A New Gauge for Computing Effective Potentials in Spontaneously Broken Gauge Theories

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Abstract

A new class of renormalizable gauges is introduced that is particularly well suited to compute effective potentials in spontaneously broken gauge theories. It allows one to keep a free gauge parameter when computing the effective potential from vacuum graphs or tadpoles without encountering mixed propagators of would-be-Goldstone bosons and longitudinal modes of the gauge field. As an illustrative example several quantities are computed within the Abelian Higgs model. The zero temperature effective potential in the new gauge is compared with that in $R_\xi$ gauge at the one-loop level and found to be not only easier to compute but also to have a more convenient analytical structure.
1 Introduction and Summary

The effective potential (EP) of a relativistic quantum field theory (QFT) [1] is a useful tool for investigating several questions of physical interest such as the vacuum structure of the theory, inflationary cosmology and finite temperature phase transitions (see e.g. [2] and references therein). Recently for instance there has been renewed interest in the details of the electroweak phase transition in the early universe since its nature is crucial in deciding whether the baryon asymmetry of the universe was created at that time. The availability of a good approximation scheme for the EP of the Higgs field is considered to be crucial in that context by many authors (see e.g. [3] and references therein).

Although the EP of spontaneously broken gauge field theories (SBGFT) is frequently employed, there is no convenient gauge that at the same time

(i) allows the EP to be computed from graphs with zero or one external line(s) (from here on to be called vacuum graphs and tadpoles, respectively),

(ii) avoids awkward-to-use mixed propagators of would-be-Goldstone bosons and the longitudinal modes of the gauge field and

(iii) keeps at least one free gauge parameter.

Usually Landau gauge (i.e. \( R_\xi \) gauge with \( \xi = 0 \)) is used because then points (i) and (ii) are fulfilled. However in this case there is no free gauge parameter left and physical results can not be checked for their gauge independence.

In this paper a class of non-linear and renormalizable gauges very similar to the \( R_\xi \) gauges is introduced that fulfills all of the above requirements. The price one has to pay is the existence of another gauge parameter \( \sigma \) which has to be set to unity to avoid mixed propagators. This causes no problems as long as no use is made of the renormalization group which would make \( \sigma \) a running quantity and again give rise to the presence of mixed propagators.

The paper is structured as follows: In section 2 why and how vacuum graphs (or tadpoles) can be used to compute the EP of a QFT is briefly reviewed. In section 3 the new class of gauges is introduced for a general gauge field theory. It is argued that the new gauge is an admissible gauge in the sense of [4]. In section 4 the Abelian Higgs model is used as an illustrative example. Its Feynman rules in the new class of gauges are given, it is renormalized at the two-loop level and the physical Higgs and gauge boson masses are computed at the one-loop level and their gauge independence checked. The one-loop EP is compared to that in \( R_\xi \) gauge and found not only to be easier to compute but also to have a more desirable analytical structure.
2 The EP and 1PI n-Point Functions

In this section the well-known connection between the EP of a QFT and vacuum graphs [5, 6], tadpoles [7, 8] and higher order functions in a shifted theory is presented in a very compact way for further reference.

The EP of a QFT is the generator for the one particle irreducible (1PI) n-point functions at zero external momenta. If we decide to expand the EP \( V(\phi) \) about some point \( \omega \), we get

\[
V(\phi) = -\sum_{n=0}^{\infty} \frac{1}{n!} \Gamma_n(\phi, p_i = 0)(\phi - \omega)^n, \tag{1}
\]

where \( \Gamma_n(\omega, p_i) \) is the 1PI n-point Greens function in the modified theory where the field \( \phi \) has been shifted by \( \omega \). Taking the derivative of this equation with respect to \( \omega \) and observing that the EP is independent of the point we expand it about, we arrive at the recursion relation

\[
\Gamma_{n+1}(\omega, p_i = 0) = \frac{\partial}{\partial \omega} \Gamma_n(\omega, p_i = 0). \tag{2}
\]

