UNIVERSE REHEATING AFTER INFLATION

Y. Shtanov,\textsuperscript{(1,2)} J. Traschen\textsuperscript{(3)} and R. Brandenberger\textsuperscript{(1)}

1) Physics Department, Brown University, Providence, Rhode Island, 02912, USA
2) Bogolyubov Institute for Theoretical Physics, Kiev, 252143, Ukraine
3) Department of Physics & Astronomy, University of Massachusetts, Amherst, Massachusetts, 01003, USA
ABSTRACT

We study the problem of scalar particle production after inflation by an inflaton field which is oscillating rapidly relative to the expansion of the universe. We use the framework of the chaotic inflation scenario with quartic and quadratic inflaton potentials. Particles produced are described by a quantum scalar field $\chi$, which is coupled to the inflaton via linear and quadratic couplings. The particle production effect is studied using the standard technique of Bogolyubov transformations. Particular attention is paid to parametric resonance phenomena which take place in the presence of the quickly oscillating inflaton field. We have found that in the region of applicability of perturbation theory the effects of parametric resonance are crucial, and estimates based on first order Born approximation often underestimate the particle production. In the case of the quartic inflaton potential $V(\varphi) = \lambda \varphi^4$, the particle production process is very efficient for either type of coupling between the inflaton field and the scalar field $\chi$ even for small values of coupling constants. The reheating temperature of the universe in this case is $[\lambda \log (1/\lambda)]^{-1}$ times larger than the corresponding estimates based on first order Born approximation. In the case of the quadratic inflaton potential the theory predicts not only the reheating temperature but also the effective energy density of the inflaton field when it freezes out. The reheating process in this case depends crucially on the type of coupling between the inflaton and the scalar field $\chi$ and on the magnitudes of the coupling constants. If the inflaton coupling to fermions and its linear (in inflaton field) coupling to scalar fields are suppressed, the inflaton field will eventually decouple from the rest of the matter. In this case of incomplete reheating, the residual inflaton oscillations may provide the (cold) dark matter of the universe.
1. Introduction

According to the simplest version of the inflationary scenario (see [1]) the universe in the past expands almost exponentially with time (such an expansion is called “inflation”) while its energy density is dominated by the effective potential energy density of a special scalar field \( \varphi \), called the inflaton. Sooner or later inflation terminates and the inflaton field starts quasiperiodic motion with slowly decreasing amplitude. Right after inflation the universe is empty of particles, i.e. “cold”. Quasiperiodic evolution of the inflaton field leads to creation of particles of various kinds, after thermalization of which due to collisions and decays the universe becomes “hot”. The role of the inflaton field in the inflationary scenario is not necessarily played by a fundamental scalar field. It can be the expectation value of a scalar operator constructed of fields of other type. It can also be the effective scalar introduced in the inflationary models based on a higher derivative theory of gravity. Almost all such scenarios have an important feature: the inflationary stage is followed by quasiperiodic evolution of the effective inflaton field that leads to particle creation.

In this work we are going to study the transition of the universe from an inflationary to a hot stage sketched above. The problem is not new, and there are many papers devoted to its study using different methods in the context of various inflationary scenarios \(^{[3-8]}\) (see also Refs.1 and 9). There is nevertheless a drawback common to most of them - the calculations were based on ordinary perturbation theory. The rates of particle production by the oscillating inflaton field were calculated in first order Born approximation. However, as it has been shown in Ref.10 this approach disregards possible parametric resonance effects which can enhance the rate of boson particle production. To detect such effects one has to go beyond standard perturbation theory. Such resonance amplification effects have been studied in \([10]\) in the context of the new inflationary scenario. It was shown that typically the oscillating inflaton field produces many particles due to parametric resonance effect even in the case of extremely small couplings, and that
estimates based on ordinary perturbation theory miss the largest effect. Parametric resonance effects as relevant to reheating of the universe were also studied in Ref.11 where it was also demonstrated that there is no resonance enhancement of fermionic particle production.

In the present paper we are going to study the problem in a systematic way. We will consider a model with inflation based on a power law potential for the inflaton field. The inflaton field \( \varphi \) throughout this paper will be regarded as classical. In Section 2 we will describe the evolution of the inflaton field after inflation without taking into account its couplings to other fields. Couplings of the inflaton field \( \varphi \) to fields describing fermionic spin-1/2 and bosonic scalar particles will be considered in the following sections. In particular, the process of production of these particles by the oscillating inflaton field will be studied. In all the cases we will assume the couplings to be sufficiently small so that perturbation theory is valid. In Section 3 we will derive the expressions for particle production rates in first order Born approximation of ordinary perturbation theory. The material of Sections 2 and 3 will closely follow a previous work [12]. In Section 4 we study general aspects of the parametric resonance effect, and in Sections 5 and 6 we apply the results obtained to the concrete cases of quartic and quadratic inflaton potentials. In Section 7 we present our general conclusions.

It will be shown that in many important cases it is not legitimate to neglect parametric resonance effects. These effects lead to a great enhancement of the particle production as compared to the estimates based on ordinary perturbation theory. The number of particles \( N_k \) in a mode with wavenumber \( k \) that has experienced the resonance turns out to be much larger than unity. The universe reheating temperature therefore can turn out to be sufficiently large even for small values of the couplings between inflaton and other fields.

The opinion of the authors of Ref.11 is that parametric resonance effects of scalar particle production are likely to be suppressed by the processes of scattering and decay of the particles produced. The reason is that particle scattering and
decay will tend to reduce the mean occupation numbers $N_k$ in each particular mode so that these numbers will grow not so fast as to be able to become large during the resonance period. This question is very important and deserves thorough study. We do not aim to touch on it in this work, but hope to consider it in future publications.

After this work has been completed we became aware of a recent paper \cite{15} in which results similar to ours have been announced. We will compare our results with those of Ref.13 in Section 7.

2. Inflaton field dynamics

In this section we will derive approximate analytic expressions for the evolution of the inflaton field as it undergoes quasiperiodic motion in the expanding universe, and compute the resulting effective equation of state of the inflaton stress-energy.

Consider the chaotic inflaton scenario based on the scalar (inflaton) field dynamics. The Lagrangian of the model is

$$L = \frac{1}{2} (\nabla \varphi)^2 - V(\varphi) - \frac{M_P^2}{16\pi} R,$$

with the inflaton field potential

$$V(\varphi) = \lambda \mu^{4-q} |\varphi|^q,$$

where $q$ is an arbitrary positive power. We are working in the standard system of units in which $\hbar = c = 1$, and $M_P$ denotes the Planck mass.

The equations of motion for a homogeneous isotropic universe and for a homogeneous scalar field look like

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi}{3M_P^2} \left( \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right),$$
\[ \ddot{\varphi} + 3H \dot{\varphi} + V'(\varphi) = 0, \]  

(4)

where \( H \equiv \dot{a}/a \) is the Hubble parameter, \( \kappa = 0 \), \( \pm 1 \) corresponds to different signs of spatial curvature, dots denote derivatives with respect to the cosmological time \( t \), and prime denotes derivative with respect to \( \varphi \).

From the equation (4) it follows that

\[ \ddot{\varphi} = a^{-3} \left( \text{const} - \int V'(\varphi) a^3 dt \right). \]  

(5)

This equation describes both the rapid evolution of \( \varphi \), for which \( \varphi \propto a^{-3} \), and the regime of slow rolling-down during inflation, when \( \varphi \approx -(1/3H) V'(\varphi) \).

According to the inflation scenario, during inflation the scalar field \( \varphi \) is rolling down the slope of its potential from its relatively large value towards its minimum. The conditions for slow rolling and for inflation are, respectively,

\[ |\ddot{\varphi}| \ll 3H |\dot{\varphi}|. \]  

(6)

and

\[ |\dot{H}| \ll H^2. \]  

(7)

During inflation the term \( \kappa/a^2 \) in the left-hand side of Eq.(3) soon becomes insignificant, and the conditions (6) and (7) read, respectively,

\[ \left( \sqrt{V(\varphi)} \right)^{''} \ll \frac{12\pi}{M_P^2} \sqrt{V(\varphi)}, \]  

(8)

\[ (V'(\varphi))^2 \ll \frac{24\pi}{M_P^2} V(\varphi). \]  

(9)

For the scalar field potential (2) these two conditions become respectively

\[ \varphi^2 \gg \frac{|q-2|}{48\pi} M_P^2, \]  

(10)
Note that conditions (10) and (11) are essentially the same.

As the scalar field evolves towards its smaller values, the conditions (10) and (11) cease to be valid, and a new regime for the scalar field begins, namely, the regime of quasiperiodic evolution with decaying amplitude. To describe this new cosmological period, it is convenient to rewrite the system of equations (3), (4) (with \( \kappa = 0 \)) as follows

\[
H^2 = \frac{8\pi}{3M_p^2} \rho, \tag{12}
\]

\[
\dot{\rho} = -3H\dot{\varphi}^2, \tag{13}
\]

where

\[
\rho = \frac{1}{2} \dot{\varphi}^2 + V(\varphi)
\]

is the scalar field energy density. Introducing the positive value \( \varphi_0(t) \) by the relation

\[
V(\varphi_0(t)) = \rho(t), \tag{15}
\]

we can present the evolution of the scalar field in the form

\[
\varphi(t) = \varphi_0(t) \cos \int W(t) \, dt, \tag{16}
\]

where \( W(t) \) is some unknown function of time. Note that the representation (16) for the scalar field evolution with \( \varphi_0 \) defined by (15) is always possible. Using the equations (2) and (12)-(15), one can then derive the following exact expression for the function \( W(t) \)

\[
W = \sqrt{\frac{2(\rho - V(\varphi))}{\varphi_0^2 - \varphi^2}} \left( 1 \pm \sqrt{\frac{48\pi}{q} \frac{\varphi}{M_p} \sqrt{1 - \frac{V(\varphi)}{\rho}}} \right). \tag{17}
\]

The signs “+” and “−” correspond respectively to the cases \( \dot{\varphi} > 0 \) and \( \dot{\varphi} < 0 \). The
second term in the brackets in (17) is much less than unity if
\[
\varphi_0^2 \ll \frac{q}{48\pi} \left( \frac{q + 2}{2} \right)^{\frac{q+2}{2}} M_p^2. \tag{18}
\]
This condition is just the opposite of (10), (11). When (18) is valid, the expression (17) for \(W(t)\) acquires the simple approximate form
\[
W \approx \sqrt{\frac{2(\rho - V(\varphi))}{\varphi_0^2 - \varphi^2}}, \tag{19}
\]
and, in addition, the following condition becomes valid
\[
|\dot{\varphi}_0/\varphi_0| \ll W, \tag{20}
\]
which allows us to regard the evolution of \(\varphi(t)\) as quasiperiodic with slowly decaying amplitude \(\varphi_0(t)\). Henceforth we assume that the condition (18) (hence also (19) and (20)) is satisfied.

