \mathcal{E}) q^{n(n+1)(h_{\beta} - h_{\alpha})/3} \right\} F_{\alpha + \beta} q^{-n h_{\alpha} + \beta} E_{\alpha}
\]

\[
E = (ad'_{q} E_{\beta})(ad'_{q} E_{\alpha}) F_{\alpha + \beta} \quad \mathcal{F} = (ad'_{q} (ad'_{q} E_{\alpha + \beta}) F_{\alpha}) F_{\beta},
\]

(\text{ad}'q \mathcal{A}) \cdot \mathcal{B} \equiv AB - Q \mathcal{B} \mathcal{A} \quad E_{n \delta}^{(i)} \text{ and } F_{n \delta}^{(i)} \text{ are determined by the equalities of formal series:}

(\alpha_i = \alpha, \beta)

\[
(q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{k=0}^{\infty} \tilde{E}_{n \delta}^{(i)}(k) u^k = \exp \left( (q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{i=1}^{\infty} \tilde{E}_{n \delta}^{(i)}(i) u^i \right) - 1
\]

\[
-(q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{k=0}^{\infty} \tilde{F}_{n \delta}^{(i)}(k) u^{-k} = \exp \left( -(q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{i=1}^{\infty} \tilde{F}_{n \delta}^{(i)}(i) u^{-i} \right) - 1
\]

in which

\[
\tilde{E}_{n \delta}^{(\alpha)} = (-1)^n [2]^{-n} (E_{\alpha} F_{\alpha} - q^{-2n} F_{\alpha} E_{\alpha}) q^{-(n-1)(h_{\alpha} - h_{\beta})/3}
\]

\[
\tilde{E}_{n \delta}^{(\beta)} = (-1)^n [2]^{-n} q^n \left\{ (ad'_{q} \mathcal{E}) q^{n(n+1)(h_{\beta} - h_{\alpha})/3} \right\} F_{\alpha + \beta} q^{-n h_{\alpha} + \beta} E_{\alpha}
\]

\[
\tilde{E}_{n \delta}^{(i)} = [2]^{-n} q^n \left\{ (ad'_{q} \mathcal{E}) q^{n(n+1)(h_{\beta} - h_{\alpha})/3} \right\} F_{\alpha + \beta} q^{-n h_{\alpha} + \beta} E_{\alpha}
\]

\[
E = E_{\alpha + \beta} F_{\alpha} - q^2 F_{\alpha} E_{\alpha + \beta}, \quad \mathcal{F} = E_{\alpha + \beta} F_{\alpha} - q^{-2} F_{\alpha + \beta} E_{\alpha}
\]

We apply (10) to \(V_{(3)} \otimes V_{(3)}\) in which \(V_{(3)}\) denotes the fundamental representation of \(U_q(A_2)\).

One can show that (10) gives rise to

\[
R_{(3), (3)}(x, y) = f_{q}(x, y) \cdot \left( \frac{e_{11} + e_{99} + q^{-1}(y - x)}{y - q^{-2} x} (e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88}) + \right.

\left. \frac{y(1 - q^{-2})}{y - q^{-2} x} (e_{24} + e_{27} + e_{68}) + \frac{x(1 - q^{-2})}{y - q^{-2} x} (e_{42} + e_{73} + e_{86}) \right)
\]

\[
R_{(3), (3)}(x, y) = q^{2/3} \cdot \exp \left( \sum_{n \geq 0} \frac{q^{2n} - q^{-2n}}{q^{2n} + 1 + q^{-2n}} \frac{(x/y)^n}{n} \right)
\]
We thus reproduce the well-known result [2], up to a scalar factor $f(x, y)$.

