Holomorphic Yang-Mills Theory
on Compact Kähler Manifolds

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Abstract

We propose $N = 2$ holomorphic Yang-Mills theory on compact Kähler manifolds and show that there exists a simple mapping from the $N = 2$ topological Yang-Mills theory. It follows that intersection parings on the moduli space of Einstein-Hermitian connections can be determined by examining the small coupling behavior of the $N = 2$ holomorphic Yang-Mills theory. This paper is a higher dimensional generalization of the Witten’s work on physical Yang-Mills theory in two dimensions.

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1. Introduction

Few years ago, Witten introduced the topological Yang-Mills (TYM) theory [1] to give a quantum field theoretical interpretation of the Donaldson polynomial invariants of smooth four-manifolds [2][3]. The construction of the Lagrangian and the various mathematical structures of the theory are understood well enough [4][5][6][7]. However, the TYM theory has not provided any new insight to the explicit computations of the invariants.

Recently, we proposed a $N = 2$ supersymmetric TYM ($N = 2$ TYM) theory on compact Kähler surfaces [8]. This theory can be easily generalized to arbitrary dimensional compact Kähler manifolds, which is a field theoretical interpretation of intersection parings on the moduli space of stable bundles. In this paper, we propose $N = 2$ holomorphic Yang-Mills (HYM) theory on compact Kähler manifolds. Our main result is a proof of the equivalence of this theory to the $N = 2$ TYM theory. The $N = 2$ HYM theory may enhance the computability of the invariants.

This paper is a generalization of the Witten’s work in two dimensions [9]. He showed that the TYM theory is equivalent to physical Yang-Mills (YM) theory in two dimensions. Since physical YM theory can be exactly solved in two dimensions [10][11][12], he was able to obtain general expressions for the intersection parings on the moduli space of flat connections\(^1\).

Classically, the $N = 2$ HYM theory is equivalent to physical YM theory after restricting the space $\mathcal{A}$ of all connections to its subspace $\mathcal{A}^{1,1}$ having curvature only of type (1, 1). In sect. 2, we discuss few aspects on the physical YM theory restricted to $\mathcal{A}^{1,1}$ (which will be called HYM theory). We compare the HYM theory with physical YM theory in two dimensions. In sect. 3, we formulate the $N = 2$ HYM theory. After a brief review on the $N = 2$ TYM theory in sect. 3.1, we show that there exists a simple mapping to the $N = 2$ HYM theory. This ensures that the $N = 2$ HYM theory is well-defined as a quantum field theory. We also show that the partition function of the $N = 2$ HYM theory can be expressed as the sum of the contributions of the critical points which can be determined by some differential-topological methods. It follows that the intersection parings on the moduli space of stable bundles can be determined by solving the $N = 2$ HYM theory. Finally, we give some remarks in sect. 4.

The basic idea of this paper was announced previously in the last section of [8]. While this work is in progress, we received a lecture notes by Thompson [14] who proposed a similar model but in a different language for compact Kähler surfaces.

\(^1\) Similar result for $SO(3)$ case was obtained independently in [13].
2. Holomorphic Yang-Mills Theory

Let \((M, \omega)\) be a complex \(n\)-dimensional compact Kähler manifold\(^2\) with Kähler form \(\omega\). Let \(E\) be a complex vector bundle over \(M\) with a reduction of structure group to \(SU(r)\). Let \(\mathfrak{su}(r)\) be the Lie algebra of \(SU(r)\). We write \(\mathfrak{g}_E = E \times_{Ad} \mathfrak{su}(r)\) for the Lie algebra bundle associated with \(E\) by adjoint representation. Let \(\mathcal{A}\) be the space of all connections on \(E\) and \(\mathcal{G}\) be the group of gauge transformations. We introduce a positive definite quadratic form \((a, b) = -\text{Tr} ab\) on \(\mathfrak{su}(r)\), where \(\text{Tr}\) denotes the trace in the \(r\) dimensional representation. The action functional of physical YM theory is given by

\[
S(A) = -\frac{1}{8\pi^2 \varepsilon} \int_M \text{Tr} F_A \wedge * F_A,
\]

where \(\varepsilon\) is a non-negative constant. An YM connection is a critical point \(d_A^* F_A = 0\) of the YM action. If we use the Bianchi identity \(d_A F_A = 0\) and the Kähler identity,

\[
\bar{\partial}_A^* = i[\partial_A, \Lambda], \quad \partial_A^* = -i[\bar{\partial}_A, \Lambda],
\]

the YM equation can be written as

\[
-\text{i}n\bar{\partial}_A f + \bar{\partial}_A^* F_A^{0,2} = 0,
\]

\[
\text{i}n\partial_A f + \partial_A^* F_A^{2,0} = 0,
\]

where \(f = \frac{1}{n} \Lambda F_A\).

For \(n = 1\), the YM action becomes

\[
S(A) = -\frac{1}{8\pi^2 \varepsilon} \int_M \omega \text{Tr} (f^2).
\]

This does not depend on metric but only on the cohomology class of the Kähler form (the volume form). Clearly, the YM action is minimized by a flat connection. The equation of motion \(d_A f = 0\) shows that any YM connection is either flat \((f = 0)\) or reducible \((f \neq 0)\). For \(n = 2\), it is well-known that the YM action can be decomposed as

\[
S(A, k) = -\frac{1}{4\pi^2 \varepsilon} \int_M \text{Tr} F_A^+ \wedge * F_A^+ + \frac{k}{\varepsilon},
\]

where

\[
k = \int_M c_2(E) = \frac{1}{8\pi^2} \int_M \text{Tr} F_A \wedge F_A \in \mathbb{Z}.
\]

\(^2\) Throughout this section we refer freely to [3][15] for mathematical details.
The positivity of the instanton number \( k \) is a topological restriction of \( E \) in order to admit an ASD connection. For \( k > 0 \), the YM action is minimized by the ASD connections.

We propose a simple higher dimensional analogue of physical YM theory on Riemann surfaces. The basic idea is to restrict \( A \) to \( A^{1,1} \). Then, the YM action becomes

\[
I(A, c_2) = -\frac{1}{8\pi^2} \int_M \text{Tr} F_A^{1,1} \wedge * F_A^{1,1} = -\frac{n^2}{8\pi^2} \int_M \frac{\omega^n}{n!} \text{Tr} f^2 + \frac{1}{8\pi^2} \int_M \text{Tr} (F_A^{1,1} \wedge F_A^{1,1}) \wedge \frac{\omega^{n-2}}{(n-2)!}.
\]

(2.7)

We may call this theory holomorphic Yang-Mills (HYM) theory, since each connection \( A \) lies in \( A^{1,1} \), i.e. \( F_A^{0,2} = \partialbar_A^2 = 0 \), and each operator \( \partialbar_A \) defines a holomorphic structure \( \mathcal{E}_A \) on \( E \) by the integrability theorem of Newlander-Nirenberg [16]. The action functional of the HYM theory is minimized by an integrable connection \( A \) satisfying \( f = 0 \) if and only if the following inequality of Lübke holds;

\[
c_2 := \frac{1}{8\pi^2} \int_M \text{Tr} (F_A^{1,1} \wedge F_A^{1,1}) \wedge \frac{\omega^{n-2}}{(n-2)!} \geq 0.
\]

(2.8)

The equality holds if and only if the connection is flat. We will assume \( c_2 > 0 \) throughout this paper. Let \( \mathcal{E} \) be a holomorphic vector bundle over \( M \), then an unitary connection \( A \) of \( \mathcal{E} \) is Einstein-Hermitian if \( \Lambda F_A = nf \) is a constant multiple of the identity endomorphism.