Setting \( \phi = \omega \) in (1) gives

\[
V(\omega) = -\Gamma_0(\omega, p_i = 0) \tag{3}
\]

and therefore the n-th derivative of the EP is given by

\[
V^{(n)}(\omega) = -\Gamma_n(\omega, p_i = 0). \tag{4}
\]

Thus instead of summing over \( n \) as in (1) we can use n-point functions in a shifted theory and integrate \( n \) times. Of course, if higher order functions than tadpoles are used, the choice of integration constants is a non-trivial problem if no additional information is available. Throughout this paper vacuum graphs (i.e. \( n = 0 \)) are used so that no integration over \( \omega \) is needed.

Note that for the above derivation to hold it is crucial that the only place where \( \omega \) enters the modified Lagrangian is through the shifted scalar field as already noted in [7]. Otherwise the derivative of the left hand side of (1) with respect to \( \omega \) does not vanish.

3 The New Gauge

Within the scalar sector of a gauge field theory under consideration let \( \Phi \) be an irreducible representation that is expected to get a vacuum expectation value (vev). Let us further assume that \( \Phi \) has been put into a real representation.
First recall the class of $R_\xi$ gauges [9]: The gauge fixing term here is

\[ L_{gf} = -\frac{1}{2\xi} \sum_a \left( \partial_\mu A^\mu_a + i\xi g_a v^T T_a \Phi' \right)^2, \]

where $\Phi = v + \Phi'$, $\langle \Phi' \rangle = 0$ and $g_a$ are the coupling constants of the gauge group with corresponding (imaginary and antisymmetric) generators $T_a$. With this gauge fixing it is not possible to satisfy all three requirements (i)-(iii) simultaneously: If the Higgs field is shifted by some $\omega$ (additionally to $v$), then point (ii) is violated. If we modify $v^T$ in the gauge fixing term so as to avoid that, then the complete Lagrangian does no longer depend only on the sum of shifted field and shift, but on those two quantities separately which invalidates the derivation in section 2 and point (i) is no longer fulfilled. If we set $\xi = 0$, it turns out that (i) and (ii) are satisfied (most easily seen by the fact that the gauge to be introduced subsequently has effectively the same Feynman rules for this case), but (iii) is violated, i.e. we have no check of gauge independence of physical quantities anymore.

Now choose a unit vector $\hat{v}$ in $\Phi$ space and consider the gauge fixing term

\[ L_{gf} = -\frac{1}{2\xi} \sum_a \left( \partial_\mu A^\mu_a + i\xi g_a \Phi^T \hat{v}^T T_a \Phi \right)^2, \]

which necessitates the further addition of the ghost term

\[ L_{gh} = \sum_{a,b} \left[ \delta_{ab} \partial_\mu \eta_a \partial^\nu \eta_b + g_c C_{abc} (\partial_\mu \eta_a) A^\mu_c \eta_b - \xi g_a g_b \Phi^T (\hat{v}^T T_b T_a - T_b \hat{v}^T T_a) \Phi \eta_a \eta_b \right], \]

where the $C_{abc}$ are defined by $[T_a, T_b] = i C_{abc} T_c$. If we now introduce some constant shift $\varphi$ by

\[ \Phi \equiv \varphi \hat{v} + \Phi', \]

then $L_{gf}$ and $L_{gh}$ still depend only on the sum of shift and remaining quantum field $\Phi'$ and the derivation in section 1 is valid and thus point (i) is fulfilled. On the other hand, it is easy to check that for any shift $\varphi$ the mixing terms between would-be-Goldstone modes and longitudinal gauge modes vanish and point (ii) is also fulfilled.