Our aim now will be to derive approximate evolution equations for the values \(\rho(t)\) and \(\varphi_0(t)\) under the condition (18). First we average the equation of motion (13) over the time period \(T\) of quasiperiodic motion of \(\varphi\). Rewriting (13) in accord with (14) as
\[
\dot{\rho} = -6H (\rho - V(\varphi)), \tag{21}
\]
and taking the average of both sides we get
\[
\frac{\Delta \rho(T)}{T} = -\frac{6}{T} \int_0^T H (\rho - V(\varphi)) \, dt, \tag{22}
\]
where \(\Delta \rho(T)\) is the change of \(\rho\) over the time period \(T\). Now take into account that according to (20) and to the equations (15) and (12) the values \(\varphi_0(t)\), \(\rho(t)\) and \(H(t)\) change only insignificantly during one period of oscillation of the \(\varphi\)-field.
This enables us to replace all variables except $\varphi$ under the integral in (22) by their averaged values. We will have then, to a good approximation

$$\frac{\Delta \rho(T)}{T} \approx -\frac{6}{T} H \int_0^T (\rho - V(\varphi)) \, dt,$$

where $H$ and $\rho$ in (23) and below denote the corresponding averaged values. The integral in (23) can be evaluated as

$$\frac{1}{T} \int_0^T (\rho - V(\varphi)) \, dt \approx \frac{\varphi_0}{\varphi_0 - \varphi} \frac{\sqrt{\rho - V(\varphi)} \, d\varphi}{\rho - V(\varphi)} \approx \frac{1}{2} \int_{-\varphi_0}^{\varphi_0} \frac{\varphi V'(\varphi) \, d\varphi}{\sqrt{\rho - V(\varphi)}} \approx \frac{q}{q + 2} \rho. \tag{24}$$

In (24) we took into account the relation (14), integrated in the numerator by parts and made use of the equation (2). In the approximation considered we can replace the finite difference expression on the left-hand-side of (23) by the derivative. Equation (23) then becomes

$$\dot{\rho} \approx -\frac{6q}{q + 2} H \rho. \tag{25}$$

All the values in (25) are now to be regarded as averaged in the sense described above. Equation (25) is valid on time scales large compared to the period of quasioscillations of $\varphi(t)$. This implies the following effective equation of state for the matter described by the scalar field $\varphi$

$$p = \frac{q - 2}{q + 2} \rho. \tag{26}$$

In particular, for the most interesting cases $q = 2$ (quadratic potential) and $q = 4$ (quadric potential) we obtain, respectively, $p = 0$ (dust) and $p = \rho/3$ (radiation).

Using the definition (15) we immediately obtain the approximate equation for the evolution of the value $\varphi_0(t)$

$$\ddot{\varphi}_0 \approx -\frac{6}{q + 2} H \varphi_0. \tag{27}$$
3. Particle production: Born approximation

In this section we will study the process of particle production by the rapidly oscillating inflaton field. We take the interaction Lagrangian to be

\[ I_{\text{int}} = -f \varphi \bar{\psi} \psi - \left( \sigma \varphi + h \varphi^2 \right) \chi^2, \]  

where \( \bar{\psi} \) and \( \psi \) describe spinor particles and \( \chi \) describes scalar particles with corresponding masses \( m_\psi \) and \( m_\chi \), \( f \) and \( h \) are dimensionless coupling constants, and \( \sigma \) is a coupling constant of dimension of mass. We will treat the scalar field \( \varphi \) as a classical external field. In this section we will employ ordinary perturbation theory in coupling constants, that is, we will work in the Born approximation. To calculate the rate of particle production we first develop the quasiperiodic evolution of the scalar field \( \varphi \) into harmonics

\[ \varphi(t) = \sum_{n=1}^{\infty} \varphi_n \cos(n \omega t), \]  

\[ \varphi^2(t) = \bar{\varphi}^2 + \sum_{n=1}^{\infty} \zeta_n \cos(2n \omega t), \]

where the value \( \bar{\varphi}^2 \approx \varphi_0^2/2 \) which is slowly varying with time is \( \varphi^2 \) averaged over the rapid oscillations of the scalar field \( \varphi \). \( \varphi_n, \zeta_n \) are amplitudes which are slowly varying with time, and

\[ \omega = \frac{2\pi}{T} = c \frac{\sqrt{b}}{\varphi_0} \]

is the leading frequency, also slowly varying with time, related to the period \( T \) of the oscillations of the field \( \varphi \). The constant \( c \) in (31) is of order 1, and is given by

\[ c = \pi \sqrt{2} / \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \]  

in the case of a scalar field potential given by (2). To derive the estimate (31) we made use of the relation (14).
Assuming the condition (18), it follows that the oscillation period of $\varphi$ is small compared to the Hubble time,

$$H \ll \omega$$  \hspace{1cm} (33)

This last condition allows us to disregard the effect of cosmological expansion in evaluating particle production rates. We will also assume that particle masses and the coupling constant $h$ are sufficiently small, so that

$$m_{\psi}^2, \quad m_{\chi}^2 + h\varphi^2 \ll \omega^2.$$  \hspace{1cm} (34)

The rates of particle production, that is, the total number of pairs produced per unit volume and unit time are then given in first order perturbation theory in coupling constants $f$, $\sigma$ and $h$ by the following relations

$$w_{\psi} = \frac{f^2}{16\pi} \sum_{n=1}^{\infty} (\varphi_n n\omega)^2,$$  \hspace{1cm} (35)

$$w_{2\chi} = \frac{1}{16\pi} \left( \sigma^2 \sum_{n=1}^{\infty} \varphi_n^2 + h^2 \sum_{n=1}^{\infty} \zeta_n^2 \right),$$  \hspace{1cm} (36)

Remembering the expansions (29) and (30) we can put the sums in the last expressions into the following form

$$\sum_{n=1}^{\infty} (\varphi_n n\omega)^2 = 2 < \varphi^2 >,$$  \hspace{1cm} (37)

$$\sum_{n=1}^{\infty} \varphi_n^2 = 2 < \varphi^2 >,$$  \hspace{1cm} (38)

$$\sum_{n=1}^{\infty} \zeta_n^2 = 2 < (\varphi^2 - \varphi^2)^2 >,$$  \hspace{1cm} (39)

where the brackets $< \ldots >$ denote the average over quasioscillations of the scalar field $\varphi(t)$. These average values in the expressions (37)-(39) can be estimated in
exactly the same manner in which the equation (24) is derived. We obtain, finally, the following relations for the particle production rates

\[ w_{\psi \bar{\psi}} = \frac{f^2}{4\pi} \frac{q}{q + 2} \rho, \tag{40} \]

\[ w_{2\chi} = \frac{1}{8\pi} \left( c_\sigma \sigma^2 \varphi_0^2 + c_h h^2 \varphi_0^4 \right), \tag{41} \]

where

\[ c_\sigma = \int_{-1}^{1} \frac{x^2 \, dx}{\sqrt{1 - x^2}} \bigg/ \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}, \tag{42} \]

\[ c_h = \int_{-1}^{1} (x^2 - \frac{c_\sigma^2}{2}) \, dx \bigg/ \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}, \tag{43} \]

are constants of order unity.

Higher order contributions to the particle production rates (40)-(41) will be negligible if

\[ f^2, h, \frac{\sigma}{\varphi_0} \ll \left( \frac{\omega}{\varphi_0} \right)^2. \tag{44} \]

Hence, our assumption (34) for the smallness of the coupling constant \( h \) is justified since \( \overline{\varphi^2} \approx \varphi_0^2/2 \). We will see below what the conditions (44) imply for the concrete inflaton potentials.

Equations (40)-(41) enable us to estimate the energy density production in the form of \( \overline{\psi}, \psi \) and \( \chi \) particles created. To do this, note that pairs of particles produced have average energy of the order \( \omega \), if we reasonably assume that the sums on the right-hand sides of (35), (36) rapidly converge. Then the equation for
\( \rho_p \), the particle energy density produced, can be written as follows

\[
\dot{\rho}_p \text{(production)} = (\Gamma_\psi + \Gamma_\chi) \rho, \tag{45}
\]

where

\[
\Gamma_\psi = \mathcal{O}(1) \frac{f^2}{4\pi} \omega, \tag{46}
\]

is the contribution corresponding to the production of fermionic particles, and

\[
\Gamma_\chi = \mathcal{O}(1) \frac{\sigma^2}{4\pi \omega} + \mathcal{O}(1) \frac{(\hbar \varphi_0)^2}{4\pi \omega}, \tag{47}
\]

is the contribution corresponding to the production of scalar particles \( \chi \). Here and below we denote by the symbol \( \mathcal{O}(1) \) a constant of order unity. The values \( \Gamma_\psi \) and \( \Gamma_\chi \) can be interpreted as the “\( \varphi \)-particle” decay rates in corresponding decay channels.