Using the Chern-Weil theory, the constant can be shown to be proportional to \( \text{deg}(\mathcal{E}) = \int_M c_1(\mathcal{E}) \wedge \frac{\omega^{n-1}}{(n-1)!} \). Note that an \( SU(r) \) connection \( A \in A^{1,1} \) satisfying \( f = 0 \) endows \( E \) with a holomorphic vector bundle \( \mathcal{E}_A \) satisfying

\[
c_1(\mathcal{E}_A) = 0, \quad \int_M c_2(\mathcal{E}_A) \wedge \frac{\omega^{n-2}}{(n-2)!} = c_2,
\]

and defines an Einstein-Hermitian structure on \( \mathcal{E}_A \). In this sense, a connection \( A \in A^{1,1} \) on \( E \) is Einstein-Hermitian (EH) or Hermitian Yang-Mills if \( f = 0 \). That is, the action of the HYM theory is minimized by the EH connections provided that the inequality (2.8) holds.

We denotes \( \mathcal{M} \) the moduli space of EH connections. One can also define the moduli space of holomorphic vector bundles as the isomorphism class of \( \mathcal{E}_A \)'s. It turns out that this moduli space can be identified with the complex quotient, \( A^{1,1}/G^C \), where \( G^C \) denotes the complexification of \( G \). Due to the theorem of Donaldson-Uhlenbeck-Yau [17], the moduli space \( \mathcal{M}^* \) of irreducible EH connections is diffeomorphic to the moduli space \( A^{1,1}_S/G^C \) of \( \omega \)-stable bundles.
Comparison With the Two Dimensional Case

The YM theory restricted to $A^{1,1}$ has great similarity with physical YM theory on Riemann surfaces [12][9]. First of all, the action functional depends only on the cohomology class of $\omega$ and $c_2(E)$ in $H^{1,1}(M)$. And, any Yang-Mills connection $A \in A^{1,1}$ (HYM connection) is either EH ($f = 0$) or reducible ($f \neq 0$)

$$i(\bar{\partial}_A - \partial_A)f = 0 \rightarrow d_A f = 0.$$ (2.9)

The HYM theory has global scaling invariance. If we scale the Kähler form by $\omega \rightarrow \omega/t$ for any positive real number $t$, the action is invariant under $\varepsilon \rightarrow t^{2-n}\varepsilon$. Note that the coupling constant is scaling invariant only for $n = 2$. Finally, the action functional HYM theory is a norm squared of moment map up to topological term $c_2$. This is the crucial property of physical YM theory on Riemann surfaces, which led Witten to determine the general expressions of the intersection pairings on the moduli space of flat connections.

Note that $A$ inherits the complex structure and the Kähler structure of $M$. For the given complex structure $J$ on $M$, we can introduce a complex structure on $A$ by identifying $T^{1,0}A$ and $T^{0,1}A$, in the decomposition of tangent space $T_A = T^{1,0}A \oplus T^{0,1}A$, with $g_E$-valued $(1,0)$-forms and $(0,1)$-forms on $M$ respectively. Then, $A$ has natural Kähler structure,

$$\tilde{\omega} = \frac{1}{4\pi^2} \int_M \text{Tr} (\delta A' \wedge \delta A'') \wedge \frac{\omega^{n-1}}{(n-1)!},$$ (2.10)

where $\delta A' \in \Omega^{1,0}(g_E)$ and $\delta A'' \in \Omega^{0,1}(g_E)$. And, $G$ acts on $A$ by isometries. If we restrict $A$ to its subspace $A^{1,1}$, the subspace $A^{1,1}$ is preserved by the action of $G$ and its smooth part has the Kähler structure given above with $\partial_A \delta A' = \bar{\partial}_A \delta A'' = 0$. Let $\text{Lie}(G)$ be the Lie algebra of $G$, which can be identified with the space of $g_E$-valued zero-form. Then, we have a moment map, $m : A^{1,1} \rightarrow \Omega^0(g_E)^*$,

$$m(A) = -\frac{1}{4\pi^2} F_A^{1,1} \wedge \frac{\omega^{n-1}}{(n-1)!} = -f \frac{n}{4\pi^2} \frac{\omega^n}{n!},$$ (2.11)

where $\Omega^0(g_E)^* = \Omega^{2n}(g_E)$ denotes dual of $\Omega^0(g_E)$. Thus, the action functional of the HYM theory is the norm squared of the moment map, $(m, m) \approx -\int_M \frac{\omega^n}{n!} \text{Tr} f^2$, with respect to a metric on $\text{Lie}(G)$ determined by the measure $\omega^n/n!$ of $M$. The reduced phase space (symplectic quotient) $m^{-1}(0)/G$ can be identified with the moduli space $\mathcal{M}$.
of EH connections. The reduced phase space also has the Kähler structure descended symplectically\(^3\) from \(\mathcal{A}^{1,1}\) by the reduction theorem of Marsden-Weinstein [18].

Predictions of the Non-Abelian Localization Theorem

We can formally define the partition function of the HYM theory by

\[
Z(\varepsilon, c_2) = \exp(-\frac{c_2}{\varepsilon}) \times Z(\varepsilon)
\]

\[
= \exp(-\frac{c_2}{\varepsilon}) \times \frac{1}{\text{vol}(G)} \int_{\mathcal{A}^{1,1}} \mathcal{D}A \exp\left[ \frac{n^2}{8\pi^2 \varepsilon} \int_M \frac{\omega^n}{n!} \text{Tr} f^2 \right].
\]

(2.12)

According to the Non-Abelian localization theorem of Witten [9], this partition function can be expressed as sum of the contributions of the critical points:

\[
Z(\varepsilon, c_2) = \exp(-\frac{c_2}{\varepsilon}) \times \sum_{\alpha \in \mathcal{N}} Z_\alpha(\varepsilon),
\]

(2.13)

where \(\mathcal{N}\) denotes the moduli space of the HYM connections of given topological type.

Due to the great similarity with the physical YM theory on Riemann surfaces, we can repeats many of the manipulations given by Witten. For example, we can introduce a first order formalism of the HYM theory defined by the action [12],

\[
I(A, \varphi, c_2) = -\frac{1}{4\pi^2} \int_M \text{Tr} (i\varphi F^{1,1}) \wedge \frac{\omega^{n-1}}{(n-1)!} \varepsilon - \frac{c_2}{8\pi^2} \int_M \frac{\omega^n}{n!} \text{Tr} \varphi^2
\]

\[
+ \frac{1}{8\pi^2 \varepsilon} \int_M \text{Tr} (F^{1,1} \wedge F^{1,1}) \wedge \frac{\omega^{n-2}}{(n-2)!},
\]

(2.14)

which is equivalent to the original one after integrating \(\varphi \in \Omega^0(g_E)\) out. This formalism is useful for studying the zero coupling limit of \(Z(\varepsilon)\), such that the only contribution to the path integral comes from the reduced phase space:

\[
Z(0) = \frac{1}{\text{vol}(G)} \int_{\mathcal{A}^{1,1}} \mathcal{D}A \mathcal{D}\varphi \exp\left( \frac{in}{4\pi^2} \int_M \frac{\omega^n}{n!} \text{Tr} \varphi \right).
\]

(2.15)

Let \(#Z\) be the number of the center \(Z(SU(r))\) of \(SU(r)\) which coincides to \(Z(G)\). If there are no reducible EH connections such that \(\tilde{G} = G/Z(SU(r))\) acts freely on the moduli

\(^3\) The Kähler structure does not descend to \(A/G\) in general.
space $\mathcal{M}$, one can expect that $Z(0)$ is related to the volume of the moduli space. One can rewrite the partition function (2.15) as

$$Z(0) = \frac{1}{\# Z} \int_{A^{1,1}/\mathcal{G}} DA' D\varphi \exp \left( \frac{i\hbar}{4\pi^2} \int_M \omega^n \Tr \varphi f \right),$$