Point (iii) is clearly satisfied, too. It turns out though (see section 4) that this gauge is really a two-parameter gauge and the gauge fixing term has to be multiplied by another gauge parameter $\sigma$ (which gets renormalized) and point (ii) seems to be threatened. However, unless we want to make use of the renormalization group which would make $\sigma$ a running quantity, we can always set the renormalized $\sigma$ equal to unity (in BPHZ renormalization) and point (ii) remains valid.
It still has to be checked that $\dot{v}$ is indeed the direction of symmetry breaking that the theory equipped with (6) and (7) picks as was assumed by choosing to shift according to (8): Through $\mathcal{L}_{gf}$ the tree-level potential for the scalar fields receives the additional contribution $\xi \sum_a (i g_a \Phi^T \dot{v} \Phi^T T_a \Phi)^2 / 2$. This term is non-negative (since the $T_a$ are imaginary) and vanishes for $\Phi \propto \dot{v}$ (since the $T_a$ are antisymmetric) and for $\Phi \perp \dot{v}$, i.e. $\Phi^T \dot{v} = 0$. Radiative corrections do not change these results: not explicitly because of BRST symmetry and not spontaneously since the tree-level result is expected to dominate for small couplings (except possibly for too small $\xi$, e.g. Landau gauge!). It has to be checked though that radiative corrections do not lead to a lower vacuum energy density in directions $\Phi \perp \dot{v}$ than in the $\dot{v}$ direction. This is done at the one-loop level for the Abelian Higgs model in the next section.

Note that the gauge (6) is admissible in the sense of [4]: The aforementioned positivity structure of the additional scalar potential part induced by the gauge fixing term together with BRST symmetry ensures that the gauge fixing operator $\partial_\mu A_\mu + i \xi g_a \Phi^T \dot{v} \Phi^T T_a \Phi$ has a vanishing expectation value and thus BRST symmetry is not spontaneously broken.

The class of gauges introduced by (6) is non-linear and renormalizable. Because of its similarity to the class of $\mathcal{R}_\xi$ gauges, it is called $\mathcal{R}_\xi$ from here on.

4 Application: The Abelian Higgs Model

In this section the use of $\mathcal{R}_\xi$ gauge is illustrated within the Abelian Higgs model. The model is given by the Lagrange density

$$\mathcal{L}_{AH} = \frac{1}{2} (D_\mu \Phi)^T (D^\mu \Phi) - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} m^2 \Phi^T \Phi - \frac{\lambda}{4} (\Phi^T \Phi)^2$$

(9)

with $F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu \Phi \equiv (\partial_\mu + i g T A_\mu) \Phi$, $T \equiv \sigma_2 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\Phi^T \equiv (\phi_1, \phi_2)$. Here $\phi_1$ and $\phi_2$ are real scalar fields and with $\lambda > 0$ and $m^2 < 0$ the U(1) gauge symmetry is spontaneously broken. If we choose $\dot{v}^T = (1, 0)$, the gauge fixing and ghost terms become

$$\mathcal{L}_{gf} = -\frac{1}{2 \xi} (\partial_\mu A_\mu + \xi g \phi_1 \phi_2),$$

(10)

$$\mathcal{L}_{gh} = \partial_\mu \bar{\eta} \partial^\mu \eta - \xi g^2 \left( \phi_1^2 - \phi_2^2 \right) \bar{\eta}.$$

(11)

Now we are ready to break the U(1) symmetry by shifting $(\phi_1, \phi_2) = (\varphi + \phi'_1, \phi'_2)$. Hereby the effective Lagrange density $\mathcal{L}_{\text{eff}} = \mathcal{L}_{AH} + \mathcal{L}_{gf} + \mathcal{L}_{gh}$, given by (9), (10) and (11), becomes the sum of the following four pieces, ordered by their dimension:

$$\mathcal{L}_0 = -\frac{1}{4} \lambda \varphi^4 - \frac{1}{2} m^2 \varphi^2,$$
\[ L_1 = - (\lambda \varphi^2 + m^2 \varphi) \phi_1^2, \]
\[ L_2 = \frac{1}{2} (\partial_\mu \phi_1^2)^2 - \frac{1}{2} (3 \lambda \varphi^2 + m^2) \phi_1^2 + \frac{1}{2} (\partial_\mu \phi_2^2)^2 - \frac{1}{2} ((\lambda + \xi \varphi^2 + m^2) \phi_2^2) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \xi (\partial_\mu A^\mu)^2 + \frac{1}{2} g^2 \varphi \phi_1 A^\mu + \partial_\mu \bar{\eta} \partial^\mu \eta - \xi \varphi^2 \bar{\eta} \eta, \]
\[ L_{3,4} = - \frac{1}{4} \lambda \phi_2^4 - \lambda \varphi \phi_1^3 - \frac{1}{2} (\lambda + \xi \varphi^2) \phi_1^2 \phi_2^2 - (\lambda + \xi \varphi^2) \phi_1^2 \phi_2^2 - \frac{1}{4} \phi_2^4 + g^2 \varphi \phi_1 A^\mu + 2 g \phi_1 A^\mu \partial^\mu \phi_1 + \frac{1}{2} g^2 \phi_1^2 A^\mu + \frac{1}{2} g^2 \phi_2^2 A^\mu A^\mu - 2 \xi \varphi^2 \phi_1 \bar{\eta} \eta - \xi \varphi^2 \bar{\eta} \eta + \xi \varphi^2 \bar{\eta} \eta. \]

The Feynman rules can immediately be read off and are given in table 1. For comparison also the Feynman rules in generalized \( R_\xi \) gauge are given ('generalized' since \( \varphi \) is not necessarily the vev of the Higgs field). Note that in Landau gauge (\( \xi = 0 \)) the Feynman rules of both classes of gauges become identical (the remaining difference in the Higgs-gauge-Goldstone vertex is immediately seen to be irrelevant because now the gauge propagator is transverse), which explains naturally why in Landau gauge among the \( R_\xi \) gauges vacuum graphs can be used to compute the EP. It can be shown that the Feynman rules effectively coincide also for unitary gauge (\( \xi \rightarrow \infty \)) in both classes of gauges.

Up to an unphysical constant term the effective Lagrange density can be renormalized as

\[ L_{\text{eff}} = \frac{1}{2} (\partial_\mu \Phi_{1B} + g_B A_{B\mu} \phi_{2B})^2 + \frac{1}{2} (\partial_\mu \Phi_{2B} - g_B A_{B\mu} \phi_{1B})^2 - \frac{1}{4} F_{B\mu\nu} F^{B\mu\nu} - \frac{1}{2} m_B^2 (\Phi_{1B}^2 + \Phi_{2B}^2) - \frac{1}{4} \lambda_B (\Phi_{1B}^2 + \Phi_{2B}^2)^2 - \frac{\sigma_B}{2 \xi_B} (\partial_\mu A_{B\mu} + \xi_B g_B \Phi_{1B} \Phi_{2B})^2 + \partial_\mu \bar{\eta}_B \partial^\mu \eta_B - \xi_B g_B (\Phi_{1B}^2 - \Phi_{2B}^2) \bar{\eta}_B \eta_B \]

with

\[ \Phi_{1B} = Z_1^B \Phi_{1R}, \quad \Phi_{2B} = Z_2^B \Phi_{2R}, \quad A_{B\mu} = Z_A^B A_{\mu R}, \quad \bar{\eta}_B \eta_B = Z_\eta \bar{\eta}_R \eta_R, \]
\[ m_B = Z_m^B m_R, \quad \lambda_B = Z_\lambda \lambda_R, \quad g_B = Z_g^B g_R, \quad \xi_B = Z_\xi \xi_R, \quad \sigma_B = Z_\sigma \sigma_R, \]

where renormalization constants \( Z_x \) have been introduced and “\( B \)” and “\( R \)” denote bare and renormalized quantities, respectively. The Ward identity requires \( Z_g = Z_A = 1 \). Using dimensional regularization [10] and the \( \overline{\text{MS}} \) scheme [11] (as for all calculations in this paper) I have computed the \( Z_x \) at the two-loop level for \( \sigma_R = 1 \). The result is given in the appendix.