### 4. Particle production: parametric resonance

The expressions (45)-(47) for the particle energy density production rates have been obtained in first order Born approximation, and in this form they have been used in all the seminal papers on reheating. However, as it has been shown in [10], this approximation does not take appropriate account of the parametric resonance phenomena which take place during the oscillation period of the scalar field \( \varphi \). Parametric resonance effects can be shown to be insignificant for fermionic fields [11]. The estimate (46) for fermion production made in the preceding section thus remains valid. However the question of the parametric resonance effect for the scalar field \( \chi \) has to be considered and this will be the aim of this chapter. We will show that the resonance occurs when the frequency \( \omega_k \) of the quantum field mode is equal to the half-integer multiples of the inflaton frequency, \( \omega_k \approx \left( \frac{n}{2} \omega \right)^2 \).

This results in exponentially enhanced production of particles in narrow resonance bands, at rates which are computed below.
The evolution of a particular mode $\chi_k$ of the quantum scalar field $\chi$ in the presence of the quickly oscillating classical scalar field $\varphi$ with the interaction (28) is described by the following equation

$$\ddot{\chi}_k + 3H \dot{\chi}_k + \left( \frac{k^2}{a^2} + m^2_k + 2\sigma \varphi + 2h \varphi^2 \right) \chi_k = 0,$$

where $\frac{k}{a} = k/a$ is the physical wavenumber of the mode under consideration, $k$ is the comoving number.

Performing the transformation

$$\chi_k = \frac{Y_k}{a^{3/2}},$$

one obtains from the equation (48) the following equation for the function $Y_k$:

$$\ddot{Y}_k + (\omega_k^2(t) + g(\omega t)) \dot{Y}_k = 0,$$

where

$$\omega_k^2(t) = \frac{k^2}{a^2} + m^2_k - \frac{9}{4} H^2 - \frac{3}{2} \dot{H} + 2h \varphi^2,$$

and

$$g(\omega t) = 2\sigma \varphi + 2h \left( \varphi^2 - \bar{\varphi}^2 \right).$$

The function $g(\omega t)$ is to a good approximation a $2\pi/\omega$-periodic function of time $t$ with two leading frequencies $\omega$ and $2\omega$, as can be seen from (52) and the expansions (29), (30). We will regard this function as a small perturbation in the equation (50), and to be able to do this we require the condition

$$2\sigma \varphi_0 + h \varphi_0^2 \ll \omega_k^2,$$

which we will assume to be valid.
The time dependence of the coefficients in the equations (48), (50) will lead to $\chi$-particle production which can be described by the standard Bogolyubov transformation technique. The problem consists in finding the solution to the equation (50) with initial conditions corresponding to the initial vacuum state and then evaluating the Bogolyubov transformation coefficients. We expect that for the equation (50) parametric resonance effect will be dominant hence we are going to study this effect in detail.

From the general theory of parametric resonance (see Appendix A for details\textsuperscript{1}) one knows that parametric resonance occurs for the equation (50) for certain values of the frequencies $\omega_k$. Namely, the resonance in the lowest frequency resonance band occurs for those values of $\omega_k$ for which

$$\omega_k^2 - \left(\frac{n}{2} \omega\right)^2 \equiv \Delta_n < |g_n|,$$  

(54)

where $n$ is an integer, and $g_n$ is the amplitude of the $n$-th Fourier harmonic of the function $g(\omega t)$. Of course, for such a conclusion to be valid in our case when the frequencies $\omega_k$ and $\omega$ depend on time we must be sure that these frequencies change with time slowly as compared to their own values, that is

$$\left|\frac{\dot{\omega}_k}{\omega_k}\right|, \left|\frac{\dot{\omega}}{\omega}\right| \ll \omega_k, \omega.$$  

(55)

But this condition immediately follows from the estimates (20), (33), and from (53) if we also take the relation (54) into account. Note that condition (53) for $\omega_k$ in the resonance band coincides with the conditions (44) on the validity of Born approximation.

In this paper we will consider only resonance in the lowest resonance frequency band for each Fourier harmonic of the function $g$ as given by equation (54). The effects of higher resonance bands, as is well known, are of higher order in the

\textsuperscript{1} In Appendix A we explicitly introduced a small dimensionless parameter $\epsilon$. In the main text we have absorbed $\epsilon$ into the definition of $g$.\textsuperscript{1}
amplitudes $|g_n|$ which we assumed to be small. Hence we expect these effects to be less efficient.

From the conditions (20), (33), (34) and (53) it also follows that part of the expression (51) satisfies

$$
\omega_k^2 - k^2 \equiv m_\chi^2 - \frac{9}{4} H^2 - \frac{3}{2} \dot{H} + 2 \hbar \phi^2 \ll \omega^2.
$$

It then follows from the equation (54) that

$$
\kappa^2 \gg m_\chi^2 - \frac{9}{4} H^2 - \frac{3}{2} \dot{H} + 2 \hbar \phi^2
$$

for the resonance values of $k^2$. This means that the redshift of the frequency $\omega_k$ due to the expansion of the universe is, to a good approximation,

$$
\omega_k(t) \propto k \propto a^{-1}(t).
$$

Due to the condition (33) we can neglect this cosmological redshift on the timescale of the scalar field $\varphi$ oscillations. The mean occupation numbers for the generated $\chi$-particles are then given by the Bogolyubov coefficients $\beta_k$ as $N_k = |\beta_k|^2$ and the expression for them can be approximately written as follows (see Appendix B for derivation)

$$
N_k \simeq \sinh^2 \left( \int \mu_+ \, dt \right),
$$

where

$$
\mu_+ = \frac{1}{m \omega} \sqrt{|g_n|^2 - \Delta_n^2}
$$

is the eigenvalue of the growing resonance mode, and $\Delta_n$ is given by (54).

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1) We must use the expression (B32) for the number of particles produced, because in our case the value $\mu_+$ varies with time.
The approximation (59) is valid only if we can treat the value $\mu_+$ as adiabatic. For our purposes it is sufficient to consider values of $\mu_+$ close to the resonance (when $\Delta n = 0$) and the variation of $\mu_+$ with time is mostly due to the expansion of the universe. Then the adiabaticity condition (see the equation (B31)) will be as follows

$$\mu_+ \gg H .$$ (61)

As time goes on, the mode with the particular wavenumber $k$ gets out of the resonance band, and the value $N_k$ ceases to grow. The reason for the frequency $\omega_k$ (initially in resonance) to get out of the resonance band is that the redshift evolution (58) of the frequency $\omega_k$ is different from the evolution of the leading frequency $\omega$ and hence the resonance frequency $n\omega/2$ is continuously passing through the spectrum of frequencies $\omega_k$ of the modes of the scalar field $\chi$. (In the case of $\lambda\varphi^4$ potential, the value $\omega$ does redshift at the same rate, and one has to take into account the back-reaction of the created particles to determine this frequency shift; see Section 5.)

Denote by $\omega_{\text{res}}$ the resonance frequency $n\omega/2$ and by $\Delta\omega_k$ denote the difference $\omega_k-\omega_{\text{res}}$. Both frequencies $\omega_{\text{res}}$ and $\omega_k$ change with time. In a small time $\delta t$ the shift between these frequencies will be $\delta \omega = |d\Delta\omega_k/dt|_{\Delta\omega_k=0}\delta t$. Hence a new region in phase space of volume $\omega_{\text{res}}^2 \delta \omega/2\pi^2$ will be filled with particles. The average number of particles in every state in this phase space region will be given by $N_k$ and each particle will have energy $\omega_{\text{res}}$. Then taking sum over all parametric resonance bands (with various numbers $n$ in (54)) we can write down the equation for the energy density $\rho_{\lambda}$ of the scalar particles produced in the following form

$$\hat{\rho}_\lambda (\text{resonance production}) = \sum_{\text{all resonance bands}} \frac{1}{2\pi^2} N_{\text{res}} \omega_{\text{res}}^3 \left| \frac{d\Delta\omega_k}{dt} \right|_{\Delta\omega_k=0} ,$$ (62)

where $N_{\text{res}}$ is the maximal value of $N_k$, for the current resonance value of $k$, achieved after the corresponding mode has passed through the resonance band and has been amplified.

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Under condition (34) particles produced will be ultrarelativistic and hence their contribution to energy density will decrease due to the cosmological expansion. Taking also this process into account we are able to write down the following complete equation for the evolution of the energy density $\rho_p$ of the particles produced (including the contribution from fermions)

$$\dot{\rho}_p = -4H\rho_p + \left( \Gamma_\psi + \Gamma_\chi^{(\text{res})} \right) \rho, \quad (63)$$

where

$$\Gamma_\chi^{(\text{res})} = \frac{1}{2\pi^2 \rho} \sum_{\text{all resonance bands}} N_{\text{res}} \omega_{\text{res}}^3 \left| \frac{d\Delta \omega_k}{dt} \right|_{\Delta \omega_k = 0} \quad (64)$$

plays the role analogous to that of $\Gamma_\chi$ in (45). The values of $\Gamma_\chi^{(\text{res})}$ given by (64) and $\Gamma_\chi$ given by (47) are to be compared in order to conclude about the significance of the parametric resonance for the process of scalar particle production. In what follows we consider two important cases of the inflaton potential $V(\varphi)$, namely, quartic and quadratic potentials.

