Folded Strings
Falling Into a Black Hole

Itzhak Bars and Jürgen Schulze

Department of Physics and Astronomy
University of Southern California
Los Angeles, CA 90089-0484, USA

Abstract

We find all the classical solutions (minimal surfaces) of open or closed strings in any two dimensional curved spacetime. As examples we consider the SL(2,R)/R two dimensional black hole, and any 4D black hole in the Schwarzschild family, provided the motion is restricted to the time-radial components. The solutions, which describe longitudinal oscillating folded strings (radial oscillations in 4D), must be given in lattice-like patches of the worldsheet, and a transfer operation analogous to a transfer matrix determines the future evolution. Then the swallowing of a string by a black hole is analyzed. We find several new features that are not shared by particle motions. The most surprising effect is the tunneling of the string into the bare singularity region that lies beyond the black hole that is classically forbidden to particles.

1 Research supported in part by the DOE Grant No. DE-FG03-84ER-40168.
2 Supported by the DAAD (Doktorandenstipendium HSP II/AUFE)
1 Introduction

The geodesics of a point particle falling into a black hole are well understood, but little is known about the “geodesics” of strings falling into a black hole. Since a string has internal degrees of freedom, its motion in the vicinity of a black hole could have new features that cannot be guessed by the study of particles. Such a string could be a model for an extended body with internal degrees of freedom, or particles in a gas with internal interactions, that are being swallowed by a black hole. Furthermore, it is possible to imagine very long strings, in definite normal modes, with one end inside the black hole and the other one outside the horizon. Can one learn more about the properties of the black hole by “fishing” with such strings?

More generally, the motion of strings in curved spacetime is basically not known. Finding the classical solutions would be helpful for understanding and interpreting the quantum theory, which is of interest for Cosmology and Unification ideas. Fortunately, there are some special curved spacetimes based on gauged WZW models (GWZW) [1] with a single time coordinate [2] for which it is possible to obtain the complete set of classical solutions [3], and in principle the quantum solutions. One of the additional purposes of our paper is to begin a study of such models in more detail and provide some interpretation of the classical solutions.

In this paper we present all the classical string solutions for any curved 2D spacetime. The two dimensions could be interpreted as the time-radial coordinates of any spherically symmetric four dimensional spacetime, with the motion occuring at constant angles $d\theta = d\phi = 0$. Examples include the $SL(2,R)/R$ black hole and the Schwarzschild type curved spaces with general metric of the form $ds^2 = -dt^2 f(r) + dr^2 / f(r)$. These examples allow us to investigate the question of how a string falls into a black hole. Some work has already been done in this direction by concentrating on special solutions involving circular strings [4]. Instead, we will investigate longitudinal motions of strings, and we will obtain all the solutions. Our classical solutions describe folded strings with the folds oscillating against each other while the whole string is being attracted by the black hole. There are many new physical features, but the main surprise is the tunneling of the string into the bare
singularity region of spacetime that lies beyond the black hole and which is forbidden to particles.

We find that naive solutions can be defined only in patches of the worldsheet while complete solutions are obtained by matching boundary conditions at the boundaries of the patches. The patches correspond to a lattice type structure on the worldsheet, with the lattice being different for each distinct solution. The lattice is dynamically defined by the initial conditions that are provided by folded string motions in the asymptotically flat spacetime. The future development of the initial condition is given in the form of a transfer operation, analogous to a transfer matrix on the lattice, but with the transfer corresponding to a period of oscillation of the folds on the string. The patching procedure is applied explicitly and a complete solution is obtained and physically analyzed.

A word of caution: In this paper the classical theory of strings in curved spacetime is studied on its own merit. However, when the string reaches the black hole the correct physics may require a full quantum treatment as well as the back reaction of the matter on the black hole. These issues which may be important are not studied here. The effects found in this paper are therefore only tentative, however we feel that they signal some new phenomena.

The paper is organized as follows. In section 2 we present the general string solution in any 2D metric (which may be the \((r,t)\) restriction of a 4D metric). In section 3 the same solution is obtained with special methods appropriate for gauged WZW models. In section 4 a cell decomposition of the worldsheet is introduced using the yo-yo solution in flat spacetime as an example. In section 5.1 the boundary matching procedure is discussed for any metric. In section 5.2 the procedure is applied explicitly on the 2D black hole based on \(SL(2,R)/R\) to obtain a complete solution and derive the transfer operation. In section 5.3 the transfer matrix is constructed and shown that a discrete version of the minimal area is one of its invariants. In section 5.4 general solutions are described. Finally the physics is discussed in section 6 where the motion is described with the help of figures and the new features are emphasized.
2 String solution in any 2D metric

Let us consider a string $x^\mu(\tau, \sigma)$ propagating in a curved spacetime manifold. The string Lagrangian in the conformal gauge is given by

$$L = \partial_+ x^\mu \partial_- x^\nu G_{\mu\nu}(x).$$

(2.1)

where $\partial_\pm = (\partial_\tau \pm \partial_\sigma)/\sqrt{2}$. The equations of motion are just the geodesic equations for a string. In addition, these are supplemented with constraints that come from the reparametrization invariance of the theory (zero energy-momentum tensor)

$$G_{\mu\nu} \partial_+ x^\mu \partial_+ x^\nu = 0 = G_{\mu\nu} \partial_- x^\mu \partial_- x^\nu.$$  

(2.2)

The general string Lagrangian may include an antisymmetric tensor, a dilaton, tachyon or other string condensates. In 2D the antisymmetric tensor can be eliminated since it corresponds to a total divergence. All of the other condensates are generated by quantum corrections. However, our purpose here is to study the classical limit of the theory in the absence of all these fields. Therefore we will define the classical propagation of strings in curved spacetime through the classical equations of motion and constraints given above.

We may always redefine the string field by a coordinate reparametrization $x^\mu(\tau, \sigma) = x^\mu(y(\tau, \sigma))$. Substituting it into the Lagrangian we see that this gives a new target space metric in the coordinates $y^\mu(\tau, \sigma)$ that is related to the old one by a general coordinate transformation $(\partial x^\lambda/\partial y^\mu)(\partial x^\sigma/\partial y^\nu) G_{\lambda\sigma}(y)$. In 2D it is always possible to choose $x^\mu(y)$ such that the new metric is conformal. Therefore, we might as well begin our analysis with the conformal metric in target space $G_{\mu\nu} = G(x) \eta_{\mu\nu}$.

It is convenient to define the lightcone coordinates

$$u = \frac{1}{\sqrt{2}}(x^0 + x^1), \quad v = \frac{1}{\sqrt{2}}(x^0 - x^1)$$

(2.3)

Then the equations of motion and constraints take the form

$$\partial_+(G \partial_- u) + \partial_- (G \partial_+ u) = \frac{\sqrt{2}}{2G} (\partial_+ u \partial_- v + \partial_- v \partial_+ u)$$

$$\partial_+(G \partial_- v) + \partial_- (G \partial_+ v) = \frac{\sqrt{2}}{2G} (\partial_+ u \partial_- v + \partial_- v \partial_+ u)$$

$$\partial_+ u \partial_+ v = 0 = \partial_- u \partial_- v.$$  

(2.4)
There are four classes of solutions that may be verified explicitly by substitution into the differential equation:

\begin{align*}
A : & \quad u = u(\sigma^+), \quad v = v(\sigma^-) \\
B : & \quad u = \tilde{u}(\sigma^-), \quad v = \tilde{v}(\sigma^+) \\
C : & \quad u = c_1, \quad v = v(a(\sigma^-), b(\sigma^+)) \\
D : & \quad u = u(\tilde{a}(\sigma^+), \tilde{b}(\sigma^-)), \quad v = c_2,
\end{align*}

where the functions \(u(\sigma^+), v(\sigma^-), \tilde{u}(\sigma^-), \tilde{v}(\sigma^+), a(\sigma^-), b(\sigma^+), \tilde{a}(\sigma^+), \tilde{b}(\sigma^-)\) are arbitrary and \(c_1, c_2\) are constants. Solutions \(A, B\) are present for any metric and do not depend on \(G\), but solutions \(C, D\) depend on the metric as follows

\begin{align*}
C : & \quad u = c_1, \quad \int u(\sigma^+) c_1 \, dv' G(c_1, v') = a(\sigma^+) + \beta(\sigma^-), \\
D : & \quad v = c_2, \quad \int u(\sigma^+) c_2 \, dv G(u', c_2) = \tilde{a}(\sigma^+) + \tilde{\beta}(\sigma^-),
\end{align*}

where the integration is performed at constant \(u = c_1\) for solution \(C\), and at constant \(v = c_2\) for solution \(D\). The functions \(a, \beta, \tilde{a}, \tilde{\beta}\) are arbitrary. The arbitrary functions \(a(\sigma^-), b(\sigma^+), \tilde{a}(\sigma^+), \tilde{b}(\sigma^-)\) that appear in (2.5) are these or their reparametrizations that may be more convenient. Taking a derivative of these integrals with respect to \(\sigma^\pm\) gives relations that solve the equations of motion.