We then apply (10) to a very interesting case: $V_{(s)} \otimes V_{(s)}$, where $V_{(s)}$ stands for the adjoint representation of $U_q(A_2)$. This is simplest nontrivial example where the tensor product is not multiplicity-free. It can be shown that spectral-dependent $R$-matrix for the adjoint representation takes the following compact and explicit form,

$$R_{(s),(s)}(x, y) = \left\{ 1 + (q - q^{-1}) \left( \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^{n} \right) \left( E'_{\alpha+\beta+n\delta} \otimes F'_{\alpha+\beta+n\delta} \right) + y^{2} f(e_{36} \otimes e_{36}) \right\}$$

$$\cdot \left\{ 1 + (q - q^{-1}) \left( \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^{n} \right) \left( E'_{\alpha+\beta+n\delta} \otimes F'_{\alpha+\beta+n\delta} \right) + y^{2} f(e_{18} \otimes e_{81}) \right\}$$

$$\cdot \left\{ 1 + (q - q^{-1}) \left( \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^{n} \right) \left( E'_{\beta+n\delta} \otimes F'_{\beta+n\delta} \right) + y^{2} f(e_{72} \otimes e_{72}) \right\}$$

$$\left\{ \text{imaginary root vectors contribution} \right\}$$

$$\cdot \left\{ 1 + (q - q^{-1}) \left( \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^{n+1} \right) \left( E'_{(\alpha-\beta)+n\delta} \otimes F'_{(\alpha-\beta)+n\delta} \right) + x^{2} f(e_{72} \otimes e_{72}) \right\}$$

$$\cdot \left\{ 1 + (q - q^{-1}) \left( \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^{n+1} \right) \left( E'_{(\alpha-\beta)+n\delta} \otimes F'_{(\alpha-\beta)+n\delta} \right) + x^{2} f(e_{36} \otimes e_{36}) \right\}$$

$$\cdot \left\{ 1 + (q - q^{-1}) \left( \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^{n+1} \right) \left( E'_{(\alpha-\beta)+n\delta} \otimes F'_{(\alpha-\beta)+n\delta} \right) + x^{2} f(e_{18} \otimes e_{18}) \right\}$$

$$\cdot \left\{ \sum_{i=1}^{8} (1 - \delta_{i4} - \delta_{i5})(e_{ii} \otimes e_{ii}) + q \left( e_{11} \otimes e_{22} \right) + e_{11} \otimes e_{33} + e_{22} \otimes e_{66} + e_{33} \otimes e_{77} + e_{66} \otimes e_{88} + \{\leftarrow\} \right) \right\}$$

$$+ q^{-1} \left( e_{11} \otimes e_{66} + e_{11} \otimes e_{77} + e_{22} \otimes e_{33} + e_{22} \otimes e_{88} + e_{33} \otimes e_{66} + e_{33} \otimes e_{77} + e_{66} \otimes e_{77} + \{\leftarrow\} \right)$$

$$+ q^{-2} \left( e_{11} \otimes e_{88} + e_{22} \otimes e_{77} + e_{33} \otimes e_{66} + \{\leftarrow\} \right)$$

(16)

where "\{\leftarrow\}" denotes the interchange of the quantities in the space $X \otimes Y$ and

$$f = [2]^2 q^{-1} (q - q^{-1})^{2} (y + q^{2} x) / ((y^{2} - x^{2}) (y - q^{-1} x))$$

$$E'_{\alpha+n\delta} = (-1)^{n} \left\{ -q^{-2n} e_{12} + [2]^{1/2} q^{-2n} e_{34} + [2]^{1/2} e_{46} + e_{38} \right\}$$

$$F'_{\alpha+n\delta} = (-1)^{n} \left\{ q^{2n} e_{21} + [2]^{1/2} q^{2n} e_{43} + [2]^{1/2} e_{64} + e_{87} \right\}$$

$$E'_{\alpha+\beta+n\delta} = (-1)^{n} \left\{ -q^{-2n} e_{14} + ([3]^{1/2} [2]^{1/2} q^{2n} e_{15} - q^{-1} e_{15} + q^{-2n} e_{37} + [2]^{1/2} q^{-1} e_{48} - ([3]^{1/2} [2]^{1/2} q^{-1} e_{69} \right\}$$