As a check on the consistency of the gauge (6) and as an illustration for its use, the one-loop radiative corrections to the Higgs and physical gauge boson masses are computed in the next two paragraphs.
With the tree-level Higgs field vev \( v_0^2 = -m^2/\lambda \) the tree-level Higgs mass obtains as \( m^2_{H_1} = 3\lambda v_0^2 + m^2 = -2m^2 \). Its one-loop correction is given by [12] (let \( k_\nu \) be the momentum flowing through the graphs)

\[
m^2_{H_1} = i \left[ \begin{array}{c} \text{1-loop} \\ \text{1PI} \end{array} \right] + \left[ \begin{array}{c} \text{1-loop} \\ \text{1PI} \end{array} \right],
\]

where in \( \overline{\text{ET}} \) gauge

\[
\begin{align*}
\text{1-loop} & \quad = \quad \quad + \quad + \quad + \quad + \quad + \quad + \\
\text{1-loop} & \quad = \quad \quad + \quad + \quad + \quad + \quad + \quad ,
\end{align*}
\]

and where the external legs have to be truncated as for all other graphs in this paper. Using the Feynman rules given in tables 1 and 2 one gets after evaluation of the momentum space integrals (let \( \mu \) be the renormalization scale)

\[
m^2_{H_1} = -m^2 \left[ \left( 6\sqrt{3}\pi - 28 \right) \lambda + 10g^2 - 6\frac{g^4}{\lambda} \right] + \left( 8\lambda - 6g^2 \right) \ln \frac{-2m^2}{\mu^2} + \left( 2\lambda - 6g^2 \right) \ln \frac{g^2}{2\lambda} + 4 \left( \lambda - 2g^2 + 3\frac{g^4}{\lambda} \right) \sqrt{\frac{2g^2}{\lambda} - 1} \arctan \frac{1}{\sqrt{\frac{2g^2}{\lambda}}} ,
\]

which as physical quantity is gauge independent as expected [12, 13, 14].

The determination of the physical gauge boson mass proceeds in close analogy: Its tree-level value is given by \( m^2_{Z^\prime} = g^2 v_0^2 = -m^2 g^2/\lambda \). If we write

\[
- i \left[ \begin{array}{c} \text{1-loop} \\ \text{1PI} \end{array} \right] \left[ \begin{array}{c} \mu \\ \nu \end{array} \right] + \left[ \begin{array}{c} \text{1-loop} \\ \text{1PI} \end{array} \right] \left[ \begin{array}{c} \mu \\ \nu \end{array} \right] = \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) A + \frac{k_\mu k_\nu}{k^2} B ,
\]

where in \( \overline{\text{ET}} \) gauge

\[
\begin{align*}
\text{1-loop} & \quad = \quad \quad + \quad + \quad + \quad + \quad + \quad + \\
\text{1-loop} & \quad = \quad \quad + \quad + \quad + \quad + \quad + \quad ,
\end{align*}
\]
and truncate the external legs, then the one-loop correction to the physical gauge 
bozon mass is easily seen to be given by

\[ m^2_{A_1} = A \mid_{k^2=m^2_{A_0}, \phi=v_0} . \]  

(25)

Using again the rules in tables 1 and 2, one gets

\[ m^2_{A_1} = -m^2 \left[ \left( -\frac{4}{3} \lambda + 10g^2 - \frac{62 g^4}{9 \lambda} + \frac{g^6}{\lambda^2} \right) + \left( 6g^2 - \frac{10 g^4}{3 \lambda} + \frac{3g^6}{\lambda^2} \right) \ln \frac{g^2 m^2}{\lambda \mu^2} \right. \]

\[ - \left( \frac{4 \lambda^2}{3 g^2} - 4 \lambda \right) \ln \frac{g^2}{2 \lambda} + \left( \frac{8 \lambda^2}{3 g^2} - \frac{16}{3} \lambda + 8 g^2 \right) \sqrt{\frac{2g^2}{\lambda} - 1} \arctan \sqrt{\frac{2g^2}{\lambda} - 1} \right] , \]

(26)

which also is gauge independent.