(2.16)

where the measure $DA'$ denotes quotient measure on $A^{1,1}/\mathcal{G}$. The quotient measure can be constructed by adopting the Faddev-Popov-BRST procedure. One can calculate $Z(0)$ by following the procedures in sect. 2.2 of [12]. The linearization of local equations which cut out the moduli space $\mathcal{M}$ of EH connections inside $A^{1,1}$ can be written as

$$QB = 0, \quad \text{where} \quad Q = P_+^{1,1}d_A \oplus d_A^* : \Omega^1(\mathfrak{g}_E) \to \Omega_+^{1,1}(\mathfrak{g}_E) \oplus \Omega^0(\mathfrak{g}_E),$$

(2.17)

where $P_+^{1,1}$ is the projection operator, acting on $\Omega^2(\mathfrak{g}_E)$, to the space $\Omega_+^{1,1}(\mathfrak{g}_E)$ of $(1,1)$-form parallel to the Kähler form. If we assume that there is an isolated irreducible EH connection $A$, we will end up with the following determinant ratio:

$$\frac{1}{\# Z} \det \Delta_0 \left/ \det Q \right|^2,$$

(2.18)

where $\Delta_0 = d_A^* d_A$ denotes the Laplacian acting on zero-forms. Clearly, the determinant $\det \Delta_0$ is given by the Gaussian integral over the standard ghost and anti-ghost. One can find that

$$|\det Q| = \sqrt{QQ^*} = \sqrt{\det \Delta_0 \det \Delta_+^{1,1}},$$

(2.19)

where $Q^*$ is the adjoint of $Q$ and $\Delta_+^{1,1} = (P_+^{1,1}d_A)(P_+^{1,1}d_A)^*$ is the Laplacian acting on $\Omega_+^{1,1}(\mathfrak{g}_E)$. Thus, eq.(2.18) becomes

$$\frac{1}{\# Z} \sqrt{\frac{\det \Delta_0}{\det \Delta_+^{1,1}}}.$$}

(2.20)

This is trivial due to the isomorphism (specific to compact Kähler manifold) between $\Omega^0(\mathfrak{g}_E)$ and $\Omega_+^{1,1}(\mathfrak{g}_E)$ by the formula

$$\Omega_+^{1,1}(\mathfrak{g}_E) = \Omega^0(\mathfrak{g}_E) \otimes \omega.$$

(2.21)

If the moduli space is a smooth manifold, one can show that $Z(0)$ reduces to the volume of $\mathcal{M}$ with the trivial correction factor $1/\# Z$,

$$Z(0) = \frac{1}{\# Z} \text{vol}(\mathcal{M}).$$

(2.22)

This should be viewed as only a formal result since the moduli space is rarely compact. Note that we assumed that there are no reducible EH connections. This is necessary to ensure a well-defined determinant ratio (2.18). It is, however, not sufficient condition (except $n = 1$ case) because there is another obstruction to have a smooth moduli space. This important issue will be discussed in the next section.
3. Mapping from $N = 2$ TYM theory

In the previous section, we defined the HYM theory as the restriction of physical Yang-Mills theory to $A^{1,1}$. However, the treatments in that section were essentially classical. The question is whether the theory is well defined as a quantum field theory. In particular, we should impose certain constraints which restricts $A$ to $A^{1,1}$ in such a way that no non-trivial quantum correction appears. Another important issue is to ensure the finiteness of the theory. In this section, we construct a $N = 2$ supersymmetric HYM ($N = 2$ HYM) theory which can be shown to be well-defined as a quantum field theory. Our arguments will be based on a simple mapping from the $N = 2$ TYM theory [8]. This is analogous to the mapping, discovered by Witten, from the $N = 1$ TYM theory to physical YM theory in two dimensions [9]. The equivalence of the $N = 2$ HYM theory to the $N = 2$ TYM theory will enable us to interpret the $N = 2$ HYM theory as a simple field theoretical model of the Donaldson invariants of compact Kähler surface and their cousins in higher dimensions.

3.1. $N=2$ TYM theory

The purpose of this subsection is to sketch the $N = 2$ TYM theory very rapidly [8]. At this time, we formulate $N = 2$ TYM theory on an arbitrary dimensional compact Kähler manifold. The basic multiplet of the theory is the $[A, (\psi, \bar{\psi}), \varphi]$ with the $N = 2$ transformation law

$$
\begin{align*}
&s A' = -\psi, \quad s\psi = 0, \\
&\bar{s} A' = 0, \quad \bar{s}\psi = -i\partial A \varphi, \quad \bar{s}\varphi = 0, \\
&s A'' = 0, \quad s\bar{\psi} = -i\bar{\partial} A \varphi, \quad s\varphi = 0, \\
&\bar{s} A'' = -\bar{\psi}, \quad \bar{s}\bar{\psi} = 0,
\end{align*}
$$

(3.1)

where $\psi \in \Omega^{1,0}(g_E), \bar{\psi} \in \Omega^{0,1}$ and $\varphi \in \Omega^{0}(g_E)$. This algebra can be obtained by decomposing the original $N = 1$ algebra of ref. [1] according to the complex structures on $M$ and $A$. The commutation relations of the fermionic symmetry generators $s, \bar{s}$ are

$$
\begin{align*}
&s^2 = 0, \quad (s\bar{s} + \bar{s}s) = id_A \varphi = -i\delta_\varphi, \quad \bar{s}^2 = 0,
\end{align*}
$$

(3.2)

where $\delta_\varphi$ is the generator of a gauge transformation with infinitesimal parameter $\varphi$. We introduce two ghost numbers $(U, R)$, which assign $(1, 1)$ to $s$ and $(1, -1)$ to $\bar{s}$. To write
a topological action, it is sufficient to introduce an anti-ghost $B \in \Omega^2(g_E)$ in the adjoint representation with $(U, R) = (-2, 0)$. Then (3.2) naturally leads us to multiplet $[B, (i\chi, -i\bar{\chi}), H]$ with transformation law

$$\begin{align*}
s B &= -i\chi, \\
\bar{s} B &= i\bar{\chi}, \\
s \bar{\chi} &= H - \frac{1}{2}[\varphi, B], \\
\bar{s} \bar{\chi} &= H + \frac{1}{2}[\varphi, B], \\
s H &= -\frac{i}{2}[\varphi, \chi], \\
\bar{s} H &= -\frac{i}{2}[\varphi, \bar{\chi}].
\end{align*}$$  
(3.3)

We assume that $B, \chi, \bar{c}, H \in \Omega^2_+(g_E)$, i.e. $B = B^{2,0} + B^{0,2} + B^0 \omega$. We further assume that $\chi^{2,0}$ and $\chi^{0,2}$ to satisfy $\partial_A \chi^{2,0} = \bar{\partial}_A \chi^{0,2} = 0$ if $n > 2$.