For a given explicit metric \(G(uv)\) one obtains a metric dependent solution \(C\) or \(D\) after performing these integrals. For example, for the flat metric \(G = 1\) one has the obvious solution \(u(\sigma^+, \sigma^-) = a(\sigma^+) + \beta(\sigma^-)\), or \(v(\sigma^+, \sigma^-) = \tilde{a}(\sigma^+) + \tilde{\beta}(\sigma^-)\). Similarly, for the 2D black hole one has \(G = (1 - uv)^{-1}\), and after performing the integrals and doing some reparametrizations one obtains the solutions

\begin{align*}
C : & \quad u = c_1, \quad v = \frac{1}{a} [1 - a(\sigma^-) \, b(\sigma^+)], \\
D : & \quad v = c_2, \quad u = \frac{1}{\tilde{a}} [1 - \tilde{a}(\sigma^+) \, \tilde{b}(\sigma^-)].
\end{align*}

The solutions for the 2D black hole metric were obtained with different methods by various groups at different stages. The first attempts were in the context of the 2D black hole [5], which later led to the general classical solution for any gauged WZW model [3] (see also next section). This approach, which gives the complete set of solutions, leads directly to the set above. In
an independent approach, the string equations for the 2D black hole were also solved directly in [6]. On the basis of their solutions, the authors of [6] came to the conclusion that there are only particle solutions (i.e., only collapsed strings) when the constraints (2.2) are taken into account. However, as we will discuss below there are an infinite number of stringy solutions in the 2D black hole background. Our conclusions are quite different because we take into account folded strings. In fact, we advertised some time ago in [7], that, in the black hole curved spacetime, we obtained the analogs of the folded string solutions that were discussed by BBHP [12] in flat spacetime.

Here we also give the string solutions for the Schwarzschild-like metrics of the form

\[ ds^2 = f(r) \, dt^2 - \frac{1}{f(r)} \, dr^2 = G(uv) \, du \, dv \quad (2.8) \]

Examples are \( f(r) = 1 - 1/r \) for the Schwarzschild black hole, and \( f(r) = 1 - 1/r + q^2/r^2 \) for the charged black hole, etc.. The transformation to Kruskal coordinates gives

\[ uv = -\exp \left( 2 \int^r \frac{dr'}{f(r')} \right) \quad , \quad G(uv) = -f(r) \exp \left( -2 \int^r \frac{dr'}{f(r')} \right) \quad , \quad (2.9) \]

and \( u/v = \exp(2t) \). In particular, for the Schwarzschild black hole case \( uv = (1 - r) \exp(r) \) and \( G = \exp(-r)/r \). Once written in Kruskal coordinates we analytically continue to the global space that includes all spacetime regions, including those outside the horizon, inside the horizon and the naked singularity region. Furthermore, it may be necessary to take multicoovers of the \( (u, v) \) space in order to have a geodesically complete manifold.

Having identified the conformal factor, the \( C, D \) type string solutions are obtained by performing the integrals in (2.6) as follows

\[ \int^{u(\sigma^+, \sigma^-)} u' G(u' c_2) = \int^{r(u_c)} r' \frac{\partial u'}{\partial r'} G(r') = \frac{1}{c_2} r(u_c) = \hat{a}(\sigma^+) + \hat{\beta}(\sigma^-) \quad (2.10) \]

where we have used, \( G(r)[\partial(u_c)/\partial r] = 1 \), so that the integral is easily done for any \( f(r) \). Replacing this result back in the relation between \( uv \) and \( r \) we
get the string solutions, e.g. for the Schwarzschild metric,

\[ C : \quad u = c_1 \quad v(a(\sigma^-), b(\sigma^+)) = \frac{1}{c_1} a(\sigma^-) b(\sigma^+) \left[ 1 - \ln(a(\sigma^-) b(\sigma^+)) \right] \]

\[ D : \quad v = c_2 \quad u(\bar{a}(\sigma^+), \bar{b}(\sigma^-)) = \frac{1}{c_2} \bar{a}(\sigma^+) \bar{b}(\sigma^-) \left[ 1 - \ln(\bar{a}(\sigma^+) \bar{b}(\sigma^-)) \right] \]

where we reparametrized the arbitrary functions to more convenient forms.

The solutions (2.5-2.11) are not yet string solutions. A string solution must be periodic in \( \sigma \), and furthermore for a valid classical description the global time coordinate \( T(\tau, \sigma) = (u + v)/\sqrt{2} \) must be an increasing function of \( \tau \) for any \( \sigma \). Backtracking solutions must be excluded from the classical theory (at least in the \( uv < 1 \) region), just as they are excluded for the free relativistic point particle, or the free string in flat spacetime \(^3\). These two requirements turn out to be incompatible for any of the solutions above, if a single solution is required to be valid everywhere on the worldsheet. Therefore we have to consider their validity in patches of the worldsheet and then learn how to combine them through boundary matching conditions that turn them into valid string solutions. For these reasons we find it necessary to split the worldsheet into cells, as defined in section 4.

\(^3\)For the relativistic point particle one usually chooses the timelike gauge \( x^0(\tau) = p^0 \tau \), with positive energy \( p^0 = \sqrt{p^2 + m^2} \), in order to insure a time coordinate that increases with proper time. Of course, there are also negative energy solutions to the constraint \( p^0 = m^2 \) that describe anti-particles. By making the gauge choice to be monotonically increasing (or monotonically decreasing) in \( \tau \) one excludes the possibility of worldlines that backtrack on themselves. This is required on the basis of physical description in the classical theory so that particles and anti-particles do not appear in the same classical solution. The same remarks apply to strings, in the sense that worldsheets are not allowed to backtrack in the physical time direction. To achieve this, the physical time coordinate must always increase monotonically as a function of the proper time \( \tau \). In this way anti-strings are excluded from appearing in the same solution with strings in the classical description in flat or curved spacetime.

\(^4\)The \( C, D \) solutions are actually valid in the entire worldsheet but, if taken on their own, they describe a massless particle, not a string. This can be seen, e.g. for the \( C \) solution, by eliminating \( \tau \) in favor of the target time coordinate from \( T = (c_1 + v(\tau, \sigma))/\sqrt{2} \), and substituting into the space coordinate \( X = (c_1 + v(\tau, \sigma))/\sqrt{2} \), in order to get the target space relation between position and time \( X = (2c_1 - T)/\sqrt{2} \). This last relation describes the motion of a free massless particle, not a string, since the \( \sigma \) dependence has disappeared.
3 Classical strings and the gauged WZW model

From the above discussion it is not evident that the solutions presented form a complete set. For this reason we would like to use the curved spacetime approach based on $G/H$ gauged WZW models [2], in particular for $SL(2, R)/R$ [8], that describes a stringy 2D black hole [9]. In these models we can discuss the completeness and other issues with more convincing methods.

In previous work [10] it was shown that for a $G/H$ model the \textit{global} sigma model variables (i.e. string coordinates) are those combinations of group parameters in $G$ that are invariant under the gauge group $H$. Therefore, we have argued that a classical solution to the \textit{gauge invariant} string coordinates can be obtained by analyzing the original gauged WZW classical equations in any convenient gauge. Using this technique, the general geodesics of a particle in the global manifold were given for any gauged WZW model. Similarly, all the string solutions were also given [3], but in the string case, the details of building the gauge invariants were left out of the explicit discussion.

The case of the 2D black hole is very simple since one can identify the global gauge invariant string coordinates as the Kruskal coordinates $(u, v)$ that parametrize the $SL(2, R)$ group element as

$$ g = \begin{pmatrix} u & a \\ -b & v \end{pmatrix}, \quad u v + a b = 1, \quad (3.1) $$

where we gauge a vector-like subgroup that acts as $g \rightarrow \Lambda g \Lambda^{-1}$, with

$$ \Lambda = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa^{-1} \end{pmatrix}. \quad (3.2) $$

Evidently $(u, v)$ are invariants under the action of the gauge group. Therefore classical solutions obtained for $(u, v)$ by using any gauge, will be valid as solutions in any other gauge. As discovered in [10], as long as one uses the gauge invariant global coordinates one captures all dual patches of the geometry, and therefore it is immaterial if one gauges a vector or an axial subgroup.

The classical equations of motion for the $G/H$ GWZW model are [3]

$$ (D_+ gg^{-1})_H = 0 = (g^{-1} D_- g)_H, \quad F_{+-} = 0, \quad D_-(D_+ gg^{-1})_{G/H} = 0. \quad (3.3) $$
Furthermore, the conformal (Virasoro) constraints require a vanishing stress tensor

\[
Tr \left[ (D_+ g^{-1})_G \right]^2 = 0 = Tr \left[ (g^{-1} D_- g)_G \right]^2.
\] (3.4)

For the \( SL(2, R)/R \) case these equations may be explicitly given in the form

\[
\begin{align*}
(v \partial_+ u + b \partial_+ a) - 2ab A_+ &= 0 \\
(u \partial_+ v + b \partial_+ a) - 2ab A_- &= 0 \\
(\partial_- + 2A_-)(v \partial_+ b + 2vb A_+) &= 0 \\
(\partial_- + 2A_-)(u \partial_+ a - 2ua A_+) &= 0 \\
\partial_- A_- - \partial_+ A_+ &= 0 \\
(v \partial_- a - 2va A_-)(u \partial_- b + 2ub A_-) &= 0 \\
(u \partial_- a - 2ua A_+)(v \partial_- b + 2vb A_+) &= 0
\end{align*}
\] (3.5)

They may be rewritten in several other forms by using the determinant condition in (3.1) and its derivative \( \partial_\pm (uv + ab) = 0 \). To solve these equations, one approach is to choose the gauge \( a = b \) (or \( a = -b \) ), solve the algebraic equations for \( A_\pm \) and recognize the remaining equations for \( u, v \) as the equations (2.4) of the string coordinates moving in the black hole. The solution to these equations are given in (2.5, 2.7). Another approach advocated in [3] is to choose the axial gauge \( A_\pm = 0 \), which leads through the equations of motion to also \( A_- = 0 \), and then analyse the remaining equations for the solution of \( (u, v, a, b) \). Since the string coordinates \( u, v \) are gauge invariant the solution obtained in one gauge is as good as the solution obtained in any other gauge.