5. Case of potential $V(\varphi) = \lambda \varphi^4$

In this case we have from (31)

$$\omega = c \sqrt{\lambda} \varphi_0, \quad (65)$$

where the constant $c$ is given by (32) with $q = 4$. The resonance frequency $\omega_{\text{res}}$ is thus proportional to $\varphi_0$ and by virtue of (27) in the absence of back reaction of the particle production process on the scalar field $\varphi$ we would have

$$\omega_{\text{res}} \propto a^{-1}. \quad (66)$$

As we also have the condition (58) this would mean that the resonance amplification takes place for all times in the same wavebands, which satisfy the condition (54).
However due to the energy loss due to particle production, the scalar field amplitude \( \varphi_0 \), and hence the frequency \( \omega_{\text{res}} \), will decrease more rapidly than given by (66).

We see that in order to make correct estimates of particle production in the model with quartic inflaton potential one has to take into account back-reaction on the inflaton evolution.

Let us then consider back-reaction of the particle production process on the evolution of the scalar field \( \varphi \). Since particle production is a random quantum mechanical phenomenon, the change in the evolution of the scalar field \( \varphi \) will be random. We can picture the back-reaction as leading to abrupt changes in the state of the scalar field at those random moments of time at which particles are produced. On sufficiently large time scales, on which many particles are produced, this change will exhibit regular features, and it is these features which we are to determine.

The evolution of the scalar field \( \varphi \) can be described by two parameters, namely, the scalar field energy density \( \rho \) and the frequency \( \omega \) of its quasioscillations. In absence of back-reaction both these parameters are changing slowly with time, when compared to the time scale \( T = 2\pi/\omega \) of the quasioscillations. The value \( \omega \) is directly related to the energy density through the relation (31). We find it possible to describe the state of the scalar field in terms of these two parameters also if we take the back-reaction mentioned above into account. Indeed, it is always possible to speak of the scalar field energy density. As far as its quasioscillation frequency is concerned we note that its very existence is necessary for particle production, with rates given by (40)-(41), or with average numbers (59) in the case of parametric resonance.

Also note that on the quasioscillation time scale there can be but few particles produced (this fairly general statement follows from the standard analysis of transition amplitudes). Hence random moments of the abrupt changes in the scalar field state will be time-separated in average by more than one scalar field quasioscillation period. Between these moments of abrupt changes the scalar field
is evolving unperturbed. From this it follows that we can assume the validity of the relation (31) between the quasioscillation frequency $\omega$ and the energy density $\rho$.

To determine in the most simple way the changes in the evolution of the scalar field energy density $\rho$ we can apply the energy conservation law. The energy is transferred from the scalar field $\varphi$ to the particles produced. Hence, given the energy production rate (63) of the latter we thereby know the energy loss of the scalar field $\varphi$. The equation for the inflaton energy density will then be

$$\dot{\rho} = - \left( 4H + \Gamma_\psi + \Gamma_\chi^{(\text{res})} \right) \rho,$$

(67)

with $\Gamma_\chi^{(\text{res})}$ given by (64). From the equation (2) with $q = 4$ and from (15) it then follows

$$\dot{\varphi}_0 = - \left( H + \frac{1}{4} \left( \Gamma_\psi + \Gamma_\chi^{(\text{res})} \right) \right) \varphi_0.$$

(68)

Now taking (58) and (65) into account we will obtain the following expression for the time derivative factor in (62)

$$\left| \frac{d\Delta\omega_k}{dt} \right|_{\Delta\omega_k=0} = \frac{1}{4} \left( \Gamma_\psi + \Gamma_\chi^{(\text{res})} \right) \omega_{\text{res}},$$

(69)

and from the equation (64) we then obtain

$$\sum_{\text{all resonance bands}} N_{\text{res}} \omega_{\text{res}}^4 = 8\pi^2 \rho \left( 1 + \frac{\Gamma_\psi}{\Gamma_\chi^{(\text{res})}} \right)^{-1}.$$

(70)

It is very remarkable that in spite of all the details of the interaction between the fields $\varphi$ and $\chi$ the maximal occupation number mean values depend through (70) only on the resonance frequencies $\omega_{\text{res}}$ in different resonance bands. This significant feature of the quartic inflaton potential makes the total analysis rather simple.
It remains to estimate the value $\Gamma^{(\text{res})}_X$. The expression for $N_k$ is given by (59) where the integration in the argument of the hyperbolic sine is taken over the time during which resonance is taking place for the mode with the wavenumber $k$. In terms of the value $\Delta \omega_k = \omega_k - \omega_{\text{res}}$, the expression (60) for $\mu_+$ can be rewritten as follows

$$\mu_+ = \frac{1}{2\omega_{\text{res}}} \sqrt{|g_n|^2 - \Delta \omega_k^2 (\Delta \omega_k + 2\omega_{\text{res}})^2}. \quad (71)$$

Then using (69) and taking into account the smallness of the value $|g_n|$ as compared to $\omega_{\text{res}}^2$ (this follows from the relations (52) and (53)) we can estimate the integral in the exponent of (59) as follows

$$\int \mu_+ dt \approx \int_{-|g_n|/2\omega_{\text{res}}}^{\frac{|g_n|}{2\omega_{\text{res}}}} \frac{1}{2\omega_{\text{res}}} \sqrt{|g_n|^2 - \Delta \omega_k^2 (2\omega_{\text{res}})^2} \frac{4}{(\Gamma_{\psi} + \Gamma^{(\text{res})}_X)} \omega_{\text{res}} \, d\Delta \omega_k \approx \frac{|g_n|^2}{(\Gamma_{\psi} + \Gamma^{(\text{res})}_X) \omega_{\text{res}}^3} \int_{-1}^{1} \sqrt{1 - x^2} \, dx = \frac{\pi}{2} \frac{|g_n|^2}{(\Gamma_{\psi} + \Gamma^{(\text{res})}_X) \omega_{\text{res}}^3}.$$ \quad (72)

In deriving (72) we considered slowly changing variables as constants. Then the expression (59) for $N_{\text{res}}$ becomes

$$N_{\text{res}} \approx \sinh^2 \left( \frac{\pi |g_n|^2}{2 (\Gamma_{\psi} + \Gamma^{(\text{res})}_X) \omega_{\text{res}}^3} \right). \quad (73)$$

Substituting this expression for $N_{\text{res}}$ into (70) we obtain the following implicit expression for the value $\Gamma^{(\text{res})}_X$

$$\sum_{\text{all resonance bands}} \omega_{\text{res}}^4 \sinh^2 \left( \frac{\pi |g_n|^2}{2 (\Gamma_{\psi} + \Gamma^{(\text{res})}_X) \omega_{\text{res}}^3} \right) \approx 8\pi^2 \rho \left( 1 + \frac{\Gamma_{\psi}}{\Gamma^{(\text{res})}_X} \right)^{-1}. \quad (74)$$

Consider first the case when energy production of scalar particles dominates.
over that of fermions so that

$$\Gamma_\psi < \Gamma^{(\text{res})}_\chi.$$  \hspace{1cm} (75)

Typically one of the terms, say, of the \(n\)-th resonance band dominates in the sum in the left-hand side of (74). In the case (75) \(\Gamma^{(\text{res})}_\chi\) is determined by this dominant term as follows

$$\Gamma^{(\text{res})}_\chi \simeq \frac{\pi |g_n|^2}{\omega^{3}_{\text{res}}} \log^{-1} \left( \frac{32 \pi^2 \rho}{\omega^{4}_{\text{res}}} \right).$$  \hspace{1cm} (76)

Two of the Fourier harmonics of the function \(g\) given by (52) will be of appreciable value, namely,

$$|g_1| \simeq 2 \sigma \varphi_0,$$

(with \(\omega_{\text{res}} = \frac{\omega}{2}\)), \hspace{1cm} (77)

and

$$|g_2| \simeq h \varphi_0^2,$$

(with \(\omega_{\text{res}} = \omega\)). \hspace{1cm} (78)

If the contribution from \(n = 1\) is dominant in (74) (that is if \(\varphi_0 < \sigma/h\)) then from (76), (77) and using the expression (65) for \(\omega\) and the explicit expression for the inflaton energy density \(\rho = \lambda \varphi_0^4\), we get

$$\Gamma^{(\text{res})}_\chi \simeq \frac{32 \pi}{e^3 \lambda^{3/2}} \frac{\sigma^2}{\varphi_0^2} \log^{-1} \left( \frac{512 \pi^2}{e^4 \lambda} \right),$$

for \(\varphi_0 < \frac{\sigma}{h}\). \hspace{1cm} (79)

In this case the adiabaticity condition (61) reads

$$\varphi_0^2 \ll \frac{M_P \sigma}{\lambda}.$$  \hspace{1cm} (80)

If the contribution from \(n = 2\) is dominant in (74) then from (76), (78) we in
a similar way obtain
\[ \Gamma^{(\text{res})}_\chi \approx \frac{\pi h^2}{e^4 \lambda^{3/2}} \varphi_0 \log^{-1} \left( \frac{32 \pi^2}{e^4 \lambda} \right), \quad \text{for } \varphi_0 > \frac{\sigma}{\hbar}. \] (81)

In this case the adiabaticity condition (61) implies
\[ \varphi_0 \ll M_p \frac{\hbar}{\lambda}. \] (82)

According to the note made at the end of the previous section, the values (79) and (81) for \( \Gamma^{(\text{res})}_\chi \) are to be compared to the values of \( \Gamma_\chi \) in the corresponding cases. The expression for \( \Gamma_\chi \) is given by (47). Calculating the ratios of \( \Gamma^{(\text{res})}_\chi \) to \( \Gamma_\chi \) in the two cases discussed just above we obtain
\[ \frac{\Gamma^{(\text{res})}_\chi}{\Gamma_\chi} \approx \frac{128 \pi^2}{e^2 \lambda} \log^{-1} \left( \frac{512 \pi^2}{e^4 \lambda} \right), \quad \text{for } \varphi_0 < \frac{\sigma}{\hbar}, \] (83)
\[ \frac{\Gamma^{(\text{res})}_\chi}{\Gamma_\chi} \approx \frac{4 \pi^2}{e^2 \lambda} \log^{-1} \left( \frac{32 \pi^2}{e^4 \lambda} \right), \quad \text{for } \varphi_0 > \frac{\sigma}{\hbar}. \] (84)