The action of $N = 2$ TYM theory can be written in the form

$$S = \frac{s \bar{s} - s \bar{s}}{2} V = \frac{s \bar{s} - s \bar{s}}{2} \left( -\frac{1}{\hbar^2} \int_M \text{Tr} B \wedge *F \right).$$  
(3.4)

Note that $V$ has $(U, R) = (-2, 0)$, such that the action has $(U, R) = (0, 0)$ \footnote{One can add $-\frac{1}{\hbar^2} \int_M \text{Tr} \chi \wedge *\bar{\chi}$ to $V$. The resulting theory will be most complete but equivalent to that considered here [8]. Our choice leads to much simpler arguments in the next subsection. I would like to thank G. Thompson for pointing this out.}. We find that

$$S = \frac{1}{\hbar^2} \int_M \text{Tr} \left[ -i H^{2,0} \wedge *F^{0,2} - i H^{0,2} \wedge *F^{2,0} + i \chi^{2,0} \wedge *\bar{\partial}_A \psi + i \bar{\chi}^{0,2} \wedge *\partial_A \psi \\
- \frac{i}{2}[\varphi, B^{2,0}] \wedge *F^{0,2} + \frac{i}{2}[\varphi, B^{0,2}] \wedge *F^{2,0} - \left( i n H^0 f - i \chi^0 A \bar{\partial}_A \psi \right) \right] \frac{\omega^n}{n!},$$  
(3.5)

where we have used $\bar{\partial}_A \bar{\partial}_A \varphi = [F^{0,2}, \varphi]$ and $\partial_A \bar{\partial}_A \varphi = [F^{2,0}, \varphi]$. The integration over $H$ gives a delta function $\delta(F^+ = 0)$. The transformation law (3.1) shows that $\varphi$ is a covariant constant, $d_A \varphi = 0$, at the fixed point locus. Thus, the path integral reduces to an integral over the moduli space of EH connections and the space of $\varphi$ zero-modes. If there are no reducible EH connection, the path integral is localized to an integral over the moduli space $\mathcal{M}^*$ of irreducible EH connections.

From (3.5), one can see the zero-modes of the fermionic variables $\psi$ and $\bar{\psi}$ are given by

$$\partial_A \psi = \bar{\partial}_A^* \psi = 0, \quad \bar{\partial}_A \bar{\psi} = \bar{\partial}_A^* \bar{\psi} = 0.$$  
(3.6)
The $\psi$ and $\bar{\psi}$ zero-modes represent the holomorphic and the anti-holomorphic (co-)tangent vectors on $\mathcal{M}$ at $A$ respectively, provided that its neighborhood [$A$] in $\mathcal{M}$ is smooth. The zero-modes of $\chi$ and $\bar{\chi}$ satisfy

$$\begin{cases}i\partial_A \chi^0 + \partial_A^* \chi^{2,0} = 0, \\
-i\bar{\partial}_A \chi^0 + \bar{\partial}_A^* \chi^{0,2} = 0,
\end{cases} \quad \Rightarrow \quad \begin{cases}\partial_A \chi^0 = \partial_A^* \chi^{2,0} = 0, \\
\bar{\partial}_A \chi^0 = \bar{\partial}_A^* \chi^{0,2} = 0.\end{cases} \tag{3.7}$$

Thus, there are no $\chi^0, \bar{\chi}^0$ zero-modes if every EH connection is irreducible. Note that the zero-mode of $\bar{\chi}^{2,0}$ satisfies $\bar{\partial}_A^* \bar{\chi}^{0,2} = \bar{\partial}_A \bar{\chi}^{0,2} = 0$. Let $\mathcal{E}_A$ be the holomorphic structure induced by an EH connection $A$ and $\text{End}_0(\mathcal{E}_A)$ be the trace-free endomorphisms of $\mathcal{E}_A$. Then a zero-mode $\bar{\chi}^{0,2}$ is an element of $H^2(\text{End}_0(\mathcal{E}_A))$. The moduli space $\mathcal{M}^*$ of irreducible EH connections is smooth at [$A$] if $H^2(\text{End}_0(\mathcal{E}_A)) = 0$ [19][20][21][15]. Thus, the moduli space $\mathcal{M}$ is a smooth Kähler manifold if there are no $\chi$ and $\bar{\chi}$ zero-modes, and its complex dimension is identical to the number of $\bar{\psi}$ zero-modes. An important point to note is that the theory has the $U$ number anomaly equals to $2(#\bar{\psi} - #\bar{\chi}^{2,0} - #\bar{\chi}^0)$ where # denotes the number of zero-modes. On the other hand, the $R$ number anomaly is absent.

The set of topological observables can be easily constructed. And, by standard analysis of cohomological theory, one can show that their expectation values are topological invariants in general circumstance [1][8]. This is valid only at formal level because the contributions of singularities and non-compactness can make the topological interpretations of those path integrals difficult [22]. To the author’s knowledge, only the existence of the EH connections are established for $n > 2$ cases. However, we will proceed our analysis assuming favorable situations. It is sufficient, for our purpose, to consider the following observables;

$$W^{2,2} = \Theta = \frac{1}{8\pi^2} \int_M \text{Tr} \left( \varphi^2 \right) \wedge \frac{\omega^n}{n!},$$

$$W^{1,1} = \tilde{\omega} = \frac{1}{4\pi^2} \int_M \text{Tr} \left( i\varphi F^{1,1} + \psi \wedge \bar{\psi} \right) \wedge \frac{\omega^{n-1}}{(n-1)!},$$

$$W^{0,0} = -\tilde{c}_2 = -\frac{1}{4\pi^2} \int_M \text{Tr} \left( F^{2,0} \wedge F^{0,2} + \frac{1}{2} F^{1,1} \wedge F^{1,1} \right) \wedge \frac{\omega^{n-2}}{(n-2)!}, \tag{3.8}$$

which are both $s$ and $\bar{s}$ invariant. A $W^{m,m}$ carrying $(U,R) = (2m,0)$ is an element of $(m,m)$-th Dolbeault cohomology group on $A/\mathcal{G}$, which depends only on the cohomology class of $\omega$ (and $c_2(E)$ for $m = 0$).
Now we assume that there are no $\chi$ and $\bar{\chi}$ zero-modes, such that the path integral is localized to the moduli space of irreducible EH connections which is a smooth Kähler manifold. The expectation value,

$$< \exp(\bar{\omega} + \varepsilon \Theta) > = \sum_{r,s} \frac{1}{\ell!s!} \varepsilon^s < \bar{\omega}^\ell \Theta^s >,$$

(3.9)

can be interpreted as the intersection pairings on the moduli space of $\omega$-stable bundles for general $n$. The right hand side of (3.9) is a non-vanishing invariant if

$$\dim_{\mathbb{C}}(\mathcal{M}) \equiv d = \ell + 2s,$$

(3.10)

which is originated from the $U$ number anomaly due to the $\bar{\psi}$ zero modes. From (2.10) and (3.6), one can identify $\bar{\omega}$ with the Kähler form on $\mathcal{M}$ after the localization of the path integral, because $F_{A}^{1,1} \wedge \omega = 0$ for an EH connection $A$. Thus, the expectation value $< \exp \bar{\omega} >$ reduces to the volume of the moduli space

$$\langle \exp \bar{\omega} \rangle = \frac{1}{\# Z} \int_{\mathcal{M}} \frac{\bar{\omega}^d}{d!} = \frac{1}{\# Z} \text{vol}(\mathcal{M}),$$

(3.11)

provided that the moduli space is compact. This implies that there is a close relation between the HYM and the $N = 2$ TYM theories.

### 3.2. Mapping to $N = 2$ HYM theory

Witten showed that there is a simple mapping to physical YM theory in two dimensions from the $N = 1$ TYM theory [9]. Using this equivalence, he was able to find general expressions for the intersection pairings on the moduli space of flat connections. To begin with, we briefly recall the basic strategy of Witten. Let $L = -i\delta_{W} W$ be the original action of the $N = 1$ TYM theory in two dimensions. Replacing $W$ by $W + tW'$, we have TYM theory with action

$$L(t) = -i\delta_{W'} (W + tW').$$

(3.12)

The deformed theory is equivalent to the original one $L(t = 0)$ if (i) $W'$ is such that $L(t)$ has nondegenerate kinetic energy for all $t$; (ii) the deformed theory does not have any new fixed points to flow in from infinity\(^5\). Witten chose of $W'$ which does not obey the

\(^5\) By the standard manipulation of cohomological theory, one can also formally show that deformed theory is independent to $t$ as long as $t$ is non-zero.
condition (ii) such that the new fixed point locus is precisely the space of YM connections. Then he compared the expectation values of certain observables computed in the original and the deformed theories by taking large imaginary $t$ limit.