In the axial gauge \( A_\pm = 0 \) the solution can be written as follows. First, the group element \( g(\tau, \sigma) \) is given as the product of left and right moving group elements \( g_L(\tau + \sigma) \) and \( g_R(\tau - \sigma) \) as

\[
g = g_L g_R^{-1}
\] (3.6)

\[
\begin{pmatrix}
 u & a \\
 -b & v
\end{pmatrix} = \begin{pmatrix}
 u_L & a_L \\
 -b_L & v_L
\end{pmatrix} \begin{pmatrix}
 v_R & -a_R \\
 b_R & u_R
\end{pmatrix}
\] (3.7)

therefore

\[
\begin{align*}
u &= u_L v_R + a_L b_R \\
a &= a_L u_R - u_L a_R \\
b &= b_L v_R - v_L b_R \\
v &= v_L u_R + b_L a_R
\end{align*}
\] (3.8)
where there are determinant constraints \( u_L v_L + a_L b_L = 1 = u_R v_R + a_R b_R \).

Then, the left and right moving currents take the form
\[
\partial_+ g_L g_L^{-1} = \begin{pmatrix}
  v_L \partial_+ u_L + b_L \partial_+ a_L & u_L \partial_+ a_L - a_L \partial_+ u_L \\
  b_L \partial_+ v_L - v_L \partial_+ b_L & u_L \partial_+ v_L + a_L \partial_+ b_L
\end{pmatrix}, \tag{3.9}
\]

and similarly for \( g_R \), with the derivative replaced by \( \partial_- \). Note that after taking into account the determinant conditions \( a_L b_L + u_L v_L = 1 \), and similarly for right movers, the diagonal entries for the currents are proportional to the Pauli matrix \( \sigma_3 \). The equations above reduce to the following:

(i) the currents that belong to the subgroup \( H \) (i.e. proportional to \( \sigma_3 \)) vanish
\[
(\partial_+ g_L g_L^{-1})_H = v_L \partial_+ u_L + b_L \partial_+ a_L = -u_L \partial_+ v_L - a_L \partial_+ b_L = 0 \tag{3.10}
\]

and (ii) the conformal constraint becomes
\[
(u_L \partial_+ a_L) (b_L \partial_+ v_L) = 0, \tag{3.11}
\]

and similarly for right movers. Thus, the equations for left and right movers separate. A common solution of (3.10, 3.11) is necessarily either \( \partial_+ u_L = \partial_+ a_L = 0 \) or \( \partial_+ v_L = \partial_+ b_L = 0 \), and similarly for the right movers. Therefore the general solution for left and right movers is of the form
\[
g_{L1} = \begin{pmatrix}
  -\frac{1}{a_{0L}} + \frac{u_{0L}}{a_{0L}} v_L(\sigma^+) & \frac{a_{0L}}{v_L(\sigma^+)} \\
  -\frac{1}{a_{0L}} + \frac{v_{0L}}{a_{0L}} u_L(\sigma^+) & \frac{v_{0L}}{u_L(\sigma^+)}
\end{pmatrix} \text{ or } g_{L2} = \begin{pmatrix}
  u_L(\sigma^+) & 1 \frac{v_{0L}}{a_{0L}} - \frac{u_{0L}}{v_{0L}} u_L(\sigma^+) \\
  -b_{0L} & -\frac{1}{b_{0L}} - \frac{v_{0L}}{b_{0L}} u_L(\sigma^+)
\end{pmatrix}
\]

\[
g_{R1}^{-1} = \begin{pmatrix}
  v_{0R} \sigma^- & u_{0R} \\
  -a_{0R} & b_{0R} \sigma^-
\end{pmatrix} \text{ or } g_{R2}^{-1} = \begin{pmatrix}
  u_{0R} & -\frac{1}{b_{0R}} - \frac{v_{0R}}{b_{0R}} u_{0R} \sigma^- \\
  -a_{0R} & b_{0R} \sigma^-
\end{pmatrix} \tag{3.12}
\]

Thus, four arbitrary functions \((u_L(\sigma^+), v_L(\sigma^+), u_R(\sigma^-), v_R(\sigma^-))\) together with initial condition constants \((u_{0L}, a_{0L}, b_{0L})\) and \((u_{0R}, v_{0R}, a_{0R}, b_{0R})\) parametrize all solutions.

Let us consider some interval of \((\sigma^+, \sigma^-)\) and build \( g(\tau, \sigma) \) by multiplying these solutions. There are four possibilities in this interval: \( g_A = g_{L1} g_{R2}^{-1} \), \( g_B = g_{L1} g_{R1}^{-1} \), \( g_C = g_{L1} g_{R2}^{-1} \), \( g_D = g_{L2} g_{R1}^{-1} \). After some rewriting these
take the form

$$
\begin{align*}
g_A(\tau, \sigma) &= \begin{pmatrix} u(\sigma^+) & \frac{1}{b_0} [1 - u(\sigma^+) v(\sigma^-)] \\ -b_0 & v(\sigma^-) \end{pmatrix} \\
g_B(\tau, \sigma) &= \begin{pmatrix} \bar{u}(\sigma^-) & a_0 \\ \frac{1}{a_0} [1 - \bar{u}(\sigma^+) \bar{v}(\sigma^+)] & \bar{v}(\sigma^+) \end{pmatrix} \\
g_C(\tau, \sigma) &= \begin{pmatrix} c_1 & a(\sigma^-) \\ -\bar{b}(\sigma^+) & \frac{1}{c_1} [1 - a(\sigma^-) \bar{b}(\sigma^+)] \end{pmatrix} \\
g_D(\tau, \sigma) &= \begin{pmatrix} \frac{1}{a_0} [1 - \bar{a}(\sigma^+) \bar{b}(\sigma^-)] & \bar{a}(\sigma^+) \\ -\bar{b}(\sigma^-) & c_2 \end{pmatrix}
\end{align*}
$$

The diagonal entries correspond to the $A, B, C, D$ solutions given in (2.5, 2.7). This method which gives a deeper insight also provides a proof that we have a complete set of solutions.

However, as already mentioned at the end of the previous section, these are not acceptable string solutions as they stand. We must take the solutions to be valid only in the patches for which $T$ increases, match boundary conditions between these patches and impose periodicity. To illustrate this we will first reexamine flat spacetime solutions.

## 4 Cells on the worldsheet

As a guide for acceptable curved spacetime string solutions we first consider the worldsheet structure for flat spacetime solutions. Since in the asymptotically flat region of any curved spacetime we must have these solutions as boundary conditions, this analysis will lead directly to the $A, B, C, D$ patterns that we are seeking on the worldsheet.

The general flat spacetime solution with a fixed center of mass was given in BBHP [12]

$$
x^0(\tau, \sigma) = \frac{M}{L} \tau, \quad x^4(\tau, \sigma) = q + \frac{M}{2L} [f(\tau + \sigma) + g(\tau - \sigma)],
$$

where the slopes of the functions $f' = \pm 1, g' = \pm 1$ may jump from one value to the other in patches of $\tau \pm \sigma$, but the functions $f, g$ are continuous, and
they are periodic. $\frac{M}{L}$ is the mass density of the string. The physical length of the string is proportional to its mass, which is $M$ for the open string and $2M$ for the closed string. The simplest yo-yo solution is

$$x^0(\tau, \sigma) = \frac{M}{L} \tau, \quad x^1(\tau, \sigma) = q + \frac{M}{2L} \left[ |\tau + \sigma|_{\text{per}} + |\tau - \sigma|_{\text{per}} \right], \quad (4.2)$$

where the absolute value is repeated periodically with a period of $\Delta \sigma = L = 4$.