We see that these ratios look similar in two cases considered, and for typical values of the inflaton coupling constant [1]
\[ \lambda \sim 10^{-12} \] (85)

they are extremely large. Hence, for the inflaton field with the potential \( \lambda \varphi^4 \) with small \( \lambda \), in the domain of applicability of perturbation theory (given by (44)), parametric resonance effects strongly dominate over the usual Born approximation estimates, so that the latter represent a serious underestimation of the particle production rate.
Let us proceed further and estimate the possible reheating temperature for the model considered.\footnote{We take the reheating temperature to be the temperature which would correspond to the energy density of the particles produced. This assumes that the particles produced will be thermalized soon after their creation. In this paper we do not touch on the theory of the thermalization.} The equation for the energy density of the particles produced is (63). We should take into account also the equation (67), modify the equation (12) as follows

\[ H^2 = \frac{8\pi}{3M_p^2} (\rho + \rho_p), \]

and then solve the system of equations (63), (67) and (86) with initial condition $\rho_p = 0$ at the moment of the end of inflation. Then the maximal value achieved in the course of evolution by $\rho_p$ would determine the reheating temperature. The system of equations mentioned can be solved exactly if we use the condition (75) to neglect $\Gamma_\psi$ and the expressions (79) or (81) for $\Gamma^{(\text{res})}_\chi$. In the case (79) the solution will be

\[ \rho = \rho_0 \left( 1 - \frac{\Gamma_0}{12H_0} (e^{3\tau} - 1) \right)^4 e^{-4\tau}, \]

and in the case (81) the solution is

\[ \rho = \rho_0 \left( 1 + \frac{\Gamma_0}{4H_0} (e^{\tau} - 1) \right)^{-4} e^{-4\tau}, \]

where $\Gamma_0$, and $H_0$ are the initial values of $\Gamma^{(\text{res})}_\chi$ and $H$ respectively, $\rho_0$ and $a_0$ are the initial values of the total energy density and the scale factor, and $\tau = \log (a/a_0)$.

In both cases (87) and (88) the solution for $\rho_p$ is given by

\[ \rho_p = \rho_0 e^{-4\tau} - \rho. \]

Until the condition $\rho_p \ll \rho$ holds, the solutions (87) and (88) can be approxi-
mately written as
\[ \rho \approx \rho_0 \left(1 - \frac{1}{\Omega} \left( \frac{\Gamma_{\chi}^{(\text{res})}}{H} - \frac{\Gamma_0}{H_0} \right) \right)^4 e^{-4\tau}, \quad (90) \]
and
\[ \rho \approx \rho_0 \left(1 + \frac{1}{\Omega} \left( \frac{\Gamma_{\chi}^{(\text{res})}}{H} - \frac{\Gamma_0}{H_0} \right) \right)^{-4} e^{-4\tau}, \quad (91) \]
respectively.

The solutions obtained determine the maximal value achieved by \( \rho_\rho \). Using the expressions (89), (90) and (91) it is easy to see that this value is achieved as soon as the condition
\[ \Gamma_{\chi}^{(\text{res})} \gtrsim 4 \, H \quad (92) \]
becomes valid. After this particle creation proceeds very effectively and the inflaton field loses all its energy in less then one Hubble time. We shall therefore use the condition (92) to estimate the reheating temperature in various cases. First, assume that reheating takes place in the region \( \varphi_0 > \sigma/h \). Then using the expression (81) for \( \Gamma_{\chi}^{(\text{res})} \) and omitting numerical factors we obtain from (92)
\[ \varphi_0 \lesssim M_\text{P} \left( \frac{h}{\lambda} \right)^2 \log^{-1} \frac{1}{\lambda}, \quad (93) \]
where we also used that \( \lambda \) is small to simplify the logarithmic factor. Now, the condition \( \varphi_0 > \sigma/h \) assumed above will read
\[ \sigma < M_\text{P} \ h \left( \frac{h}{\lambda} \right)^2 \log^{-1} \frac{1}{\lambda}. \quad (94) \]
Hence if the condition (94) is valid, the value of the inflaton field amplitude \( \varphi_0 \) at the moment of reheating is given by the right-hand side of (93). Note that the adiabaticity condition (82) is valid for the values of \( \varphi_0 \) given by (93) since the ratio \( h/\lambda \) has to be small according to the conditions (44).
If the condition (94) is not valid then reheating takes place in the region \( \varphi_0 < \sigma/h \) and we must use the expression (79) for \( \Gamma^{(\text{res})}_{\chi} \). In this case from (92) we obtain the estimate

\[
\varphi_0 \lesssim \frac{M_p^{1/3} \sigma^{2/3}}{\lambda^{2/3}} \log^{-1/3} \frac{1}{\lambda}, \tag{95}
\]

and the condition \( \varphi_0 < \sigma/h \) would imply, as it should, the opposite of (94), namely

\[
\sigma > M_p \left( \frac{h}{\lambda} \right)^2 \log^{-1} \frac{1}{\lambda}. \tag{96}
\]

In this case the last inequality together with (95) leads to the estimate

\[
\varphi_0 > M_p \left( \frac{h}{\lambda} \right)^2 \log^{-1} \frac{1}{\lambda}, \tag{97}
\]

for the boundary value of \( \varphi_0 \) at which effective reheating starts.

Due to the conditions (44) the adiabaticity condition (80) can again be seen to be valid for the values (95) of \( \varphi_0 \).

We see that the expression in the right-hand side of (93) gives the minimum possible value for the inflaton field amplitude \( \varphi_0 \) at the moment of reheating. We stress again that for the values of \( \varphi_0 \) obeying (93) the condition \( 4H \lesssim \Gamma^{(\text{res})}_{\chi} \) holds and reheating is extremely effective. To estimate the numerical value of the threshold (93) note, that according to the conditions (44) (which we remember to be necessary in order that perturbation theory is valid) we have the condition \( h \ll \lambda \). So taking for example

\[
\frac{h}{\lambda} \simeq 10^{-1} \tag{98}
\]

and taking into account (85) we will get from (93)

\[
\varphi_0 \lesssim 10^{-3} M_p. \tag{99}
\]

The reheating temperature \( T_{\text{rh}} \) is then estimated from the condition that the value of \( \rho_p \) at the moment of reheating is of the order of the value of \( \rho \) at the moment
when the condition (92) is first achieved, and from the relation

$$T_{rh} \simeq \rho_p^{1/4}. \tag{100}$$

Finally we obtain

$$T_{rh} \sim \lambda^{1/4} \varphi_0 \sim M_p \lambda^{1/4} \left( \frac{h}{\lambda} \right)^2 \log^{-1} \frac{1}{\lambda}. \tag{101}$$

In case of the values (85), (98) for the couplings we get

$$T_{rh} \sim 10^{-6} M_P \simeq 10^{13} \text{GeV}. \tag{102}$$

Now let us see what our condition (75) means. In the region $\varphi_0 > \sigma/h$ using the expressions (46) and (81) we obtain the condition for the coupling $f$ (again simplifying logarithmic factor by using smallness of the coupling constant $\lambda$)

$$f^2 < \mathcal{O}(1) \left( \frac{h}{\lambda} \right)^2 \log^{-1} \frac{1}{\lambda}, \tag{103}$$

which does not depend on the value of $\varphi_0$. For the values (98) of the ratio $h/\lambda$ the condition (103) is not very restrictive.

In the region $\varphi_0 < \sigma/h$ using expression (79) we obtain from (75) the estimate

$$f^2 < \mathcal{O}(1) \left( \frac{1}{\lambda^2} \left( \frac{\sigma}{\varphi_0} \right)^2 \log^{-1} \frac{1}{\lambda}, \tag{104}$$

which depends on the value of $\varphi_0$ but certainly is satisfied for $\varphi_0$ sufficiently small. For the condition (104) to be valid for our estimate (95) of the inflaton field amplitude at the moment of reheating the coupling $f$ is to be

$$f^2 < \mathcal{O}(1) \left( \frac{\sigma}{\lambda M_P} \right)^{2/3} \log^{-1/3} \frac{1}{\lambda}. \tag{105}$$

Thus, to conclude this part, if the value $\sigma$ satisfies (94) and the coupling $f$ satisfies (103) then the value of the inflaton field amplitude $\varphi_0$ at the moment of
reheating is given by the right-hand side of (93). If the value $\sigma$ satisfies (96) and the coupling $f$ satisfies (105) then the value of the inflaton field amplitude $\varphi_0$ at the moment of reheating is given by the right-hand side of (95) and satisfies (97). In both cases the reheating temperature is not less then given by (101), with the numerical estimates (102) for the coupling values (85) and (98).