Now we proceed to compute the EP at the one-loop level in both \( R_\xi \) and \( \bar{R}_\xi \) 
gauges. The tree-level potential is just \( V_0 = \lambda \phi^4/4 + m^2 \phi^2/2 \). In \( R_\xi \) gauge I 
have computed its one-loop correction by summing up graphs with all numbers 
of external lines (i.e. using (1) at \( \omega = 0 \)) as well as from vacuum graphs as in 
[6, 14, 15], but as stated earlier the price one pays in the latter case is the use of 
mixed propagators between longitudinal gauge boson modes and would-be-
Goldstone bosons. The result is of course the same:

\[ V_{1,R_\xi}(\phi) = \frac{1}{4(4\pi)^2} \left[ m_H^4 \left( \ln \frac{m_H^2}{\mu^2} - \frac{3}{2} \right) + m_A^4 \left( 3 \ln \frac{m_A^2}{\mu^2} - \frac{5}{2} \right) \right. \]

\[ + m_H^4 \left( \ln \frac{m_H^2}{\mu^2} - \frac{3}{2} \right) + m_A^4 \left( \ln \frac{m_A^2}{\mu^2} - \frac{3}{2} \right) - 2m_\xi^4 \left( \ln \frac{m_\xi^2}{\mu^2} - \frac{3}{2} \right) \]  

(27)

with (the tree-level values for the squares of) the Higgs mass \( m_H^2 = 3 \lambda \phi^2 + m^2 \) 
and the physical gauge boson mass \( m_A^2 = g^2 \phi^2 \) and with \( m_{a,h}^2 = (\lambda \phi^2 + m^2)/2 + \xi g^2 \phi v_0 \) 
\( \pm \sqrt{(\lambda \phi^2 + m^2)((\lambda \phi^2 + m^2) - 4 \xi g^2 \phi (\phi - v_0))}/2 \) and \( m_\xi^2 = \xi g^2 \phi v_0 \).

In \( \bar{R}_\xi \) gauge, we simply have

\[ V_{1,\bar{R}_\xi}(\phi) = i \left[ \begin{array}{c} \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \end{array} \right] \]

\[ = \frac{1}{4(4\pi)^2} \left[ m_H^4 \left( \ln \frac{m_H^2}{\mu^2} - \frac{3}{2} \right) + m_A^4 \left( 3 \ln \frac{m_A^2}{\mu^2} - \frac{5}{2} \right) \right. \]

\[ + m_H^4 \left( \ln \frac{m_H^2}{\mu^2} - \frac{3}{2} \right) + m_A^4 \left( \ln \frac{m_A^2}{\mu^2} - \frac{3}{2} \right) \]  

(28)

with the would-be-Goldstone mass \( m_G^2 = (\lambda + \xi g^2) \phi^2 + m^2 \), the ghost (and 
longitudinal gauge boson) mass \( m_{g,h}^2 = \xi g^2 \phi^2 \) and \( m_H^2 \) and \( m_A^2 \) as above. Clearly, 
in \( \bar{R}_\xi \) gauge \( V_1 \) is not only easier to compute but also has a much more convenient
analytical structure due to the simple expressions for the appearing masses which contain no awkward square roots anymore.