These solutions manifestly satisfy the physical requirements of periodicity in $\sigma$ and forward propagation in $\tau$ (i.e. $x^0(\tau, \sigma)$ monotonically increases with $\tau$, for all $\sigma$). Our aim in this section is to clarify how these properties together with the $A, B, C, D$ solutions that are valid only in patches. For this purpose it is convenient to rewrite (4.2) in the form

$$u(\sigma^+, \sigma^-) = u_0 + \frac{p^+}{2} \left( (\sigma^+ + |\sigma^+|_{\text{per}}) + (\sigma^- + |\sigma^-|_{\text{per}}) \right)$$

$$v(\sigma^+, \sigma^-) = v_0 + \frac{p^-}{2} \left( (\sigma^+ - |\sigma^+|_{\text{per}}) + (\sigma^- - |\sigma^-|_{\text{per}}) \right) \quad (4.3)$$

For $p^+ = p^- = \sqrt{2}M/L$ the two expressions agree, however (4.3) is slightly more general in that it also includes the motion of the center of mass when $p^+ \neq p^-$. The periodic absolute value function $|\sigma^+|_{\text{per}}$ is given by $(\pm \sigma^+ - \text{const.})$ where the value of the constant and the $\pm$ signs depend on regular intervals in $\sigma^+$, and similarly for $\sigma^-$. Therefore, the two sets of signs lead to four possible types of intervals in which the expressions for $u(\sigma^+, \sigma^-)$ and $v(\sigma^+, \sigma^-)$ take four different looking forms. These forms are precisely of the $A, B, C, D$ types, specialized to flat spacetime

$$\begin{align*}
(+, +) & : A: \quad u = u_A + p^+_A \sigma^+, \quad v = v_A + p^-_A \sigma^- \\
(-, +) & : B: \quad u = u_B + p^+_B \sigma^-, \quad v = v_B + p^-_B \sigma^+ \\
(-, -) & : C: \quad u = u_C, \quad v = v_C + p^+_C \sigma^+ + p^-_C \sigma^- \\
(+, -) & : D: \quad v = v_C, \quad u = u_D + p^+_D \sigma^+ + p^-_D \sigma^-
\end{align*} \quad (4.4)$$

where the momenta are all related $p^+_A = p^+_B = p^+_C = p^+_D = p^-$.

To be more precise, we need to define a cell decomposition of the worldsheet and give the value of $u, v$ in each cell as a map from the worldsheet to target spacetime. The patterns of the $A, B, C, D$ solutions assigned to these
cells is the key for obtaining the physical solutions that satisfy the periodicity and forward propagation requirements. Therefore, we now turn to the definition of the cells and the patterns.

The solution given in (4.3) decomposes the worldsheet in the following way. The worldsheet labelled by $\sigma$ horizontally and by $\tau$ vertically is sliced by $45^\circ$ lines that form a light-cone lattice with equal spacings in $\sigma^\pm$. The crosses in figure (4.5) below represent the corners of the cells on the worldsheet.

Each cell on the worldsheet is labelled by the values of $(\sigma^+, \sigma^-)$ at the center of the cell, divided by a factor of $\sqrt{2}$. For example at the center of the cell labelled by $(1, 2)$ the worldsheet coordinates are $\sigma^+ = \sqrt{2}$, $\sigma^- = 2\sqrt{2}$. Furthermore the values of $(\tau, \sigma)$ at any point on the sheet are $\tau = (\sigma^+ + \sigma^-)/\sqrt{2}$ and $\sigma = (\sigma^+ - \sigma^-)/\sqrt{2}$. So, at the center of the cell labelled as $(m, n)$ the worldsheet coordinates are $\tau = m + n$, $\sigma = m - n$. The points inside the cell $(m, n)$ are parametrized by $\sigma^\pm$ in the ranges

\[
(m - \frac{1}{2})\sqrt{2} < \sigma^+ < (m + \frac{1}{2})\sqrt{2}, \quad (n - \frac{1}{2})\sqrt{2} < \sigma^- < (n + \frac{1}{2})\sqrt{2}. \quad (4.6)
\]

We define the periodic functions $\sigma^\pm_{\text{per}}$ that take the following values in cell $(m, n)$

\[
\sigma^+_{\text{per}} = \sigma^+ - m\sqrt{2}, \quad \sigma^-_{\text{per}} = \sigma^- - n\sqrt{2}, \quad \text{periodic.} \quad (4.7)
\]
This is the periodic saw-tooth function that has slope $+1$.

With these definitions, the solution (4.2-4.4) can be rewritten in the following form:

\[
\begin{align*}
A(0, 0) : & \quad u_{00} = u_0 + p^+ \sigma^+_{\text{per}}, & \quad v_{00} = v_0 + p^- \sigma^-_{\text{per}} \\
B(1, -1) : & \quad u_{1,-1} = u_0 + p^+ \sigma^-_{\text{per}}, & \quad v_{1,-1} = v_0 + p^- \sigma^+_{\text{per}} \\
D(0, 1) : & \quad u_{01} = (u_0 + e_{01}^+ + p^+ (\sigma^+_{\text{per}} + \sigma^-_{\text{per}})), & \quad v_{01} = v_0 + \frac{p^-}{\sqrt{2}} \\
C(1, 0) : & \quad u_{10} = u_0 + \frac{p^-}{\sqrt{2}}, & \quad v_{10} = (v_0 + \frac{p^-}{\sqrt{2}}) + p^- (\sigma^+_{\text{per}} + \sigma^-_{\text{per}}) \\
B(1, 1) : & \quad u_{11} = (u_0 + 2 \frac{p^-}{\sqrt{2}}) + p^+ \sigma^+_{\text{per}}, & \quad v_{11} = (v_0 + 2 \frac{p^-}{\sqrt{2}}) + p^- \sigma^+_{\text{per}} \\
A(2, 0) : & \quad u_{20} = (u_0 + 2 \frac{p^-}{\sqrt{2}}) + p^+ \sigma^-_{\text{per}}, & \quad v_{20} = (v_0 + 2 \frac{p^-}{\sqrt{2}}) + p^- \sigma^-_{\text{per}} \\
D(2, 1) : & \quad u_{21} = (u_0 + 3 \frac{p^-}{\sqrt{2}}) + p^+ (\sigma^+_{\text{per}} + \sigma^-_{\text{per}}), & \quad v_{21} = v_0 + 3 \frac{p^-}{\sqrt{2}} \\
C(1, 2) : & \quad u_{12} = u_0 + 3 \frac{p^-}{\sqrt{2}}, & \quad v_{12} = (v_0 + 3 \frac{p^-}{\sqrt{2}}) + p^- (\sigma^+_{\text{per}} + \sigma^-_{\text{per}}) \\
A(2, 2) : & \quad u_{22} = (u_0 + 4 \frac{p^-}{\sqrt{2}}) + p^+ \sigma^+_{\text{per}}, & \quad v_{22} = (v_0 + 4 \frac{p^-}{\sqrt{2}}) + p^- \sigma^+_{\text{per}} \\
B(3, 1) : & \quad u_{31} = (u_0 + 4 \frac{p^-}{\sqrt{2}}) + p^+ \sigma^-_{\text{per}}, & \quad v_{31} = (v_0 + 4 \frac{p^-}{\sqrt{2}}) + p^- \sigma^-_{\text{per}}
\end{align*}
\]

This list is the same solution as (4.3-4.4) and it illustrates the procedure for building the solution. Namely, first assign the patterns to insure forward propagation, then impose periodicity at a fixed value of $\tau$, e.g. at $\tau = 0$, and then obtain the propagation into the future by simply matching boundary conditions while increasing $\tau$.

The general solution in (4.1), that has more folds on the string, defines a corresponding pattern of $A, B, C, D$ on the worldsheet. Given correct boundary conditions in the asymptotic region, these patterns are guarantied to correspond to strings that propagate forward in time. This is understood intuitively by considering that curved spacetime is a continuous deformation of flat spacetime except for singularities, and that the flat solution is manifestly periodic and forward propagating. These decompositions of the worldsheet into cells is the key for building the physical string solutions in curved spacetime. We emphasize that we must consider precisely these patterns if we want to describe the motion of strings prepared by observers in the asymptotic region.
5 Boundary matching

5.1 General approach for any metric

We have seen that the four types of solutions are given in the cells defined naturally by the periodic saw-tooth functions $\sigma_{\text{per}}^\pm$:

\[
\begin{align*}
A : & \quad u = u(\sigma_{\text{per}}^+), \quad v = v(\sigma_{\text{per}}^-) \\
B : & \quad u = \bar{u}(\sigma_{\text{per}}^-), \quad v = \bar{v}(\sigma_{\text{per}}^+) \\
C : & \quad u = c_1, \quad v = v(\bar{a}(\sigma_{\text{per}}^+), \bar{b}(\sigma_{\text{per}}^-)) \\
D : & \quad u = u(a(\sigma_{\text{per}}^-), b(\sigma_{\text{per}}^+)), \quad v = c_2.
\end{align*}
\] (5.1)

For a classical description of a string the time coordinate $T(\tau, \sigma) = (u + v)/\sqrt{2}$ must increase as a function of $\tau$ for all $\sigma$. To achieve this we must start from a cell decomposition of the worldsheet such that in each cell one of the solutions is valid, and then connect them by continuity across the boundaries. The patching of the solutions must respect two important requirements:

\[(i) \text{ periodicity in } \sigma, \text{ and } (ii) \text{ forward propagation.} \] (5.2)

Satisfying these two requirements turns out to be rather non-trivial. However, the folded string solutions of [12] provide a guide to construct the curved spacetime solutions. Without this guide it seems bewildering what solution corresponds to each cell. As an example we start with the simplest yo-yo solution (4.2) in the asymptotic region, and use it to assign the $A, B, C, D$ solutions to various patches according to (4.5).