In this subsection, we will show that an analogous mapping exists from the $N = 2$ TYM theory to the $N = 2$ HYM theory on a compact Kähler manifold. Our basic observation is that the localization of $N = 2$ TYM theory can be realized by following two steps: i) localization to $A_{1,1}$ (not to $A_{1,1}/G$) by integrating $H^{2,0}, H^{0,2}, \chi^{2,0},$ and $\bar{\chi}^{0,2}$ out, provided that there are no $\chi^{2,0}$ and $\bar{\chi}^{0,2}$ zero-modes. ii) localization of $A_{1,1}$ to $M$ by integrating out $H^0, \chi^0, B^0,$ provided that there are no reducible EH connections. The arena of the HYM theory is precisely the intermediate state, $F^{2,0} = F^{0,2} = \partial_A \psi = \bar{\partial}_A \bar{\psi} = 0,$ after the first localization. We will deform the $N = 2$ TYM theory to obtain a new $N = 2$ TYM theory whose fixed point of locus is the space of HYM connections. Since both theories have the same $N = 2$ supersymmetry and topological observables, we can compare expectation values of observables evaluated in the two theories. We, then, interpret the expectation value of a combination of observables evaluated in the deformed theory as the partition function of the $N = 2$ HYM theory.

**Deformation to a New $N = 2$ TYM theory**

We consider one parameter family of the $N = 2$ TYM theories with the action

$$S(t) = \frac{ss - \bar{s}s}{2} (V + tV').$$

A suitable choice of $V'$ is

$$V' = -\frac{n}{\hbar^2} \int_M \omega^n/n! \operatorname{Tr} B^0 B^0.$$

Note that the global $U$-number symmetry of the original ($t = 0$) theory is no longer maintained in the deformed ($t \neq 0$) theory, since $(U,R)$ numbers of $V'$ are $(-4,0)$ so those of $ssV'$ are $(-2,0)$. The upshot is that we can integrate $B_0$ out from the action,

$$S(t) = \frac{ss - \bar{s}s}{2} \left( -\frac{1}{\hbar^2} \int_M \operatorname{Tr} (B^{2,0} \wedge *F^{0,2} + B^{0,2} \wedge *F^{0,2}) - \frac{n}{\hbar^2} \int_M \omega^n/n! \operatorname{Tr} (B_0 f + tB_0 B_0) \right) ,$$

leaving a local action

$$S'(t) = -\frac{ss - \bar{s}s}{2} \left( \frac{1}{\hbar^2} \int_M \operatorname{Tr} (B^{2,0} \wedge *F^{0,2} + B^{0,2} \wedge *F^{0,2}) \right) + \frac{ss - \bar{s}s}{2} \left( \frac{n}{4\hbar^2 t} \int_M \omega^n/n! \operatorname{Tr} f^2 \right).$$

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We can check this explicitly by integrating out $H^0, B^0, \chi^0$ and $\bar{\chi}^0$ from the action

\[
S(t) = \frac{1}{\hbar^2} \int_M \text{Tr} \left[ -iH^{2,0} \wedge \ast F^{0,2} - iH^{0,2} \wedge \ast F^{2,0} - \frac{i}{2} [\varphi, B^{2,0}] \wedge \ast F^{0,2} + \frac{i}{2} [\varphi, B^{0,2}] \wedge \ast F^{2,0} \\
+ i\chi^{2,0} \wedge \ast \bar{\partial} \bar{\varphi} + i\chi^{0,2} \wedge \ast \partial A \varphi - \left( niH^0 (f + 2tB^0) - i\bar{\chi}^0 \Lambda \bar{\partial} \bar{\varphi} - i\chi^0 \Lambda \partial A \bar{\varphi}\right) \right] \omega^n,
\]

which leads to

\[
S'(t) = \frac{1}{\hbar^2} \int_M \text{Tr} \left[ -iH^{2,0} \wedge \ast F^{0,2} - iH^{0,2} \wedge \ast F^{2,0} - \frac{i}{2} [\varphi, B^{2,0}] \wedge \ast F^{0,2} + \frac{i}{2} [\varphi, B^{0,2}] \wedge \ast F^{2,0} \\
+ i\chi^{2,0} \wedge \ast \bar{\partial} \bar{\varphi} + i\chi^{0,2} \wedge \ast \partial A \varphi + \frac{1}{4t} \left( f \Lambda((i\partial A \bar{\partial} A - i\bar{\partial} A \partial A)\varphi - 2[\psi, \bar{\psi}]) \right) \frac{\omega^n}{n!} \right],
\]

(3.17)

This is identical to (3.16).

Now we examine what kind of localization governs the deformed $N = 2$ TYM theory. The $t$ independent part of $S'(t)$ is a cohomological theory which localize the theory to $TA^{1,1}$. Integrations over $H^{2,0}$ and $H^{0,2}$ gives delta function support to $F^{2,0} = F^{0,2} = 0$ and the $\chi^{2,0}$ and $\bar{\chi}^{0,2}$ integrals show that $\psi$ and $\bar{\psi}$ are tangent to $A^{1,1}$,

\[
\partial A \psi = 0, \quad \bar{\partial} \bar{\psi} = 0.
\]

(3.19)

The $\varphi$ equation of motion in the $t$-dependent part gives

\[
(\bar{\partial}^A \bar{\partial} A + \partial^A \partial A)f = d_A^* d_A f = 0,
\]

(3.20)

where we have used the Kähler identities. Thus, we have

\[
0 = \int_M \text{Tr} f \ast d_A^* d_A f = \int M \text{Tr} d_A f \wedge \ast d_A f \longrightarrow d_A f = 0.
\]

(3.21)

This is the classical equation of motion for the HYM theory. Note that the $\psi$ and $\bar{\psi}$ equations of motion are

\[
\frac{1}{n} \partial A \Lambda(\partial A \psi) + [\psi, f] = 0, \quad \frac{1}{n} \bar{\partial} A \Lambda(\partial A \bar{\psi}) + [\bar{\psi}, f] = 0.
\]

(3.22)
Thus, (the bosonic parts of) the locus $N \subset A^{1,1}/G$ of the fixed points of the deformed theory is the disjoint unions of the moduli space of EH connections ($f = 0$) and the moduli space of the higher critical points of HYM theory (reducible connections with $f \neq 0$). Note that the moduli space of EH connections is identical to the symplectic quotient $m^{-1}(0)/G$. On the other hand, the moduli space $U_\beta$ of the higher critical points with a constant value of $f_\beta$ is not isomorphic to the symplectic quotient $m^{-1}(f_\beta \omega^n/4\pi^2(n-1))/G = M_\beta$. The space $U_\beta$ is at most a set of finite points, since open dense subset of $M_\beta$ consists of irreducible connections. Put it differently, the fixed points locus $N$ consists of the moduli space $M^*$ of irreducible EH connections and the gauge equivalence classes of reducible connections in $A^{1,1}$.