Since the EP is not itself a physical quantity it can be and indeed is gauge dependent. However its value at points where $V' = 0$ is a physical energy density and should therefore be gauge independent [13, 14, 16]. It is easy to see that the one-loop correction to the energy density at the symmetry breaking solution of $V' = 0$ is given by $V_1(v_0)$. By substituting $v_0$ into (27) and (28) one finds $V_{1,\pi_i}(v_0) = V_{1,R_i}(v_0) = \text{gauge independent as expected although the one-loop correction to the Higgs field vev, given by } v_1 = V'_1(v_0)/(2m^2), \text{ turns out to be gauge dependent as is expected for the location of the vev [13].}$

Finally we check at the one-loop level that radiative corrections do not cause a lower minimum in the $\phi_2$ direction than in the $\phi_1$ direction. Shifting $(\phi_1, \phi_2) = (\phi_1, \varphi + \phi_2)$, I have computed the one-loop correction to the EP in $\overline{\text{MS}}$ gauge from vacuum graphs, where now the propagators of the $\phi_1$ field and the longitudinal gauge mode mix. The result is

$$V_{1,\overline{\text{MS}}} \kern -9pt (-) (\varphi) = \frac{1}{4(4\pi)^2} \left[ m_H^4 \left( \ln \frac{m_H^2}{\mu^2} - \frac{3}{2} \right) + m_A^4 \left( 3 \ln \frac{m_A^2}{\mu^2} - \frac{5}{2} \right) ight]$$

$$+ m^4_a \left( \ln \frac{m_a^2}{\mu^2} - \frac{3}{2} \right) + m^4_\beta \left( \ln \frac{m_\beta^2}{\mu^2} - \frac{3}{2} \right) - 2m_\gamma \left( \ln \frac{m_\gamma^2}{\mu^2} - \frac{3}{2} \right),$$

with $m_{a, \beta}^2 = \left( \lambda \varphi^2 + m^2 \right)/2 - \xi g^2 \varphi^2 \pm \sqrt{\left( \lambda \varphi^2 + m^2 \right)^2 - \xi g^2 \varphi^2 \left( \lambda - 8(\xi g^2) \varphi^2 + m^2 \right)}/2$, $m_\gamma^2 = -\xi g^2 \varphi^2$ and $m_H^2$ and $m_A^2$ as above. Substituting in $v_0$ gives the same result for $V_1(v_0)$ as before.

It is expected that this property is not restricted to the one-loop level or to the Abelian Higgs model: i.e. in directions in functional space where there is no tree-level contribution from the gauge fixing term to the scalar field potential (that is for $\Phi \propto \hat{\varphi}$ or $\Phi^T \hat{\varphi} = 0$), the shape of the EP differs due to different renormalizations of the scalar field components but its value at stationary points is unique.

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Appendix

Using dimensional regularization and the MS scheme the $Z_x$ up to two loops are for $\sigma_R = 1$ (all quantities are the renormalized ones):

$$Z_1 = \mu^{-\epsilon} \left( 1 + \frac{(6 + 2\xi)g^2}{(4\pi)^2\epsilon} - \frac{-4\lambda^2 - 4\lambda\xi g^2 + (-\frac{10}{3} + 2\xi - 3\xi^2)g^4}{(4\pi)^4\epsilon} + \frac{8\lambda\xi g^2 + (20 + 12\xi^2)g^4}{(4\pi)^4\epsilon^2} \right)$$

$$Z_2 = \mu^{-\epsilon} \left( 1 + \frac{(6 - 6\xi)g^2}{(4\pi)^2\epsilon} - \frac{-4\lambda^2 - 4\lambda\xi g^2 + (-\frac{10}{3} - 2\xi - 11\xi^2)g^4}{(4\pi)^4\epsilon} + \frac{-8\lambda\xi g^2 + (20 - 24\xi + 12\xi^2)g^4}{(4\pi)^4\epsilon^2} \right)$$

$$Z_m = 1 + \frac{8\lambda - 6g^2}{(4\pi)^2\epsilon} + \frac{-20\lambda^2 + 32\lambda g^2 + \frac{43}{3}g^4}{(4\pi)^4\epsilon} + \frac{112\lambda^2 - 96\lambda g^2 + 40g^4}{(4\pi)^4\epsilon^2}$$