If neither of the two mutual conditions just mentioned holds then at the moment of reheating the condition (75) is not valid, and the reheating temperature is determined by fermionic particle production. In this latter case at the moment of reheating we have

$$\Gamma_{\chi}^{(\text{res})} < \Gamma_\psi,$$  \hspace{1cm} (106)

and in the condition (92) for reheating we should replace the value $\Gamma_{\chi}^{(\text{res})}$ by $\Gamma_\psi$. Then using expression (46) for $\Gamma_\psi$ we obtain for the value of the inflaton field amplitude $\varphi_0$ at the moment of effective reheating the well-known expression

$$\varphi_0 \lesssim f^2 M_P,$$  \hspace{1cm} (107)

and for the reheating temperature we get

$$T_{\text{rh}} \sim \lambda^{1/4} f^2 M_P.$$  \hspace{1cm} (108)

In the case (106) from the equation (74) we obtain the following expression for $\Gamma_{\chi}^{(\text{res})}$

$$\Gamma_{\chi}^{(\text{res})} \simeq \frac{\Gamma_\psi}{8 \pi^2 \rho} \sum_{\text{all resonance bands}} \omega_{\text{res}}^4 \sinh^2 \left( \frac{\pi |g_\chi|^2}{2 \Gamma_\psi \omega_{\text{res}}^3} \right).$$  \hspace{1cm} (109)

It is useful to compare this value of $\Gamma_{\chi}^{(\text{res})}$ to the value of $\Gamma_\chi$ given by (47). To do this note that mainly two resonance bands contribute to the sum (109), namely,
those given by (77) and (78). Therefore we can write

$$\Gamma^{(\text{res})}_\chi = \Gamma^{(\text{res})}_1(\sigma) + \Gamma^{(\text{res})}_2(h),$$

(110)

where $\Gamma^{(\text{res})}_1$ depends only on the coupling $\sigma$, and $\Gamma^{(\text{res})}_2$ only on the coupling $h$. These two terms in the sum (110) correspond to the two terms in the expression (47) for $\Gamma_\chi$. We then rewrite the expression for $\Gamma_\chi$ in the similar way

$$\Gamma_\chi = \Gamma_1(\sigma) + \Gamma_2(h).$$

(111)

Now using the expressions (46), (47), (65), (77) and (78) we can easily estimate the ratios of each of the two terms in (110) to its corresponding term in (111) as

$$\frac{\Gamma^{(\text{res})}_n}{\Gamma_n} \simeq \frac{1}{x_n} \sinh^2 x_n,$$

(112)

where

$$x_n = \frac{\pi |g_n|^2}{2\Gamma_\psi \omega_{\text{res}}^2}.$$

(113)

We can see that parametric resonance effects for the scalar field $\chi$ would be significant in the case (106) if the values of $x_n$ given by (113) were larger than unity. This last condition however can be shown to be incompatible with the assumption that fermionic particle production dominates at the moment of efficient reheating. Hence at this moment the value $\Gamma_\chi$ given by (47) is larger than the resonance value $\Gamma^{(\text{res})}_\chi$ given under the conditions considered by (109).

All the results obtained in this section for the inflaton field amplitude during effective reheating can be written in a single equation

$$\varphi_0 \lesssim M_P \max \left\{ f^2, \left(\frac{h}{\lambda}\right)^2 \log^{-1} \frac{1}{\lambda}, \left(\frac{\sigma}{M_P \lambda}\right)^{2/3} \log^{-1/3} \frac{1}{\lambda}\right\},$$

(114)

from which also all the necessary inequalities between coupling constants can be easily derived. We stress once again that for the values of $\varphi_0$ given by (114) the condition $4H < \Gamma_\psi + \Gamma^{(\text{res})}_\chi$ is valid so the reheating process is very effective.
The expression for the temperature of reheating is then

$$T_{rh} \sim \lambda^{1/4} \varphi_0 \sim$$

$$\sim M_p \lambda^{1/4} \max \left\{ f^2, \left( \frac{\hbar}{\lambda} \right)^2 \log^{-1} \frac{1}{\lambda}, \left( \frac{\sigma}{M_p \lambda} \right)^{2/3} \log^{-1/3} \frac{1}{\lambda} \right\}. \quad (115)$$

It is very instructive to compare our results (114), (115) to those which would be obtained by making use of Born approximation (described in Section 3) for the scalar particles $\chi$ production rates. If instead of $\Gamma^{(\text{res})}_{\chi}$ we used the value $\Gamma_{\chi}$ given by (47) then we would arrive at the following reheating conditions

$$\varphi_0 \lessapprox M_p \max \left\{ f^2, \lambda \left( \frac{\hbar}{\lambda} \right)^2, \lambda \left( \frac{\sigma}{M_p \lambda} \right)^{2/3} \right\}, \quad (116)$$

and

$$T_{rh} \sim M_p \lambda^{1/4} \max \left\{ f^2, \lambda \left( \frac{\hbar}{\lambda} \right)^2, \lambda \left( \frac{\sigma}{M_p \lambda} \right)^{2/3} \right\}. \quad (117)$$

One can see that for the scalar particle couplings the values (116), (117) are smaller by an enormous factor $[\lambda \log(1/\lambda)]^{-1}$ than the corresponding correct estimates (114) and (115).

6. Case of potential $V(\varphi) = \frac{1}{2} m^2 \varphi^2$ 

In this section the analysis will proceed along the same lines as in the previous one. In the case of a quadratic inflaton potential we get from (31), (32)

$$\omega = m,$$  \quad (118)

so the inflaton field oscillates with constant frequency. The resonance frequency $\omega_{\text{res}} = m \omega / 2$ will thus be constant in time. For the time derivative factor in (62)
we then obtain, taking into account (58)

$$\left| \frac{d \Delta \omega_k}{dt} \right|_{\Delta \omega_k=0} = H \omega_{\text{res}}, \quad (119)$$

and from (64) we will have

$$\Gamma_{\chi}^{(\text{res})} = \frac{H}{2 \pi^2 \rho} \sum_{\text{all resonance bands}} N_{\text{res}} \omega_{\text{res}}^4, \quad (120)$$

where all the relevant variables have been explained in Section 4. The expressions (59), (71) remain valid in the case considered but the estimate analogous to (72) will be modified as follows

$$\int \mu_+ dt \approx \int_{-|g_n|/2\omega_{\text{res}}}^{|g_n|/2\omega_{\text{res}}} \frac{1}{2\omega_{\text{res}}} \sqrt{|g_n|^2 - \Delta \omega_k^2 (2\omega_{\text{res}})^2} \frac{1}{H \omega_{\text{res}}} d \Delta \omega_k \approx \frac{|g_n|^2}{4 H \omega_{\text{res}}^3} \int_{-1}^{1} \sqrt{1 - x^2} dx = \frac{\pi}{8} \frac{|g_n|^2}{H \omega_{\text{res}}^3}, \quad (121)$$

so that the occupation numbers $N_{\text{res}}$ will be given by

$$N_{\text{res}} \simeq \sinh^2 \left( \frac{\pi}{8} \frac{|g_n|^2}{H \omega_{\text{res}}^3} \right), \quad (122)$$

and the equation (120) for $\Gamma_{\chi}^{(\text{res})}$ will read

$$\Gamma_{\chi}^{(\text{res})} \simeq \frac{H}{2 \pi^2 \rho} \sum_{\text{all resonance bands}} \omega_{\text{res}}^4 \sinh^2 \left( \frac{\pi}{8} \frac{|g_n|^2}{H \omega_{\text{res}}^3} \right). \quad (123)$$

We must compare this value to the value of $\Gamma_{\chi}$ obtained in the Born approximation. Writing $\Gamma_{\chi}^{(\text{res})}$ and $\Gamma_{\chi}$ in the form (110) and (111) and using the expressions
(47), (77) and (78), we will arrive at an estimate of the same form as (112) but with different $x_n$:

$$x_n = \frac{\pi}{8} \frac{|g_n|^2}{H \omega_{\text{res}}^3}.$$  \hspace{1cm} (124)

Again we see that parametric resonance effects for the scalar field $\chi$ are significant if the values $x_n$ are large.

As usual, one of the terms in the sum (123) will dominate. In the case $\varphi_0 < \sigma/h$ the harmonic with $n = 1$ dominates, whereas for $\varphi_0 > \sigma/h$ the harmonic corresponding to $n = 2$ is more important. The relevant expressions for the amplitudes in these two cases are given by (77) and (78).

In the case $\varphi_0 < \sigma/h$ we get

$$\Gamma^{(\text{res})}_\chi \sim \frac{H m^2}{16 \pi^2 \varphi_0^2} \sinh^2 \left( \frac{4 \pi \sigma^2 \varphi_0^2}{H m^3} \right),$$  \hspace{1cm} (125)

and the condition of effective particle production

$$\Gamma^{(\text{res})}_\chi \gtrsim 4H,$$  \hspace{1cm} (126)

in this case will read (omitting numerical factors)

$$\frac{\varphi_0}{m} \gtrsim \frac{m^3}{\sigma^2 M_P} \log \frac{m^3}{\sigma^2 M_P}.$$  \hspace{1cm} (127)

This estimate is valid only if the coupling constant $\sigma$ is in the range given by

$$\left( \frac{m}{M_P} \right)^2 \ll \left( \frac{\sigma}{m} \right)^2 \ll \frac{m}{M_P}.$$  \hspace{1cm} (128)

The first inequality in (128) is the condition for perturbation theory to be valid, it stems from (44). The second inequality in (128) is the criterion of large occupation numbers of the particles produced, that is, of effectiveness of the resonance particle production (in other terms, it is the condition for the value $x_1$ given by (124) to be larger then unity). The validity of the adiabaticity condition (61) in this case follows from (127) and from the first inequality in (128).
In any case there is also another region in which particle production is effective so that condition (125) holds. This region is given by

$$\frac{\varphi_0}{m} \lesssim \frac{\sigma^2 M_P}{m^3}, \quad (129)$$

and it can be obtained by using the first term in the expression (47) for the value $\Gamma_\chi$ since parametric resonance in this region is not effective.