$$F'_{\alpha+\beta+n\delta} = (-1)^{n} \left\{ -q^{-2n} e_{14} + ([3]^{1/2} [2]^{1/2} q^{2n} e_{37} + q^{-2n} e_{37} + [2]^{1/2} q^{-1} e_{48} - ([3]^{1/2} [2]^{1/2} q^{-1} e_{69} \right\}$$

$$E'_{\beta+n\delta} = q^{n} e_{13} + [2]^{1/2} q^{n} e_{24} + ([3]^{1/2} [2]^{1/2} q^{2n} e_{25} +
\[ F'_{\beta + n} = q^{-n}e_{31} + [2]_{q}^{-1/2}q^{-n}e_{42} + ([3]_{q}/[2]_{q})^{1/2}q^{3n}e_{52} + \\
+ [2]_{q}^{-1/2}q^{3n}e_{74} + ([3]_{q}/[2]_{q})^{1/2}q^{-n}e_{75} + q^{3n}e_{86} \]

\[ E'_{(\beta - \beta) + n} = -q^{n+2}e_{31} - [2]_{q}^{-1/2}q^{n+2}e_{42} - ([3]_{q}/[2]_{q})^{1/2}q^{-3n-1}e_{52} \]

\[ -[2]_{q}^{-1/2}q^{-3n-3}e_{74} - ([3]_{q}/[2]_{q})^{1/2}q^{n+1}e_{75} - q^{-3n-2}e_{86} \]

\[ F'_{(\beta - \beta) + n} = -q^{-n-2}e_{13} - [2]_{q}^{-1/2}q^{-n-2}e_{24} - ([3]_{q}/[2]_{q})^{1/2}q^{3n+1}e_{25} \]

\[ -[2]_{q}^{-1/2}q^{3n+3}e_{47} - ([3]_{q}/[2]_{q})^{1/2}q^{-n-1}e_{57} - q^{3n+2}e_{68} \]

\[ E'_{(\beta - \beta) + n} = (1)^{n} \left\{ -[2]_{q}^{-1/2}q^{-2n+2}e_{14} + ([3]_{q}/[2]_{q})^{1/2}q^{-2n}e_{51} - q^{2}e_{62} \right. \]

\[ + q^{-2n-1}e_{73} + [2]_{q}^{-1/2}q^{-1}e_{84} - ([3]_{q}/[2]_{q})^{1/2}q^{2}e_{85} \right. \]

\[ F'_{(\beta - \beta) + n} = (1)^{n} \left\{ -[2]_{q}^{-1/2}q^{-2n}e_{14} + ([3]_{q}/[2]_{q})^{1/2}q^{2n}e_{15} - q^{-2}e_{26} \right. \]

\[ + q^{2n+1}e_{37} + [2]_{q}^{-1/2}q^{-1}e_{48} - ([3]_{q}/[2]_{q})^{1/2}q^{-2}e_{58} \right. \]