The $A$ and $B$ solutions are present for any metric in target spacetime, while the form of the $v(a, b)$ or $u(\bar{a}, \bar{b})$ in the $C, D$ solutions depend on the metric $G(u, v)$ in target spacetime. Since there still is the freedom to fix the remaining conformal gauge invariance, we fix it by concentrating on solutions of type $A$ and $B$ that are independent of the target spacetime metric. It is convenient to fix the gauge so as to reproduce the well known solutions in flat spacetime in the asymptotic region where the metric is flat anyway. Therefore for solutions of type $A$ and $B$ we take the gauge fixed form

\[
\begin{align*}
A : & \quad u = p_A^+ \sigma_{\text{per}}^+ + q_A^+, \quad v = p_A^- \sigma_{\text{per}}^- + q_A^- \\
B : & \quad u = p_B^+ \sigma_{\text{per}}^+ + q_B^+, \quad v = p_B^- \sigma_{\text{per}}^- + q_B^-.
\end{align*}
\] (5.3)

The constants depend on the cell, and they can be chosen as initial conditions on only one cell. It turns out that there is no more remaining gauge degrees
of freedom, and the functions $a, b, \bar{a}, \bar{b}$ in solutions $C, D$ get fixed completely
by the forward propagation in $\tau$, and take a form which depends on the spacetime
metric.

We start by assigning initial conditions in cell $(0,0)$ by taking the $A$ type solution
\begin{equation}
  u_{00} = u_0 + p^+ \sigma^+_{\text{per}}, \quad v_{00} = v_0 + p^- \sigma^-_{\text{per}}
\end{equation}
Then in the neighboring cell, by continuity in $\sigma$, the $B(1,-1)$ solution must be
\begin{equation}
  u_{1,-1} = u_0 + p^+ \sigma^+_{\text{per}}, \quad v_{1,-1} = v_0 + p^- \sigma^+_{\text{per}}.
\end{equation}
This is sufficient to fix all the even and odd cells horizontally at $\tau = 0$ because of
periodicity in $\sigma$
\begin{equation}
  u_{2k,-2k} = u_0 + p^+ \sigma^+_{\text{per}}, \quad v_{2k,-2k} = v_0 + p^- \sigma^+_{\text{per}}
  \quad u_{2k+1,-2k-1} = u_0 + p^+ \sigma^-_{\text{per}}, \quad v_{2k+1,-2k-1} = v_0 + p^- \sigma^-_{\text{per}}.
\end{equation}
Next we go one level up to cells $(1,0)$ and $(0,1)$. Consider the boundary
between $(0,0)$ and $(1,0)$ which is at $\sigma^+_{\text{per}} = \frac{1}{\sqrt{2}}$, $\sigma^-_{\text{per}} = \text{any}$. Matching the
solutions of type $A$ and $C$ we find
\begin{equation}
  u_{10} = u_0 + \frac{p^+}{\sqrt{2}}, \quad v_{10}(a(\frac{1}{\sqrt{2}}), b(\sigma^-_{\text{per}})) = v_0 + p^- \sigma^-_{\text{per}}.
\end{equation}
A similar matching of solutions of type $B$ and $C$ at the boundary between
$(1,1)$ and $(1,0)$ gives
\begin{equation}
  u_{10} = u_0 + \frac{p^+}{\sqrt{2}}, \quad v_{10}(\bar{a}(\sigma^+_{\text{per}}), \bar{b}(-\frac{1}{\sqrt{2}})) = v_0 + p^- \sigma^+_{\text{per}}.
\end{equation}
Taken together these two conditions completely fix the functions $a_{10}(\sigma^+_{\text{per}})$
and $b_{10}(\sigma^-_{\text{per}})$, once the function $v(a, b)$ is given for some target spacetime
metric. Replacing these functions back into $v(a, b)$ one completely fixes $v_{10}$.
To make further progress we must specify the metric and the corresponding
solutions $u(\bar{a}, \bar{b})$, $v(a, b)$.
5.2 2D black hole metric

So, for the $SL(2,R)/R$ two dimensional black hole metric we have the form

$$v_{10} = (1 - a_{10}b_{10})/u_{10}.$$ Using the above procedure we find

$$u_{10} = u_0 + \frac{r^+}{\sqrt{2}},$$

$$v_{10} = \frac{1}{u_0 + \frac{r^+}{\sqrt{2}}} \left\{ 1 - \frac{(1 - (\omega_0 + \frac{r^+}{\sqrt{2}})(\omega_0 + p^+\sigma^+_{per})\times(1 - (\omega_0 + \frac{r^+}{\sqrt{2}})(\omega_0 + p^+\sigma^+_{per}))}{[1 - (\omega_0 + \frac{r^+}{\sqrt{2}})(\omega_0 + \frac{r^+}{\sqrt{2}})]} \right\} \quad (5.9)$$

A similar matching procedure for $B(-1,1) \nearrow D(0,1) \searrow A(0,0)$ gives

$$v_{01} = v_0 + \frac{r^-}{\sqrt{2}},$$

$$u_{01}(\sigma^+_{per}, \sigma^-_{per}) = \frac{1}{u_0 + \frac{r^-}{\sqrt{2}}} \left\{ 1 - \frac{(1 - (\omega_0 + \frac{r^-}{\sqrt{2}})(\omega_0 + p^+\sigma^+_{per})\times(1 - (\omega_0 + \frac{r^-}{\sqrt{2}})(\omega_0 + p^+\sigma^+_{per}))}{[1 - (\omega_0 + \frac{r^-}{\sqrt{2}})(\omega_0 - \frac{r^-}{\sqrt{2}})]} \right\} \quad (5.10)$$

By periodicity these solutions are extended to the rest of the even and odd cells horizontally at the same value of $\tau$. So, we may write

$$u_{2k+1,-2k} = u_0 + \frac{r^+}{\sqrt{2}}, \quad v_{2k+1,-2k} = v_{10}(\sigma^+_{per}, \sigma^-_{per}),$$

$$u_{-2k,2k+1} = u_{01}(\sigma^+_{per}, \sigma^-_{per}), \quad v_{-2k,2k+1} = v_0 + \frac{r^-}{\sqrt{2}} \quad (5.11)$$

We next climb one more level in $\tau$ and consider the cells $B(1,1)$, $A(2,0)$. Matching boundaries $D(0,1) \nearrow B(1,1) \searrow C(1,0)$ and $C(1,0) \nearrow A(2,0) \searrow D(2,-1)$ we get the solution

$$u_{11}(\sigma^-_{per}) = \frac{1}{u_0 + \frac{r^-}{\sqrt{2}}} \left\{ 1 - \frac{(1 - (\omega_0 + \frac{r^-}{\sqrt{2}})(\omega_0 + p^+\sigma^+_{per})\times(1 - (\omega_0 + \frac{r^-}{\sqrt{2}})(\omega_0 + p^+\sigma^+_{per}))}{[1 - (\omega_0 + \frac{r^-}{\sqrt{2}})(\omega_0 + \frac{r^-}{\sqrt{2}})]} \right\} \quad (5.12)$$

$$v_{11}(\sigma^+_{per}) = \text{similar to } u_{11} \text{ but replace } u_0 \leftrightarrow v_0, \quad p^+ \leftrightarrow p^-, \quad \sigma^-_{per} \rightarrow \sigma^+_{per},$$

$$u_{20}(\sigma^-_{per}) = \text{same as } u_{11} \text{ but replace } \sigma^-_{per} \rightarrow \sigma^+_{per},$$

$$v_{20}(\sigma^-_{per}) = \text{same as } v_{11} \text{ but replace } \sigma^+_{per} \rightarrow \sigma^-_{per} \quad (5.12)$$