Mapping to $N = 2$ HYM theory

Now we have a new $N = 2$ TYM theory with the enlarged fixed point locus $N$. We can study the expectation values of the topological observables evaluated in this new theory and compare them with those of the original one. With a suitable choice of a set of observables, we may be able to cleverly extract the contribution of $M$ from the path integral performed over $N$. The observables $\Theta, \tilde{\omega}$ and $\tilde{c}_2$ are very special among the set of all observables. They are nontrivial and non-degenerate because $\omega$ is nowhere vanishing $(1,1)$-form in the non-trivial second cohomology. We consider a expectation value $< \exp(\tilde{\omega} + \varepsilon \Theta - \tilde{c}_2/\varepsilon)'>$ of the deformed theory,

$$< \exp(\tilde{\omega} + \varepsilon \Theta - \tilde{c}_2/\varepsilon)'> = \frac{1}{\text{vol}(G)} \int_A \mathcal{D}A' \mathcal{D}A'' \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\varphi \mathcal{D}B^{2.0} \mathcal{D}B^{0.2} \mathcal{D}\chi^{2.0} \mathcal{D}\chi^{0.2} \mathcal{D}H^{2.0} \mathcal{D}H^{0.2} \times \exp\left(\frac{n}{8h^2} \int_M \omega^n / n! \text{Tr} f^2 + \frac{1}{2h^2} \int_M \text{Tr} (B^{2.0} \wedge *F^{0.2} + B^{0.2} \wedge *F^{2.0})\right) + \frac{1}{4\pi^2} \int_M \text{Tr}(i\varphi F^{1,1} + \psi \wedge \bar{\psi}) \wedge \frac{\omega^{n-1}}{(n-1)!} + \frac{\varepsilon}{8\pi^2} \int_M \omega^n / n! \text{Tr} \varphi^2 \right. \right. \left. \left. - \frac{1}{4\pi^2 \varepsilon} \int_M \text{Tr}(F^{2.0} \wedge F^{0.2} + \frac{1}{2} F^{1,1} \wedge F^{1,1}) \wedge \frac{\omega^{n-2}}{(n-2)!} \right) \right] \right),

(3.23)

The above correlation function is formally independent of $t$ for $t \neq 0$. It is really independent of $t$ even at $t = \infty$ as long as, in varying $t$, the Lagrangian remains non-degenerate and with a good behavior at infinity in field space [9]. The above choice ensures that these conditions are obeyed as in the two-dimensional ($n = 1$) case. Thus we can simply
discard the \( u \)-dependent part in (3.23) by setting \( t = \infty \). Then, the expectation value
\[
< \exp(\tilde{\omega} + \varepsilon \Theta - \tilde{c}_2/\varepsilon) >'
\]
can be interpreted as the partition function \( Z(\varepsilon, \tilde{c}_2) \) of a \( N = 2 \) supersymmetric theory defined by the action functional
\[
I = \frac{1}{\hbar^2} \int_M \text{Tr} \left[ -iH^{2,0} \wedge \ast F^{0,2} - iH^{0,2} \wedge \ast F^{2,0} - \frac{i}{2} [\varphi, B^{2,0}] \wedge \ast F^{0,2} + \frac{i}{2} [\varphi, B^{0,2}] \wedge \ast F^{2,0} \right.
+ i\bar{\chi}^{2,0} \wedge \ast \bar{\partial}_A \bar{\psi} + i\bar{\chi}^{0,2} \wedge \ast \bar{\partial}_A \bar{\psi} \left. \right] - \frac{1}{4\pi^2} \int_M \text{Tr} \left( i\varphi F^{1,1} + \psi \wedge \bar{\psi} \right) \wedge \frac{\omega^{n-1}}{(n-1)!}
- \frac{\varepsilon}{8\pi^2} \int_M \frac{\omega^n}{n!} \text{Tr} \varphi^2 + \frac{1}{4\pi^2 \varepsilon} \int_M \text{Tr} \left( F^{2,0} \wedge F^{-0,2} + \frac{1}{2} F^{1,1} \wedge F^{1,1} \right) \wedge \frac{\omega^{n-2}}{(n-2)!} \right].
\]
(3.24)

It will be useful to show the equivalence of \( Z(\varepsilon, \tilde{c}_2) \) and the expectation value (3.23) explicitly. Since both theories have the same \( N = 2 \) supersymmetry, it is sufficient to check that they have the same fixed points. Provided that the localization of the both theory to \( TA^{1,1} \) is understood, we find the fixed point locus of the latter theory is
\[
\begin{align*}
sp = \partial_A \varphi, \\
\bar{s}\bar{\psi} = \bar{\partial}_A \varphi, \\
in \varphi + \varepsilon \varphi = 0,
\end{align*}
\]
which are identical to (3.21) and (3.22).

The \( N = 2 \) supersymmetric theory defined by the action functional (3.24) (the \( N = 2 \) HYM theory) is the desired quantum field theoretical setting of the HYM theory discussed in the previous section. Before showing this, we should comment that the \( N = 2 \) HYM theory is well defined as a quantum field theory. Obviously, It is a finite theory in the standard criteria, since its partition function \( Z(\varepsilon, \tilde{c}_2) \) is identical to the expectation value
\[
< \exp(\tilde{\omega} + \varepsilon \Theta - \tilde{c}_2/\varepsilon) >'
\]
of the topological observables evaluated in a cohomological theory. The subtle point is that even the cohomological field theory has difficulties in dealing with the non-compactness and the singularities in the moduli space. The path integral of the \( N = 2 \) HYM theory has two kinds of contributions, from the moduli space \( \mathcal{M} \) of EH connections and from the higher critical points \( U_\beta \). Even though the path integral contributed from \( \mathcal{M} \) is identical to the path integral of the original \( (t = 0) \) \( N = 2 \) TYM theory, a new problem may appear due to the contributions of higher critical points.

However, the contributions of \( U_\alpha \) can be precisely determined, since \( U_\alpha \) consists of, at most, finite collection of points.
We also have a partial resolution of the singularities in the moduli space. Since we have already integrated out \( \chi^0 \) and \( \bar{\chi}^0 \), there are no fermionic zero-modes due to reducible EH connections. The (bosonic) \( \varphi \)-zero-modes due to reducible EH connections is not so harmful because of the Tr \( \varphi^2 \) term in the action.

**Final Reduction**

We assume that the cohomology \( H^2(\text{End}_0(E_A)) \) is trivial everywhere. Then, we can integrate \( H^{2,0}, H^{0,2}, B^{2,0}, B^{0,2}, \chi^{2,0} \) and \( \bar{\chi}^{0,2} \) out and the partition function \( Z(\varepsilon, \bar{c}_2) \) of the \( N = 2 \) HYM theory becomes

\[
\exp(-\frac{c_2}{\varepsilon}) \times \frac{1}{\text{vol}(G)} \int_A DA' DA'' D\psi D\bar{\psi} D\varphi \cdot \prod_{x \in M} \delta(F^{2,0}(x)) \delta(F^{0,2}(x)) \delta(\partial_A \psi(x)) \delta(\bar{\partial}_A \bar{\psi}(x))
\times \exp \left( \frac{1}{4\pi^2} \int_M \text{Tr} \left( i\varphi F^{1,1} + \psi \wedge \bar{\psi} \right) \wedge \frac{\omega^{n-1}}{(n-1)!} + \frac{\varepsilon}{8\pi^2} \int_M \frac{\omega^n}{n!} \text{Tr} \varphi^2 \right).
\]

The measure of the above path integral is equivalent to

\[
\int_A DA' DA'' D\psi D\bar{\psi} D\varphi \prod_{x} \delta(F^{2,0})(x) \delta(F^{0,2})(x) \delta(\partial_A \psi(x)) \delta(\bar{\partial}_A \bar{\psi}(x)) \cdots \equiv \int_{T^A^{1,1}} DA' DA'' D\psi D\bar{\psi} D\varphi \cdots,
\]

since there are no loop corrections of the delta function constraints, \( \prod \delta(F^{2,0}(x)) \delta(F^{0,2}(x)) \) and \( \prod \delta(\partial_A \psi(x)) \delta(\bar{\partial}_A \bar{\psi}(x)) \), due to the \( N = 2 \) fermionic symmetry,

\[
sF^{2,0} = \partial_A \psi, \quad sF^{2,0} = 0, \quad sF^{0,2} = \bar{\partial}_A \bar{\psi}, \quad sF^{0,2} = 0.
\]