\[ \{ \text{imaginary root vectors contribution} \} = a'/a \sum_{i=1}^{8} \left( 1 + (b/b' - 1)\delta_{i4} + (c/c' - 1)\delta_{i5} \right) (e_{ii} \otimes e_{ii}) + \\
+ d'(e_{11} \otimes e_{22} + e_{11} \otimes e_{33}) + aad'(e_{11} \otimes e_{44}) + a'd'(e_{11} \otimes e_{65}) + a(e_{11} \otimes e_{65}) + \\
+ aad'(e_{11} \otimes e_{77}) + ac(e_{11} \otimes e_{88}) + 1/a(e_{22} \otimes e_{11}) + 1/b'(e_{22} \otimes e_{33}) + \\
+ d'/b'(e_{22} \otimes e_{44}) + d'(e_{22} \otimes e_{55}) + d'(e_{22} \otimes e_{66}) + aa'd'/b'(e_{22} \otimes e_{77}) + \\
+ aad'(e_{22} \otimes e_{88}) + 1/a(e_{33} \otimes e_{11}) + b(e_{33} \otimes e_{22}) + a'b(e_{33} \otimes e_{44}) + \\
+ ab(e_{33} \otimes e_{66}) + a'(e_{33} \otimes e_{77}) + a(e_{33} \otimes e_{88}) + 1/(a'')(e_{44} \otimes e_{11}) + \\
+ b/a(e_{44} \otimes e_{22}) + 1/(a'b')(e_{44} \otimes e_{33}) + a'b(e_{44} \otimes e_{66}) + a'/b'(e_{44} \otimes e_{77}) + \\
+ aad'(e_{44} \otimes e_{88}) + 1/(a'')(e_{55} \otimes e_{11}) + e_{55} \otimes e_{22}) + d'(e_{55} \otimes e_{77} + e_{55} \otimes e_{88}) + \\
+ 1/a'(e_{66} \otimes e_{11}) + 1/a(e_{66} \otimes e_{22}) + 1/(a'b')(e_{66} \otimes e_{33}) + 1/(a'b')(e_{66} \otimes e_{44}) + \\
+ 1/b'(e_{66} \otimes e_{77}) + d'(e_{66} \otimes e_{88}) + 1/(a'')(e_{77} \otimes e_{11}) + b/(aa'd')(e_{77} \otimes e_{22}) + \\
+ 1/a(e_{77} \otimes e_{11}) + b/a(e_{77} \otimes e_{44}) + 1/(a')(e_{77} \otimes e_{55}) + b(e_{77} \otimes e_{66}) + \\
+ d'(e_{77} \otimes e_{88}) + 1/(a'')(e_{88} \otimes e_{11}) + 1/(a'')(e_{88} \otimes e_{22}) + 1/a'(e_{88} \otimes e_{33}) + \\
+ 1/(a'')(e_{88} \otimes e_{44}) + 1/(a')(e_{88} \otimes e_{55}) + 1/a(e_{88} \otimes e_{66} + e_{88} \otimes e_{77})\] (17)
in which we have defined \( a = (y - q^{-2}x)/(y - x) \),  
\( a' = (y - q^{-2}x)/(y - x) \),  
\( b = (y - q^2x)/(y - q^2x) \),  
\( b' = (y - q^{4}x)/(y - q^{4}x) \),  
\( c = (y - q^{6}x)/(y - q^{6}x) \),  
\( c' = (y - q^{-4}x)/(y - q^{-4}x) \). We remark that (16) is an extremly explicit formula: the sums in (16) can be easily worked out.

For compactness, we leave them as the present form.

4. Concluding Remarks: In this letter, we have obtained the spectral-dependent universal \( R \)-matrix for \( U_q(A_1) \) and \( U_q(A_2) \). As their applications, we reproduce from them some well-known results and obtain an extremly explicit and compact formula for spectral-dependent \( R \)-matrix in the adjoint representation of \( U_q(A_2) \). We have only announced the results and the details will be published elsewhere [8].

We mention here some further applications for our formula. Firstly, we wonder if there is some kind of "universal" lattice models which have our spectral-dependent \( R \)-matrix as their Boltzmann weights. Secondly, one may consider the possibility of finding and computing eigenvalues of Casimir operators constructed from these spectral-dependent \( R \)-matrix which are expected to play some role in one dimensional open spin chains [9]. Thirdly, we believe our formula will be useful in quantizing the conformal affine Liouville (and Toda) theories [10] and in the recently-developed \( q \)-deformed WZNW CFT’s [11]. All these issues are now under consideration.

Acknowledgements: Y.Z.Z. would like to thank Anthony John Bracken for continous interest, suggestions and discussions, to thank V.N.Tolstoy for communication of his papers on quantum groups and to thank R.Cuerno for email correspondences. The financial support from Australian Research Council is gratefully acknowledged.

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