$$Z_c = 1 - \frac{2\xi^2g^4}{(4\pi)^4\epsilon}$$

$$Z_\lambda = \mu^\epsilon \left( 1 + \frac{20\lambda - 12g^2 + 6g^4/\lambda}{(4\pi)^2\epsilon} + \frac{-120\lambda^2 + 56\lambda g^2 + \frac{158}{3}g^4 - \frac{104}{3}g^6/\lambda}{(4\pi)^4\epsilon} + \frac{400\lambda^2 - 360\lambda g^2 + 188g^4 - 32g^6/\lambda}{(4\pi)^4\epsilon^2} \right)$$

$$Z_A = \mu^{-\epsilon} \left( 1 - \frac{\frac{2}{3}g^2}{(4\pi)^2\epsilon} - \frac{4g^4}{(4\pi)^4\epsilon} \right)$$

$$Z_g = \mu^\epsilon \left( 1 + \frac{\frac{2}{3}g^2}{(4\pi)^2\epsilon} + \frac{4g^4}{(4\pi)^4\epsilon} + \frac{\frac{4}{3}g^4}{(4\pi)^4\epsilon^2} \right)$$

$$Z_\zeta = 1 + \frac{4\lambda + (-\frac{20}{3} + 4\xi)g^2}{(4\pi)^2\epsilon} + \frac{12\lambda^2 + 8\lambda g^2 + (-\frac{25}{3} + 8\xi - 12\xi^2)g^4}{(4\pi)^4\epsilon} + \frac{-48\lambda^2 - (\frac{8}{3} + 32\xi)\lambda g^2 + (48 - \frac{80}{3}\xi - 20\xi^2)g^4}{(4\pi)^4\epsilon^2}$$

$$Z_\sigma = 1 + \frac{4\lambda + (-6 + 6\xi)g^2}{(4\pi)^2\epsilon} + \frac{12\lambda^2 + 8\lambda g^2 + (-\frac{13}{3} + 4\xi - 12\xi^2)g^4}{(4\pi)^4\epsilon} + \frac{-48\lambda^2 - 16\lambda\xi g^2 + (44 - 48\xi)g^4}{(4\pi)^4\epsilon^2}$$

Note that $Z_gZ_A = 1$ up to terms of higher than two-loop order as required by the Ward identity.

The one- and two-loop counterterms can be reconstructed from the $Z_x$ above. For illustrative purposes and because some of them are used in the text, the one-loop counterterms are given in table 2.
References


<table>
<thead>
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<th>constant:</th>
<th>( \overline{R}_\xi )</th>
<th>( R_\xi )</th>
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</thead>
<tbody>
<tr>
<td>tadpole:</td>
<td>(-i\left(\frac{1}{4}\lambda\varphi^4 + \frac{1}{2}m^2\varphi^2\right))</td>
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<td>propagators:</td>
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<tr>
<td>Higgs:</td>
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<td>(\frac{i}{k^2 - m_G^2})</td>
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<td>Goldstone:</td>
<td>(\frac{i}{k^2 - m^2_{th}})</td>
<td>same</td>
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Table 1: Feynman rules for \( \overline{R}_\xi \) gauge with \( m_H^2 = 3\lambda\varphi^2 + m^2 \), \( m_G^2 = (\lambda + \xi g^2)\varphi^2 + m^2 \), \( m_\lambda^2 = g^2\varphi^2 \), \( m_{gh}^2 = \xi g^2\varphi^2 \). \( k_\mu \) is the momentum flowing through the propagators. For comparison also the rules for generalized \( R_\xi \) gauge, i.e. \( \mathcal{L}_{gf} = -1/(2\xi)(\partial_\mu A^\mu + \xi g\varphi\phi_2')^2 \) and \( \mathcal{L}_{gh} = \partial_\mu \eta \partial^\mu \eta - \xi g^2\varphi\phi_2'\eta \) are given.
Table 2: One-loop counterterms in $T_\xi$ gauge. $k_\mu$ is the momentum flowing through the two-point functions.