Thus, the partition function (3.25) can be written as

\[
\exp(-\frac{c_2}{\varepsilon}) \times \frac{1}{\text{vol}(G)} \int_{T^A^{1,1}} DA' DA'' D\psi D\bar{\psi} D\varphi
\exp \left( \frac{1}{4\pi^2} \int_M \text{Tr} \left( i\varphi F^{1,1} + \psi \wedge \bar{\psi} \right) \wedge \frac{\omega^{n-1}}{(n-1)!} + \frac{\varepsilon}{8\pi^2} \int_M \frac{\omega^n}{n!} \text{Tr} \varphi^2 \right).
\]

This can be viewed as the partition function of the HYM theory after integrating \( \varphi \) out. The role of the decoupled fields \( \psi \) and \( \bar{\psi} \) is to give a symplectic (Kähler) measure on \( A^{1,1} \), which will be denoted as \( DA \), and to ensure the \( N = 2 \) supersymmetry. Now we can extract one important conclusion that the partition function of the HYM theory can be expressed as a sum of contribution of critical points \( \mathcal{N} \)

\[
Z(\varepsilon, c_2) = \exp(-c_2/\varepsilon) \times \sum_{\alpha \in \mathcal{N}} Z_{\alpha}(\varepsilon).
\]

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This coincides to the prediction of the Witten’s non-Abelian localization theorem. In sect. 3.1, we showed that \(<\exp \tilde{\omega}>\) can be identified with the volume of \(\mathcal{M}\) under suitable topological conditions. From (3.23) and (3.28), one can write \(Z(0) = <\exp \tilde{\omega}>'\). The \(\varphi\) integral gives delta function support on EH connections and there are no additional critical points contributing to the path integral. This leads that the deformed theory is equivalent to the original theory in this particular case. Thus, we can recover (2.22) from (3.11),

\[ <\exp \tilde{\omega}>' = <\exp \tilde{\omega}> = \frac{1}{\#Z \text{vol}(\mathcal{M})}. \]  

(3.30)

Now the key step is to find the relations between the original \(N = 2\) TYM and the \(N = 2\) HYM theories in general situation, \(\varepsilon \neq 0\). By integrating \(\varphi\) and \(\psi, \bar{\psi}\) out in (3.28), we have

\[ \langle \exp(\tilde{\omega} + \varepsilon \Theta - \tilde{c}_2/\varepsilon) \rangle' = \exp(-\frac{c_2}{\varepsilon}) \times \frac{1}{\text{vol}(G)} \int_{\mathcal{M}} \mathcal{D}A \exp \left( \frac{n}{8\pi^2 \varepsilon} \int_{\mathcal{M}} \omega^n n! \text{Tr} f^2 \right). \]  

(3.31)

Now, the two expectation values \(<\exp(\tilde{\omega} + \varepsilon \Theta - \tilde{c}_2/\varepsilon)>\) and \(<\exp(\tilde{\omega} + \varepsilon \Theta - \tilde{c}_2/\varepsilon)>'\) are no longer identical. The later has contributions from the higher critical points \(\{U_\beta\}\). Since a \(U_\beta\) consists of finite set of points, the relation (3.31) clearly shows that their contributions are exponentially small, involving the relevant values of the action of the HYM. Thus

\[ \exp(-\frac{c_2}{\varepsilon}) \times \frac{1}{\text{vol}(G)} \int_{\mathcal{M}} \mathcal{D}A \exp \left( \frac{n}{8\pi^2 \varepsilon} \int_{\mathcal{M}} \omega^n n! \text{Tr} f^2 \right) \]

\[ = \langle \exp(\tilde{\omega} + \varepsilon \Theta - \tilde{c}_2/\varepsilon) \rangle + O(\exp(-c_2/\varepsilon - c/\varepsilon)), \]

where \(c\) is the smallest value of \(\varepsilon I(A) = -\frac{n}{4\pi^2} \int_{\mathcal{M}} \omega^n n! \text{Tr} f^2\) at the higher critical points.

Higher Critical Points

The critical points of the HYM theory are the solutions of the classical equations of motion \(d_A f = 0\). One obvious solution is \(f = 0\) which corresponds to the EH connections. If there is a higher critical point \(f \neq 0\), it means that the connection is reducible. For simplicity, we consider the \(SU(2)\) case only. We further assume that \(M\) is a projective algebraic manifold\(^6\) such that we can pick a Hodge metric and associated Kähler form \(\omega \in H^{1,1}(M, \mathbb{Z})\).

\(6\) Note that every compact Riemann surface is projective algebraic.
If there is a reducible $SU(2)$ connection $A \in \mathcal{A}^{1,1}$, the associated holomorphic vector bundle $\mathcal{E}_A$ is decomposed into the direct sum of $U(1)$ holomorphic line bundles,

$$\mathcal{E}_A = \mathcal{U} \oplus \mathcal{U}^{-1},$$

(3.33)

whose curvature $F_A \in \Omega^{1,1}(\text{End}_0(\mathcal{E}_A))$ is constant in each higher critical point,

$$F_A = \begin{pmatrix} F_c & 0 \\ 0 & -F_c \end{pmatrix} \in \mathfrak{su}(2).$$

(3.34)

A holomorphic line bundle $\mathcal{U}$ is classified by its first Chern class $c_1(\mathcal{U})$

$$c_1(\mathcal{U}) = \frac{i}{2\pi} F_c \in H^{1,1}(M, \mathbb{Z}).$$

(3.35)

We can decompose $c_1(\mathcal{U})$ as

$$\frac{i}{2\pi} F_c = \frac{i}{2\pi} f_c \omega + \frac{i}{2\pi} S, \quad S \wedge \omega^{n-1} = 0,$$

(3.36)

where the part orthogonal to $\omega$ is an element of the primitive cohomology class $P^{1,1}(M, \mathbb{Z})$. The degree of $\mathcal{U}$, $\text{deg}(\mathcal{U})$, is defined by

$$\text{deg}(\mathcal{U}) = \int_M c_1(\mathcal{U}) \wedge \omega^{n-1} = \frac{i}{2\pi} \int_M F_c \wedge \omega^{n-1} = \frac{if_c}{2\pi} \int_M \omega^n \in \mathbb{Z},$$

(3.37)

The $\text{deg}(\mathcal{U})$ is a (real) topological invariant depending only on $c_1(\mathcal{U})$ and the de Rahm cohomology class of $\omega$. The constant value $f_c$ is determined by

$$\frac{if_c}{2\pi} = \frac{\text{deg}(\mathcal{U})}{\int_M \omega^n}.$$ (3.38)

If we consider the $n = 1$ case, the first Chern number $\langle c_1(\mathcal{U}), M \rangle$ is identical to $\text{deg}(\mathcal{U})$, which can be an arbitrary integer. This is sufficient to determine all of the allowed values of $f_c$.