In the case opposite to the one just considered, namely, when $\varphi_0 > \sigma/h$ we obtain

$$\Gamma_\chi^{(\text{ms})} \approx \frac{H m^2}{\pi^2 \varphi_0^2} \sinh^2 \left( \frac{\pi h^2 \varphi_0^4}{8 H m^3} \right), \quad (130)$$

and the condition (126) will give the estimate

$$\left( \frac{\varphi_0}{m} \right)^3 \gtrsim \frac{m}{h^2 M_P} \log \frac{m}{h^2 M_P}. \quad (131)$$

The theory which led to the result (131) is self-consistent if the coupling $h$ is in the range

$$\left( \frac{m}{M_P} \right)^2 \ll h^2 \ll \frac{m}{M_P}. \quad (132)$$

Again, the first inequality in (132) is the requirement that perturbation theory be valid and the second one is the requirement of large occupation numbers of the particles produced. If this last inequality is not valid then the region of effective particle production is given by the condition

$$\frac{\varphi_0}{m} \gtrsim \frac{m}{h^2 M_P}, \quad (133)$$

which can be obtained by making use of the second term in the expression (47) for the value $\Gamma_\chi$. For the values of $\varphi_0$ (131) the adiabaticity condition (61) can be shown to hold.
The estimates (127), (131) and (133) are remarkable in that they determine not only the possible reheating temperature but also the “freeze-out” boundary for the inflaton field $\varphi$: below the values of $\varphi_0$ given by these expressions the particle creation process becomes ineffective, the energy density of ultrarelativistic particles created decreases more rapidly than that of the inflaton field and the universe can well become again dominated by the inflaton field $\varphi$ until the condition (129) is satisfied. It is easy to see that in all the cases considered above the minimal possible “freeze-out” and/or the maximal possible reheating temperature is

$$T \sim m.$$  \hspace{1cm} (134)

For the values of the inflaton mass consistent with the cosmological bounds [1],

$$m \sim 10^{-6} \, M_P \approx 10^{13} \, \text{GeV},$$  \hspace{1cm} (135)

the maximal possible reheating temperature also will be of the same order.

If interactions of the inflaton $\varphi$ with fermions take place, we obtain, using Equation (46), the following estimate

$$\varphi_0 \lesssim f^2 \, M_P$$  \hspace{1cm} (136)

for the reheating condition, which coincides with the analogous estimate (107) in the case of the potential $V(\varphi) = \lambda \varphi^4$ which we considered in the previous section. It can be shown, using the expression (31) for the value $\omega$ and the expression (46) for the value of $\Gamma_\varphi$, that the estimate (136) does not depend on the power $q$ of the inflaton potential $V(\varphi)$.
7. Discussion

In this paper we considered the problem of scalar particle production after inflation by a quickly oscillating inflaton field. We were using the framework of the chaotic inflation scenario with quartic and quadratic inflaton potentials, and we considered linear and quadratic coupling of the inflaton field $\phi$ to a quantum scalar field $\chi$ (see formula (28) for the interaction Lagrangian). The particle production effect has been studied using the standard technique of Bogolyubov transformations. Specific attention has been paid to parametric resonance phenomena which take place in the presence of a quickly oscillating inflaton field.

We have found that in the region of applicability of perturbation theory (when inequalities (44) are valid) the effects of parametric resonance are crucial, and estimates based on first order Born approximation are often not correct.

In the case of the quartic inflaton potential $V(\phi) = \lambda \phi^4$ the reheating process is very efficient for any type of coupling between the inflaton field and the scalar field $\chi$ even for small values of coupling constants. The reheating temperature in this case is $[\lambda \log (1/\lambda)]^{-1}$ times larger than the corresponding estimates based on first order Born approximation (see the expressions (115), (117)).

In the case of a quadratic inflaton potential the situation is more complicated. The theory predicts not only the possible reheating temperature but also the effective energy density of the inflaton field “freezing out” below which the inflaton field energy density decrease is dominated by the cosmological redshift rather then by particle production. The inflaton, so to say, decouples from the matter it creates. The reheating process depends crucially on the type of coupling between the inflaton and the scalar field $\chi$ and on the magnitudes of the coupling constants $\sigma$ and $h$. With an interaction linear in the inflaton field there is, besides the freeze-out boundary (127) which exists for couplings in the range (128), also a reheating boundary (129). For an interaction quadratic in the inflaton field the freeze-out boundary (131) exists in the range (132) of the values of coupling $h$. If the coupling
is not in this range then this boundary is given by the expression (133) obtained in the Born approximation.

If the inflaton potential is quadratic then one can see that in the case of absence of inflaton coupling to fermions and without linear (in inflaton field) coupling to scalar particles there will be no efficient particle creation below the threshold (132) or (133). In this case of incomplete reheating, the residual inflaton oscillations may provide the (cold) dark matter of the universe. In the presence of an interaction linear in the inflaton field, eventual reheating will take place below the threshold (129) which will be small for a small value of the coupling $\sigma$.

The situation is more complex for more complicated potentials. For example, one may wish to consider the inflaton potential of shape

$$V(\varphi) = \lambda (\varphi^2 - \eta^2)^2,$$

with spontaneous symmetry breaking at a scale $\eta$. In the chaotic inflation scenario the inflaton field rolls from large values of $|\varphi|$ to the minimum of its potential. After inflation it starts quasiperiodic motion. For values of the amplitude $\varphi_0 \gg \eta$ the behaviour of the inflaton is like for a $\lambda \varphi^4$ potential, and reheating proceeds as described in Section 5. But as soon as the magnitude of $\varphi_0$ becomes close to $\eta$ the dynamics changes and the inflaton field starts oscillating around one of the minima of its potential at $|\varphi| = \eta$. The inflaton potential in the vicinity of its minimum can be approximated as quadratic, hence the reheating (freeze-out) will be described by the theory of Section 6.

After this work was completed we became aware of a recent preprint [13] in which the same class of problems are considered. While our paper deals only with the case of small couplings of the inflaton field to the other fields (so that perturbation theory is valid), both the cases of small and large couplings are considered in [13]. In the case of large values of the coupling constants, higher resonance bands dominate the contribution to the particle production. After the inflaton field amplitude decreases sufficiently one enters into the regime which for certain
types of inflaton potentials and for certain couplings can be accurately described by perturbation theory. The results announced in [13] for the case when perturbation theory is valid in general agree with those obtained in our paper in Chapters 5 and 6. There are some results for the case of large couplings which we would like to compare with our results for the case of small couplings. Recall that the couplings are considered “small” if the conditions (44) are valid. They are “large” in the opposite case.

In the case of the quadratic inflaton potential\(^1\) \(V(\varphi) = \frac{1}{2}m^2\varphi^2\) and for coupling quadratic in the inflaton field \(L_{\text{int}} = -h\varphi^2\chi^2\), the authors of [13] find that growth of the modes in the higher instability bands leads to the estimate for the number of particles \(N_k(t) \propto \exp(0.5mt)\) in the case \(h\varphi_0^2 \gg m^2\) (opposite to (44)). In the case (44), i.e. when \(h\varphi_0^2 \ll m^2\), we can see using the expressions (59), (60) that \(N_k(t) \propto \exp(2\mu_+t) \simeq \exp\left((2h\varphi_0^2/m^2)mt\right)\) which for \(h\varphi_0^2 \simeq 0.25m^2\) (in our case) would lead to the same numerical estimate for \(N_k\) as in the case of large coupling. These two cases seem to differ in the widths of the corresponding resonance bands, which in the case of small coupling is more narrow than in the case of large coupling.

In the case of quartic inflaton potential the estimate of [13] \(N_k(t) \propto \exp\left(\frac{1}{15}\sqrt{\lambda}\varphi_0t\right)\) for the case of coupling \(h = \lambda\) (self-excitation of the inflaton) can be compared with our estimate \(N_k(t) \propto \exp(2\mu_+t) \simeq \exp\left(\frac{2h}{\lambda}\sqrt{\lambda}\varphi_0t\right)\) for the case of small coupling \((h \ll \lambda)\). We see again that these estimates will be numerically the same if we choose our small coupling to be \(h \simeq 0.05\lambda\). For the values of \(h \simeq 0.07\lambda\) our estimate (93) for the beginning of reheating will be numerically the same as the estimate of [13] \(\varphi_0 \lesssim 0.005M_P\log^{-1}(1/\lambda)\) for \(h = \lambda\). Therefore in the case of validity of perturbation theory (equations (44)) with coupling constant \(h\) not too small, for the conditions of effective particle production we can obtain numerical estimates larger than those obtained in [13] in the case of coupling \(h = \lambda\). This shows that self-excitation of the inflaton field is not necessarily the leading effect in the case of coupling \(h \ll \lambda\). For it to be leading additional smallness of the

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\(^1\) We use the notation of the present paper which differs somewhat from that of paper [13].
couplings $h$ is required, in the case considered, $h \ll 0.07\lambda$. In the range of still small couplings $h$ such that, say, $0.07 < h/\lambda < 0.3$ the resonance effects appear to be more efficient than in the case $h = \lambda$. This seems to be due to the fact that in the case of small coupling the resonance is taking place in the lowest resonance band in which it is the strongest whereas for couplings $h \gtrsim \lambda$ the frequency $\omega_k$ can take its numerical values only in the higher resonance bands in which the resonance effect is known to be weaker.

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Appendix A

In this appendix we present necessary expressions which describe the effect of parametric resonance in the lowest resonance band.

Consider the equation for the function $Y(t)$ of the following form

$$
\ddot{Y} + \left( \omega_0^2 + \epsilon g(\omega t) \right) Y = 0,
$$

where $\omega_0$ and $\omega$ are constant parameters, $g(x)$ is a $2\pi$-periodic function, and $\epsilon$ is a small number. Our task is to find the approximate solution to the equation (A1) in the first instability band of the frequencies $\omega_0$. To do this we will use Bogolyubov method of averaging \cite{14} which implies perturbation theory of certain type in the small parameter $\epsilon$.