By periodicity in $\sigma$ this solution is extended horizontally to all cells at the same $\tau$. Thus

$$u_{2k+1,-2k+1} = u_{11}(\sigma^-_{per}),$$

$$u_{2k+2,-2k} = u_{20} = u_{11}(\sigma^+_{per}),$$

$$v_{2k+1,-2k+1} = v_{11}(\sigma^+_{per}),$$

$$v_{2k+2,-2k} = v_{20} = v_{11}(\sigma^-_{per}), \quad (5.13)$$
In the next step we match $A(0, 2)$ \(\rightarrow\) $C(1, 2)$ \(\setminus\) $B(1, 1)$ and $B(1, 1)$ \(\rightarrow\) $D(2, 1)$ \(\setminus\) $A(2, 0)$. The result must be the same as starting in the cell $A(2, 2)$ and matching boundaries backward in $\tau$ with cell $C(1, 2)$. Therefore the result in $C(1, 2)$ can be written in the form that relates to the constants in cell $A(2, 2)$, that is

\[
\begin{align*}
  u_{12} &= \tilde{u}_0 - \frac{\hat{p}^+}{\sqrt{2}} \\
  v_{12}(\sigma^+_{\text{per}}, \sigma^-_{\text{per}}) &= \frac{1}{\tilde{u}_0 - \frac{\hat{p}^+}{\sqrt{2}}} \left( 1 - \frac{[1 - (u_0 + \frac{\hat{p}^+}{\sqrt{2}})(v_0 + \frac{\hat{p}^-}{\sqrt{2}})]^2}{[1 - (u_0 - \frac{\hat{p}^+}{\sqrt{2}})(v_0 + \frac{\hat{p}^-}{\sqrt{2}})]} \right) \\
  v_{21} &= \tilde{v}_0 - \frac{\hat{p}^-}{\sqrt{2}} \\
  u_{21}(\sigma^+_{\text{per}}, \sigma^-_{\text{per}}) &= \text{similar to } v_{12} \text{ but substitute } \tilde{u}_0 \leftrightarrow \tilde{v}_0, \quad \hat{p}^+ \leftrightarrow \hat{p}^- \quad (5.14)
\end{align*}
\]

where we have defined $A(2, 2)$ by $u_{22} = \tilde{u}_0 + \hat{p}^+ \sigma^+_{\text{per}}$ and $v_{22} = \tilde{v}_0 + \hat{p}^- \sigma^-_{\text{per}}$. By comparing this with the results of $A(0, 2)$ \(\rightarrow\) $C(1, 2)$ \(\setminus\) $B(1, 1)$ and $B(1, 1)$ \(\rightarrow\) $D(2, 1)$ \(\setminus\) $A(2, 0)$, the constants $\tilde{u}_0$, $\tilde{v}_0$, $\hat{p}^+$, $\hat{p}^-$ are related to the original initial conditions at $\tau = 0$:

\[
\begin{align*}
  \tilde{u}_0 - \frac{\hat{p}^+}{\sqrt{2}} &= \frac{1}{v_0 + \frac{\hat{p}^-}{\sqrt{2}}} \left( 1 - \frac{[1 - (u_0 + \frac{\hat{p}^+}{\sqrt{2}})(v_0 + \frac{\hat{p}^-}{\sqrt{2}})]^2}{[1 - (u_0 - \frac{\hat{p}^+}{\sqrt{2}})(v_0 + \frac{\hat{p}^-}{\sqrt{2}})]} \right) \\
  \tilde{v}_0 - \frac{\hat{p}^-}{\sqrt{2}} &= \text{same form but interchange } u_0 \leftrightarrow v_0, \quad p^+ \leftrightarrow p^- \quad (5.15) \\
  \hat{p}^+ &= \left\{ \frac{[(1 - (u_0 + \frac{\hat{p}^+}{\sqrt{2}})(v_0 + \frac{\hat{p}^-}{\sqrt{2}})]^2 - 2p^+ p^-}{[1 - (u_0 - \frac{\hat{p}^+}{\sqrt{2}})(v_0 + \frac{\hat{p}^-}{\sqrt{2}})]^2} \right\} p^+ \\
  \hat{p}^- &= \text{same form but interchange } u_0 \leftrightarrow v_0, \quad p^+ \leftrightarrow p^- 
\end{align*}
\]

The result in cell $A(2, 2)$ has the same form as the cell $A(0, 0)$ given in (5.4) except for replacing $\tilde{u}_0$, $\tilde{v}_0$, $\hat{p}^+$, $\hat{p}^-$ instead of $u_0, v_0, p^+, p^-$. Therefore the procedure can now be repeated by using the constants $\tilde{u}_0$, $\tilde{v}_0$, $\hat{p}^+$, $\hat{p}^-$ as new initial conditions. So, the forward propagation of the solution to future times $\tau \rightarrow \tau + 4$ is done through the formulas in (5.15) which serve the role of a transfer operation analogous to the transfer matrix of a lattice theory.

In the asymptotic region $u_0 \rightarrow \infty$, $v_0 \rightarrow -\infty$ (where $G \rightarrow \text{flat}$) the
formulas reduce to the results for a flat metric. Namely,

\[ p_0^\pm = p_0^\pm, \quad \tilde{v}_0 = u_0 + 4 \frac{p_0^+}{\sqrt{2}}, \quad \tilde{v}_0 = v_0 + 4 \frac{p_0^-}{\sqrt{2}} \quad (5.16) \]

and the propagation in flat space coincides with (4.8) as given in the previous section.

5.3 Transfer operator and invariant minimal area

Although we have derived this result by choosing a gauge in eq. (5.3) it is easy to verify that the constants \( u_{10}, v_{01}, u_{12}, v_{21} \) in the \( C, D \) type cells are independent of the gauge. That is, even if we had started from arbitrary functions in cell \( A(0,0) \), i.e. \( u_{00} = u(\sigma_{per}^+) \) and \( v_{00} = v(\sigma_{per}^-) \), the values of these functions at the boundaries of the cell is what defines \( u_0 \pm p_0^+ / \sqrt{2}, v_0 \pm p_0^- / \sqrt{2} \). Then the values of the constants

\[ u_{\pm 1,0} = u_0 \pm p_0^+ / \sqrt{2}, \quad v_{0,\pm 1} = v_0 \pm p_0^- / \sqrt{2} \quad (5.17) \]

have identical expressions, independent of the gauge. As we will see below, these boundary constants and their transfer to future times via the transfer operation completely determine the gauge invariant motion of the folds (or end points for an open string). The physical motion of the remainder of the string in between the folds is inferred from the motion of the folds. Since we can choose \( u(\sigma_{per}^+) \) and \( v(\sigma_{per}^-) \) arbitrarily, it is evident that the motion of the intermediate points is gauge dependent, and therefore has no physical meaning. The physically significant motion is the motion of the folds (or end points). These issues were understood for the flat case by BBHP.

In order to give a convenient expression for the transfer operation it is more convenient to concentrate on the gauge invariant constants in the \( C, D \) type patches. The \( C \) patches with constant values of \( u \) are isomorphic to the cells \( (2k + 1, 0) \) by periodicity at \( \tau = 2k + 1 \), and similarly the \( D \) patches with constant values of \( v \) are isomorphic to the cells \( (0, 2k + 1) \). Therefore we may write

\[ u_{2k+1} \equiv u_{2k+1+2l, -2l}, \quad v_{2k+1} \equiv v_{-2l, 2k+1+2l}, \quad k, l = 0, \pm 1, \pm 2, \pm 3, \ldots \quad (5.18) \]

The four constants \( u_{\pm 1}, v_{\pm 1} \) for \( k = -1, 0 \) and \( l = 0 \) are equivalent to the four constants that define the initial conditions \( u_0, v_0, p_0^\pm \). The transfer operation
given in (5.15) relate the initial values \((u_{-1}, v_{-1})\) to the future values \((u_3, v_3)\), at the later time \(\tau \rightarrow \tau + 2\), etc. It is not difficult to rewrite the transfer operation as a recursion relation for any \(k\). Thus, given four constants \((u_{2k-1}, v_{2k-1})\) and \((u_{2k+1}, v_{2k+1})\) at some \(k\) we find the action of the transfer operation by deriving the future constants for \(k \rightarrow k + 1\), i.e. \((u_{2k+3}, v_{2k+3})\):

\[
(u_{2k+3} - u_{2k+1}) = \frac{(1 - u_{2k+1}v_{2k+1})}{(1 - u_{2k-1}v_{2k+1})}(u_{2k+1} - u_{2k-1})
\]

\[
(v_{2k+3} - v_{2k+1}) = \frac{(1 - u_{2k+1}v_{2k+1})}{(1 - u_{2k+1}v_{2k-1})}(v_{2k+1} - v_{2k-1})
\]

(5.19)

This provides a recursion relation that gives the progress of the folds in discrete steps. We will call this map the \textit{transfer matrix}.

The motion of fold-1 at \(\sigma = -1\) is obtained as follows. At \(\tau = 0\) fold-1 is identified as the cross (\(\times\)) between cells \(C(-1,0)\) and \(D(0,1)\). From the constants in those cells we infer that the spacetime location of fold-1 is \((u_{-1,0} = u_{-1}, v_{01} = v_1)\). At \(\tau = 2\) fold-1 is identified as the cross between \(D(0,1)\) and \(C(1,2)\). Therefore, the spacetime location has moved to \((u_{12} = u_3, v_{01} = v_1)\). At \(\tau = 4\) it is at the cross between \(C(1,2)\) and \(D(2,3)\). Therefore it has moved to the spacetime point \((u_{12} = u_3, v_{23} = v_5)\), and so on. In a similar way we can find the spacetime location of fold-2 at \(\sigma = 1\), as well as the location of the midpoint between the two folds at \(\sigma = 0, 2\) that represents the center of mass of the string. The successive spacetime locations of these points are as follows

\[
\begin{align*}
\text{fold-1 (}\sigma = -1\text{)} & \quad \text{midpoint (}\sigma = 0, 2\text{)} & \quad \text{fold-2 (}\sigma = 1\text{)} \\
(u_{-1,0} = u_{-1}, v_{01} = v_1)_{\tau=0} & \quad (u_{10} = u_1, v_{01} = v_1)_{\tau=1} & \quad (u_{10} = u_1, v_{01} = v_1)_{\tau=0} \\
(u_{12} = u_3, v_{01} = v_1)_{\tau=2} & \quad (u_{12} = u_3, v_{21} = v_3)_{\tau=3} & \quad (u_{10} = u_1, v_{21} = v_3)_{\tau=2} \\
(u_{12} = u_3, v_{23} = v_5)_{\tau=4} & \quad (u_{32} = u_5, v_{23} = v_5)_{\tau=5} & \quad (u_{32} = u_5, v_{21} = v_3)_{\tau=4} \\
& \quad \vdots & \quad \vdots \\
& \quad \vdots & \quad \vdots
\end{align*}
\]