For $n > 1$, we have another important condition

$$\langle c_2(\mathcal{E}_A) \sim \omega^{n-2}, [M] \rangle = -\langle c_1(\mathcal{U}) \sim c_1(\mathcal{U}) \sim \omega^{n-2}, [M] \rangle,$$

(3.39)

that is

$$(n - 2)! c_2 = \frac{1}{8\pi^2} \int_M \text{Tr}(F_A \wedge F_A) \wedge \omega^{n-2}$$

$$= -\int_M c_1(\mathcal{U}) \wedge c_1(\mathcal{U}) \wedge \omega^{n-2}$$

$$= -\int_M \left( \frac{i}{2\pi} f_c \right) \left( \frac{i}{2\pi} f_c \right) \omega^n - \int_M \frac{i}{2\pi} S \wedge \frac{i}{2\pi} S \wedge \omega^{n-2} \in \mathbb{Z}^+.$$

(3.40)
This is precisely the Hodge-Riemann bilinear form $Q$ among $H^{1,1}(M, \mathbb{Z})$ [23],

$$Q(\eta, \eta) = \int_M \eta \wedge \eta \wedge \omega^{n-2} = -(n-2)!c_2. \tag{3.41}$$

Thus, we should find the every solution $\pm \eta \in H^{1,1}(M, \mathbb{Z})$ of the bilinear form to determine the values $f_c$ at the critical points. If $c_1(\mathcal{U})$ is a solution, the value of $I(A)$ is given by

$$I(A) = -\frac{n^2}{8\pi^2\varepsilon} \int_M \frac{\omega^n}{n!} \text{Tr} f^2 = \frac{n}{\varepsilon(n-1)!} \left( \frac{\deg(\mathcal{U})^2}{\int \omega^n} \right) = \frac{n}{\varepsilon(n-1)!} m^2 \tag{3.42}$$

where $m$ is a certain integer which can be determined if we can solve (3.41). Eq.(3.40) shows that a reducible EH connection is a solution of

$$Q(\eta_\perp, \eta_\perp) = -(n-2)!c_2, \tag{3.43}$$

where $\eta_\perp \in P^{1,1}(M, \mathbb{Z})$. From the Hodge-Riemann bilinear relations, we have

$$(-1)^{(2n-2)(2n-3)/2} Q(\eta_\perp, \eta_\perp) > 0. \tag{3.44}$$

Since $c_2 > 0$ to admit the EH connections, we can conclude that there are no reducible EH connections if $(2n-2)(2n-3)/2$ is an non-zero even integer.

**Application to the Donaldson Invariants**

The most interesting mathematical application of the $N = 2$ HYM theory is to the Donaldson polynomial invariants. Let $S$ be a simply connected algebraic surface with a Hodge metric and associated Kähler form $\omega \in H^{1,1}(S, \mathbb{Z})$. We further assume that the geometric genus $p_g(S) > 0$ is strictly positive. Let $H$ be an algebraic cycle Poincaré dual to $\omega$. Then $H$ is an ample divisor that some integer multiple of $H$ is the hyperplane section of an suitable embedding $S \subset \mathbb{CP}^m$. Let $E$ be a (complex) vector bundle over $S$ with reduction of structure group to $SU(2)$. The bundle $E$ is classified by the instanton number $k = < c_2(E), S >$ which will be always assumed to be strictly positive. The topological observables $\tilde{\omega}$ and $\Theta$ define the Donaldson $\mu$-maps

$$\mu : H_2(S, \mathbb{Z}) \to H^{1,1}(A_k^+ / \mathcal{G}, \mathbb{Z}),$$

$$\mu : H_0(S, \mathbb{Z}) \to H^{2,2}(A_k^+ / \mathcal{G}, \mathbb{Z}). \tag{3.45}$$
The expectation value \(< \exp(\tilde{\omega} + \varepsilon\Theta) >\) evaluated in the original \(N = 2\) TYM theory corresponds to the Donaldson invariants

\[
< \exp(\tilde{\omega} + \varepsilon\Theta) > = \frac{1}{\#Z} \sum_{r,s}^{r+2s=d} \varepsilon^s q_k,s(H_{1}, \ldots, H_{r}, pt_{1}, \ldots, pt_{s})
\]

\[
= \frac{1}{\#Z} \sum_{r,s}^{r+2s=d} \frac{\varepsilon^s}{r!s!} < \mu(H) \cdots \mu(H') \cdots \mu(pt) \cdots \mu(pt), [\mathcal{M}_k] > .
\]

(3.46)

where \(d = 4k - 3(1 + p_g(S))\) denotes the complex dimension of the moduli space \(\mathcal{M}\) of anti-self-dual connections. Mathematically, the Donaldson invariants \(q_k,s(H_{1}, \ldots, H_{r}, pt_{1}, \ldots, pt_{s})\) is well defined for \(p_g(S) > 0\) and for large enough \(k\). The former condition is that there are no reducible instantons and the latter one ensures that the Hodge metric behaves as a generic metric so the \(H^2(\text{End}_0(E_A))\) cohomology is trivial.

The expectation value \(< \exp(\tilde{\omega} + \varepsilon\Theta) >'\) evaluated by the deformed \(N = 2\) TYM theory, which is identical to (3.46) up to exponentially small term, can be represented by the partition function \(Z(\varepsilon)_{k}\) of the \(N = 2\) HYM theory without the topological term \(k/\varepsilon\) in the action. With the conditions we have stated above, we have a reduction similar to (3.32),

\[
Z(\varepsilon)_{k} = \frac{1}{\text{vol}(G)} \int_{\mathcal{A}^{1,1}} \text{DA} \exp \left( \frac{n}{8\pi^2\varepsilon} \int_{M} \omega^n \frac{n}{n!} \text{Tr} f^2 \right) = \langle \exp(\tilde{\omega} + \varepsilon\Theta) \rangle + O(\exp(-c/\varepsilon)).
\]

(3.47)

Thus, the partition function \(Z(\varepsilon)_{k}\) in a small \(\varepsilon\) limit becomes

\[
Z(\varepsilon)_{k} = \frac{1}{\#Z} \sum_{r,s}^{r+2s=d} \frac{\varepsilon^s}{r!s!} q_k,s(H_{1}, \ldots, H_{r}, pt_{1}, \ldots, pt_{s}) + \text{exponentially small terms}.
\]

(3.48)

4. Further Studies

We have seen that the \(N = 2\) TYM theory is equivalent to the \(N = 2\) HYM theory. Using this equivalence, we can obtain general expressions of certain intersection parings on the moduli space of EH connections on a compact Kähler manifold by solving the \(N = 2\) HYM theory. The question is whether we can evaluate the partition function exactly even for \(n > 1\) cases. We may use the fixed point theorem of Witten [24] in evaluating the partition function, since the \(N = 2\) HYM theory has global fermionic symmetry. To do actual calculation, we should be able to find every HYM connections. However, this will be difficult in general for the \(n > 2\) cases. Furthermore, we do not understand many
properties of the moduli space of EH connections in those cases. The most tractable case is the $N = 2$ HYM theory on algebraic surfaces with the structure group $SU(2)$ [25].

We have also seen that the $N = 2$ HYM theory is a natural higher dimensional analogue of physical Yang-Mills theory in two dimensions. It will be interesting to examine whether other noble properties of the two dimensional Yang-Mills theory not studied in this paper remain valid in the $N = 2$ HYM theory; i) It is known that the action functional of Yang-Mills theory in two dimensions is an equivariantly perfect Morse functional [26]. This is almost followed from the fact that the action functional is the norm squared of the moment map [27]. Thus, we can expect that the action functional of the HYM theory is a perfect Morse functional even for the higher dimensional cases. Then, it will be possible to obtain the Poincaré polynomials of the moduli space of EH connections. ii) Recently, Gross et. al. showed that two dimensional Yang-Mills theory has a simple interpretation as a string theory[28]. It will be interesting to explore the possible stringy behavior of the $N = 2$ HYM theory, which will be much easier than the case of physical Yang-Mills theory.

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