(5.20)

From these values we can construct one full oscillation of the string. Note that we have the freedom to choose four initial constants. The four initial constants already determine half of the motion up to \(\tau = 2\). We then iterate according to (5.19) to compute the future motion. The resulting motion of the string is plotted in the figures with the help of a computer. We will describe it in section 6.
We now make an important observation. If we consider the rectangular shapes traced by the minimal surfaces, we may compute approximately the invariant area of rectangle $k$ as

$$A_k = \frac{(\Delta u)_k (\Delta v)_k}{(1 - u_0v_0)_k}$$  \hspace{1cm} (5.21)

where $(\Delta u)_k$, $(\Delta v)_k$ are the sides of the squares and $(u_0, v_0)_k$ are the values of $u, v$ in the middle of rectangle $k$. We will call this quantity the lattice minimal area to distinguish it from the actual minimal area $\int d\tau d\sigma \left( \partial_+ u \partial_- v + \partial_- u \partial_+ v \right) / (1 - u v)$. It is evident that the lattice minimal area is an approximation to the actual minimal area. The remarkable observation is that the lattice minimal area $A_k$ as defined in (5.21) is actually an invariant of the motion. That is we can show that

$$A = 4 \frac{(u_{2k+1} - u_{2k-1})(v_{2k+1} - v_{2k-1})}{4 - (u_{2k+1} + u_{2k-1})(v_{2k+1} + v_{2k-1})}$$  \hspace{1cm} (5.22)

is an invariant under the action of the transfer matrix. To prove it we only need to consider two neighboring values of $k$ and compare $A_k$ to $A_{k+1}$ that are related to each other by the transfer matrix. By using the formulas in (5.19) it takes a little algebra to verify our assertion.

The lattice minimal area as well as the transfer matrix are reparametrization invariants (i.e. independent of the conformal gauge) since they are functions of only the positions of the folds (or end points).

### 5.4 Generalizations

The general flat spacetime solution generates a more complicated periodic $A, B, C, D$ pattern than (4.5), and that is the pattern that must be used for the general curved spacetime solution that connects to the flat spacetime motion as an initial condition. The role of the $A, B, C, D$ pattern is to insure that the global time coordinate $T = (u + v)/\sqrt{2}$ is an increasing function of $\tau$ for all $\sigma$. The pattern is defined by the flat solution written in the form

$$
\begin{align*}
  u(\sigma^+, \sigma^-) &= u_0 + \frac{p}{2} \left[ (\sigma^+ + f(\sigma^+)) + (\sigma^- - g(\sigma^-)) \right] \\
  v(\sigma^+, \sigma^-) &= v_0 + \frac{p}{2} \left[ (\sigma^+ - f(\sigma^+)) + (\sigma^- + g(\sigma^-)) \right]
\end{align*}
$$  \hspace{1cm} (5.23)

where $f(\sigma^+)$ and $g(\sigma^-)$ are any periodic functions with slopes $f'(\sigma^+) = \pm 1$ and $g'(\sigma^-) = \pm 1$. The slope can change discontinuously any number of times.
and at arbitrary locations $\sigma_i^+, \sigma_j^-$ within the basic intervals $-\frac{1}{\sqrt{2}} \leq \sigma^\pm \leq \frac{1}{\sqrt{2}}$. For example we can define the analog of $\sin(k \sigma^\pm)$, as $s_k(\sigma^+) = (-1)^m \sigma^\pm_{\text{per}}$, and $s_k(\sigma^-) = (-1)^m \sigma^-_{\text{per}}$, with periods of $(\sqrt{2}/k, \sqrt{2}/k)$ for $(\sigma^+_{\text{per}}, \sigma^-_{\text{per}})$ respectively. By the arguments of BBHP one sees that the choice $f(\sigma^+) = s_k(\sigma^+)$ and $g(\sigma^-) = s_k(\sigma^-)$ gives the flat spacetime solution that corresponds to normal modes for folded strings. This choice of functions describes strings that are folded in equal lengths, $2k$ times for left movers and $2k$ times for right movers. One could discuss the normal modes or any arbitrary motion.

The curved spacetime solution based on the $A, B, C, D$ patterns generated by the form (5.23) will have this expression as the approximate initial condition in the asymptotically flat region. Therefore, the procedure for obtaining the curved spacetime solution is to first build the worldsheet lattice defined by the functions $f(\sigma^+)$ and $g(\sigma^-)$. This means drawing the $45^\circ$ lines along $\sigma^\pm$ according to the patterns defined by these functions. Next rewrite the solution (5.23) in the forms $A, B, C, D$ as in the example (4.8) and figure out the $A, B, C, D$ pattern on the lattice. Then, instead of the flat $A, B, C, D$ functions substitute the functions (2.5) for the appropriate curved spacetime metric and demand periodicity in $\sigma \rightarrow \sigma + 4$ at $\tau = 0$. This defines the initial condition at $\tau = 0$ and the expected patterns for all future values of $\tau$. Then by matching the solutions at the boundaries one fixes the remaining boundary parameters. It is best to rewrite these parameters as the $(u, v)$ locations of the folds at successive switching points as described in the discussion before (5.19). By carrying out this procedure for a complete periodic motion of all the folds one obtains the transfer operation.

These arguments show that a complete classification of all the physical classical solutions is obtained by the complete classification of all the boundary conditions in the asymptotically flat region, and those are given by (5.23). Each classical solution for a given metric defines a transfer matrix.

Furthermore, it is evident that for each distinct spacetime metric $G_{\mu\nu}(x)$ we obtain a different transfer matrix. This introduces an interesting idea that connects geometry on the one hand and lattice-like transfer matrices for minimal surfaces on the other hand.
6 String falling into the black hole

Using the solution obtained above we now describe what happens to a relativistic string which is falling into the black hole. We will discuss explicitly only the case of the closed string with two folds (or the open string without folds), for which we have given detailed expressions in the previous section. This is sufficient to understand the main physical phenomena.

The two dimensional motion described here corresponds to a motion in 4 dimensions in the radial and time coordinates \((r, t)\). Thus, we may imagine a string stretched along the radial direction at fixed angles \((\theta, \phi)\), and then falling into the black hole in the radial direction while performing longitudinal oscillations. This may approximate an extended object with internal degrees of freedom, such as a star or gas, that oscillates under the influence of internal forces while falling into the black hole.

1 – Starting with a stretched closed string in the asymptotic region, we can now follow its development in time by using our classical solution in the 2D black hole metric. The solution is plotted in target space \((u, v)\) coordinates in Figs.1,2,3. These three figures, taken together correspond to the motion resulting from the same initial condition. Note the change in scale from one figure to the next. The results are not what we would have guessed a priori. At first the string begins to oscillate as in the flat spacetime case, but the gravitational attraction of the black hole also forces it to move towards it, as expected. The combined oscillatory and dragging motion is almost as in eq.(4.2) as long as the string remains far away from the horizon. This produces a minimal surface in curved spacetime similar to the one in flat spacetime. As the string passes the horizon and approaches the black hole, it shrinks in average size, and the invariant lattice minimal area that it sweeps in one oscillation stays constant.

2 – The string with generic initial conditions does not stop at the black hole singularity, but rather, it continues its journey into a second sheet of spacetime that is provided by the \(SL(2, \mathbb{R})/\mathbb{R}\) manifold. Once it passes to the second sheet it begins to expand again and resumes its oscillatory motion while moving away from the singularity. Thus, in the second sheet the singularity is interpreted as a white hole. This motion continues until the string reaches the other branch of the singularity to which it is attracted. It shrinks in size while passing trough it and emerges back into the first sheet (or a third sheet, depending on interpretation). Thus, the second branch
of the singularity, which is a black hole in the second sheet, is a white hole in the first sheet (or third sheet). The motion continues, and the string can come back to the region outside the horizons of both branches of the singularity, where it started initially. If there are only two sheets there is the possibility of making closed timelike curves. However, in the quantum theory it is not necessary to interpret the emergence from the white hole as coming back to the first sheet. Rather, it would be interpreted as a third sheet if the wavefunction $\psi(u, v)$ does not come back to the same value. In fact, in a quantum state that has an irrational “magnetic” quantum number $m$ of $SL(2, R)$, the wavefunction $\psi_{jm}(u, v)$ never comes back to the same value. For such states we have to interpret the covering manifold as having an infinite number of sheets, which is analogous to the Reissner-Nordstrom black hole manifold. The multisheet structure is further clarified in item 6 below.

3 – What happens precisely at the black hole? Does the string shrink to a point before falling into the black hole, as it seems to be the case in Figs.1,2? The answer is no, as seen in Fig.3. The string is smoothly swallowed by the black hole like a spagetti, but the end inside reaches into the forbidden region beyond the singularity! When the second end of the string catches up as the string shrinks to a point inside the forbidden region, the string snaps back and begins to expand into the second sheet of the black hole. This is a strange behavior that point particles cannot perform. Its origin must be related to the wave nature of the string. Like wave phenomena in quantum mechanics, the wave motion of the string seems to permit it to penetrate a barrier that cannot be reached by classical motion of point particles. This is perhaps the most interesting surprise of our analysis.

4 – As seen in Fig.3 the four corners of the ingoing rectangle are sufficient to infer the motion of the string as it is absorbed by the black hole. Is it possible to eliminate the strange behavior by having the far corner coincide with the black hole? This requires fixing two constants to special values. Recall that the initial conditions for the yo-yo solution is specified by just four numbers (the positions and velocities of the end points). Therefore, generic initial conditions used by an observer far away from the black hole will not satisfy the special requirements, and the string will generically perform the

---

5 The wavefunction is simply the diagonal entries in the representation of the group element $D^m_{m'}(g)$, with $m + m' = 0$ because of gauge invariance [11].
strange behavior.

5 – From the remarks in number 4 we can see that we have the ability of choosing initial conditions to arrange for a sufficiently long string, such that, while it is being swallowed by the black hole, one of its ends is inside the black hole and its other end is outside the horizon. This raises questions on whether one might be able to extract information about the black hole by using a long string as a probe, instead of relying only on light rays (which cannot escape the black hole).

6 – One may also start with a string whose initial conditions are specified in the “bare singularity region”. We find that such a string never hits the singularity. An example of its motion is given in Fig.4. Its general average behavior is similar to the motion of the massive particle, for which an analytic expression is given in (6.3) below. As seen from these expressions a massless particle can reach the singularity, but a massive particle cannot do it. The string is a massive state, and this is the explanation for not hitting the singularity. Of course, in the bare singularity region that adjoins the white hole the story is different. In that region the massive particle does fall into the singularity and then moves into the second sheet. Similarly, a string follows a similar route, but at the singularity, its minimal area tunnels into the physical region outside of the white hole. This tunneling is similar to the one described in items 3, 4, 5 above, but now occurring from the other side.

7 – In order to understand the multi-sheet nature of the manifold, as well as the general behaviour of the solution in the various black hole regions, it is useful to investigate the motion of a string of zero size, or a particle. Its equations of motion correspond to the geodesic equations of a point particle, and for any conformal metric it is given by

$$\partial_{x}(x^{\mu}G) = \frac{1}{2} \partial_{x}x \cdot \partial_{x}x \frac{\partial G}{\partial x^{\mu}} \eta^{\mu\nu}$$

(6.1)

This equation follows from the string equations by specializing to the center of mass motion, by doing dimensional reduction (ignoring the $\sigma$ dependence). For the 2D black hole we use the lightcone coordinates $x^{\mu} = (u, v)$ and $G(u, v) = (1 - uv)^{-1}$. These equations are solved by applying the techniques of [10] for $SL(2, R)/R$ which give the time dependence of the $SL(2, R)$ group element for a point particle.
\[
g(\tau) = \begin{pmatrix}
u(\tau) & a(\tau) \\ -b(\tau) & v(\tau)
\end{pmatrix}
\]

where
\[
\begin{align*}
u(\tau) &= e^{\alpha \tau} \left\{ u_0 \cosh(\gamma \tau) - [u_0 \alpha - a_0 p^+] \frac{1}{\gamma} \sinh(\gamma \tau) \right\} \\
v(\tau) &= e^{-\alpha \tau} \left\{ v_0 \cosh(\gamma \tau) + [v_0 \alpha + b_0 p^-] \frac{1}{\gamma} \sinh(\gamma \tau) \right\} \\
a(\tau) &= e^{\alpha \tau} \left\{ a_0 \cosh(\gamma \tau) + [a_0 \alpha - u_0 p^-] \frac{1}{\gamma} \sinh(\gamma \tau) \right\} \\
b(\tau) &= e^{-\alpha \tau} \left\{ b_0 \cosh(\gamma \tau) - [b_0 \alpha + v_0 p^-] \frac{1}{\gamma} \sinh(\gamma \tau) \right\} \\
\alpha &\equiv \frac{1}{2} \left( \frac{w_0}{w_0} p^+ - \frac{w_0}{w_0} p^- \right), \quad \gamma \equiv \left[ \alpha^2 - p^+ p^- \right]^{1/2}, \\
ds^2/d\tau^2 &= (1 - uv)^{-1} \dot{u} \dot{v} = p^+ p^-.
\]

The diagonal entries in \(g(\tau)\) give the desired solutions to the geodesic equations. This can be checked directly by substitution in the geodesic equation (6.1). The constants \(u_0, v_0, p^\pm\) define the initial conditions, and \(u_0 v_0 + a_0 b_0 = 1\) is the \(SL(2, R)\) determinant condition. When \(p^+ p^-\) is positive the geodesic represents the motion of a massive particle. When either \(p^+ \) or \(p^-\) is zero, it corresponds to a massless particle that moves along light-like trajectories (either \(u\) or \(v\) reduces to constant).

As seen from the explicit solution, for the massive particle, in the \(u_0 v_0 < 1\) physical region, there are initial conditions that correspond to an imaginary \(\gamma\). In that case \(\nu(\tau), v(\tau)\) are oscillating functions, indicating backward propagation in time for certain parts of the motion. This is understood as moving from sheet to sheet in the \(SL(2, R)/R\) manifold.

These pointlike solutions are useful to understand the general trend of the motion of the string in all regions \(uv < 1\) or \(uv > 1\). The overall string behaves like a massive state, and therefore we should expect its motion to have some similarities to the motion of massive particles, except for the wave-like phenomena that became manifest in the vicinity of the black hole. Indeed the global trends of the motion plotted in the figures are similar to the trends given by the analytic expressions in (6.3) for the massive particle case. This is so despite the fact that the folds (or end points) on the string move just like massless particles (always parallel to the \(u\) or \(v\) axes). However, the folds are not free particles and they change direction abruptly under the influence of the string.
It is also possible to consider the sigma model directly without relating it to the $SL(2, R)/R$ manifold. In this case one is not necessarily committed to a given number of sheets. In particular one might want to insist that the motion of Figs. 1,2,3 are all on the same sheet. Let us suppose the string reaches the black hole when $\tau = \tau_0$, i.e. $u(\tau_0, \sigma) v(\tau_0, \sigma) = 1$. Then for $\tau > \tau_0$ the time coordinate $T(\tau, \sigma) = (u + v)/\sqrt{2}$ decreases since there is only one sheet. To interpret the backward motion physically we need to invoke an anti-string moving forward in time on the same sheet, and annihilating the original string in the vicinity of the black hole. Similarly pairs of strings are emitted at the white hole. If this interpretation is adopted, it turns out that there can be no solutions with single strings moving only forward in time for all parts of the string in the $uv < 1$ region. We find always anti-strings together with strings in parts of the motion. This situation is analogous to the Klein paradox in the interacting Klein-Gordon or Dirac equations. As is well known in those cases, when strong fields are present the classical equations produce phenomena that seem to violate physical intuition. The physics behind these phenomena is the production of pairs of particles and anti-particles due to the strong fields. When such strong fields are present the Klein paradox is the signal that the correct physics can no longer be fully described by the corresponding classical equations, and that the proper formalism is quantum field theory. However, classical solutions are still useful in obtaining at least qualitative physical information.

We may repeat the analysis for the string motion in any spherically symmetric curved spacetime in the the time-radial subspace at fixed $d\theta = d\phi = 0$. Interesting possibilities include cosmological spacetimes as well as black hole type spacetimes. As discussed in section 2, the string solutions have the same $A, B, C, D$ structure for every spacetime metric in 2D. The dependence on the metric appears through the functions $v(a, b)$, $u(a, b)$. In particular, as an example, one can now analyze the classical string motions falling into the Schwarzschild black hole. We expect general phenomena that are similar to the ones described in this paper, but the details near the singularity may be somewhat different. If one admits a second sheet (defined with negative $r$), the string goes into the second sheet, otherwise with a single sheet there must be pairs of strings and anti-strings annihilated at the black hole, as argued in item 8.

Is there message from all this for the information paradox in black holes? Evidently, we seem to be learning something that was not suspected by
studying only point particles. Given these string phenomena, the discussions about the information paradox in black hole physics may also need revision.

References


6.1 Figure captions

Fig. 1. String approaching the black hole. The invariant minimal area per period is a constant of motion.

Fig. 2. The ingoing string on the first sheet meets the black hole and moves out to the second sheet.

Fig. 3. The string minimal area tunnels to the forbidden region beyond the black hole.

Fig. 4. The string in the dual region (naked singularity region) does not hit the singularity.
Fig. 1. String approaching black hole. Invariant minimal area per period is constant. Average string length shrinks.
Fig. 2. Ingoing string on 1st sheet meets black hole, moves out to 2nd sheet.
Fig. 3. String minimal area tunnels to forbidden region beyond black hole. Arrows along trajectories of midpoint.
Fig. 4. String in dual region (naked singularity region). It does not hit the singularity.