The function $g(x)$ being periodic, it can be developed in Fourier series as follows

$$
g(x) = \sum_{n=-\infty}^{\infty} g_n e^{inx},
$$

with the amplitudes $g_n$ satisfying

$$
g_n^* = g_{-n}.
$$

With no loss of generality (for a small parameter $\epsilon$) we can put $g_0 = 0$ (redefining $\omega_0$ if necessary). For the following purposes it will be convenient to introduce the phases $\alpha_n$ as follows

$$
g_n = |g_n| e^{i\alpha_n}.
$$

Let us introduce the value $\Delta$ by the following relation

$$
\omega_0^2 = \left( \frac{p}{q} \omega \right)^2 + \epsilon \Delta,
$$

where $p/q$ is a rational non-contractible number ($p$ and $q$ are integers). The equa-
tion \((A1)\) then becomes

\[
\ddot{Y} + \left(\frac{p}{q} \omega\right)^2 Y = -\epsilon (g(\omega t) + \Delta) Y. \tag{A6}
\]

The Bogolyubov method \([14]\) consists in looking for a solution to the equation \((A6)\) in the form of the following expansion in powers of \(\epsilon\)

\[
Y = a \cos \psi + \sum_{s=1}^{\infty} \epsilon^s u^{(s)}(a, \theta, \frac{\omega}{q} t), \tag{A7}
\]

where

\[
\psi = \frac{p}{q} \omega t + \theta, \tag{A8}
\]

the \(u^{(s)}\) are functions periodic in their second and third arguments, \(a\) is the amplitude and \(\theta\) is the phase of the solution \((A7)\). The values of \(a\) and \(\theta\) are not constant in the Bogolyubov approach. Rather, they are functions of time \(t\).

We will be concerned explicitly only with the first order approximation in the small parameter \(\epsilon\). For the function \(u^{(1)}\) we have

\[
u^{(1)}(a, \theta, \frac{\omega}{q} t) = \sum_{n=-\infty}^{\infty} u_n(a, \theta) e^{in\frac{\omega}{q} t}, \tag{A9}
\]

with coefficients \(u_n(a, \theta)\) periodic in \(\theta\).

Clearly, in the case of \(\epsilon = 0\) the solution to \((A6)\) will be just the first term in the right-hand side of \((A7)\), with arbitrary constant amplitude \(a\) and phase \(\theta\), so in this case we have \(\dot{a} = 0, \dot{\theta} = 0\). For \(\epsilon \neq 0\) according to the Bogolyubov method we regard the values \(\dot{a}\) and \(\dot{\theta}\) as functions of \(a\) and \(\theta\) and we expand these functions in powers of \(\epsilon\)

\[
\dot{a} = \epsilon A(a, \theta) + O(\epsilon^2), \tag{A10}
\]

\[
\dot{\theta} = \epsilon B(a, \theta) + O(\epsilon^2).
\]

Putting the anzatz \((A7)\) into the equation \((A6)\) and using \((A10)\), after collecting
terms of first order in \( \epsilon \) we will get the following equation for the function \( u^{(1)} \)

\[
\ddot{u}^{(1)} + \left( \frac{p}{q} \right)^2 u^{(1)} = -(g + \Delta) a \cos \psi + 2 \frac{p}{q} \omega A \sin \psi + 2 \frac{p}{q} \omega B a \cos \psi . 
\]  

(A11)

Using the expansions (A2) and (A9) we can develop both sides of the equation (A11) in harmonics \( \exp(i \frac{\pi}{q} \omega t) \). On the left-hand side of (A11) we will have, to the zeroth order in \( \epsilon \),

\[
\sum_{n = -\infty}^{\infty} \left[ \left( \frac{n}{q} \right)^2 - \left( \frac{n}{q} \right) ^2 \right] \omega^2 u_n \epsilon^{i \frac{\pi}{q} \omega t} ,
\]

(A12)

and we see that terms with \( n = \pm p \) in this expansion are equal to zero. The corresponding terms in the development of the right-hand side of (A11) should also vanish. These conditions allow us to determine the functions \( A(a, \theta) \) and \( B(a, \theta) \). If we denote by \( s \) the number

\[
s = \frac{2p}{q} ,
\]

(A13)

then the expressions for these functions will look as follows

\[
A(a, \theta) = \frac{a |g_s|}{\omega} \sin(2\theta - \alpha_s) ,
\]

(A14)

\[
B(a, \theta) = \frac{1}{\omega} (\Delta + |g_s| \cos(2\theta - \alpha_s)) ,
\]

with the phases \( \alpha_s \) defined in (A4) for \( s \) integer. The expressions (A14) are valid for arbitrary \( s \), in the case of \( s \) non-integer we should simply put \( g_s \) to zero, and the (unspecified) value of \( \alpha_s \) is of no importance.

If we make the change of variables

\[
x = a \cos \left( \theta - \frac{\alpha_s}{2} \right) ,
\]

(A15)

\[
y = a \sin \left( \theta - \frac{\alpha_s}{2} \right) ,
\]
then the system (A10) in terms of new variables $x$ and $y$ will look as follows

$$
\dot{x} = \frac{c}{s\omega} (|g_s| - \Delta) y + O(\epsilon^2),
$$

$$
\dot{y} = \frac{c}{s\omega} (|g_s| + \Delta) x + O(\epsilon^2).
$$

(A16)

To first order in $\epsilon$ the system (A16) is linear and its eigenvalues are

$$
\mu_\pm = \pm \frac{c}{s\omega} \sqrt{|g_s|^2 - \Delta^2}.
$$

(A17)

From this expression one can see that $\mu_\pm$ can be real only if $g_s \neq 0$, that is if $s$ is integer. Hence unstable growth of the solution $Y$ to the equation (A6) can take place only for the values of $\Delta$ ($\Delta$ is defined in (A5))

$$
\Delta < |g_s|,
$$

(A18)

and the growth (decay) rate is exponential, $Y \sim e^{\mu_\pm t}$, with $\mu_\pm$ given by (A17).

For every integer $s$ for which $g_s$ is non-zero the expression (A17) gives the growth rate of the solution $Y$ in the first instability band determined by (A5) and (A18). There are also infinite number of instability bands whose width and growth rates are higher order in $\epsilon$ (to all orders of $\epsilon$). Relevant expressions can be obtained to any desirable order in $\epsilon$ by keeping track of the higher order terms in the expansions (A7) and (A10). For the purpose of this paper these higher order expressions are not required.

**Appendix B**

The aim of this Appendix is to provide necessary details of the resonance particle production effect studied in Section 3 of this paper, in particular, to derive the formula (60). We first consider the general problem of a harmonic oscillator with time-dependent frequency, and after that the problem of resonant scalar particle production.
a) Harmonic oscillator with time-dependent frequency.

Consider an oscillator with coordinate $Q$, conjugate momentum $P$, whose frequency $\Omega$ depends on time so that the Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \left( P^2 + \Omega^2(t) Q^2 \right). \quad (B1)$$

For a quantum oscillator the values $Q$ and $P$ are Hermitian operators with standard commutation relations (we put $\hbar = 1$)

$$[Q, P] = i. \quad (B2)$$

In the Schrödinger representation, these operators are time-independent.

Our aim will be to go to a convenient time-dependent frame in Hilbert space, in which the Hamiltonian (B1) is diagonal at every moment of time. To do this define time-dependent operators $a(t)$ and $a^\dagger(t)$ by

$$a = \frac{e^{i \int \Omega \, dt}}{\sqrt{2 \Omega}} \left( \Omega Q + iP \right), \quad (B3)$$

$$a^\dagger = \frac{e^{-i \int \Omega \, dt}}{\sqrt{2 \Omega}} \left( \Omega Q - iP \right). \quad (B4)$$

These operators are mutually Hermitian conjugate and have the standard commutation relations of creation-annihilation operators

$$[a, a^\dagger] = 1. \quad (B4)$$

In terms of these operators the Hamiltonian is expressed as follows

$$\mathcal{H} = \Omega \left( \frac{1}{2} + a^\dagger a \right). \quad (B5)$$

The orthonormal frame in Hilbert space which diagonalizes the Hamiltonian
The Hamiltonian \( \mathcal{H} \) is given by time-dependent states

\[
|n_t> = \frac{(a^\dagger(t))^n}{\sqrt{n!}} |0_t>, \tag{B6}
\]

counted from the time-dependent vacuum state \(|0_t>\) which is annihilated by the operator \(a(t)\).

The operators \(a(t)\) and \(a^\dagger(t)\) obey the following equations of motion

\[
\dot{a} = \frac{\dot{\Omega}}{2\Omega} e^{2i \int \Omega dt} a^\dagger, \tag{B7}
\]

\[
\dot{a}^\dagger = \frac{\dot{\Omega}}{2\Omega} e^{-2i \int \Omega dt} a,
\]

so these operators vary with time only when \(\Omega\) varies \(^1\).

The solution to the equations (B7) can be written in terms of constant creation and annihilation operators \(a_0^\dagger\) and \(a_0\) as follows

\[
a(t) = \alpha(t) a_0 + \beta^*(t) a_0^\dagger, \tag{B8}
\]

\[
a^\dagger(t) = \beta(t) a_0 + \alpha^*(t) a_0^\dagger,
\]

with the complex functions \(\alpha(t)\) and \(\beta(t)\) obeying the system of equations similar to (B7)

\[
\dot{\alpha} = \frac{\dot{\Omega}}{2\Omega} e^{2i \int \Omega dt} \beta, \tag{B9}
\]

\[
\dot{\beta} = \frac{\dot{\Omega}}{2\Omega} e^{-2i \int \Omega dt} \alpha.
\]

The system (B8) represents what is called Bogolyubov transformation between two pairs of creation - annihilation operators.

---

1) This property is achieved by a convenient choice of the time-dependent phase factors in the definition (B3) of the operators \(a\) and \(a^\dagger\).
If the oscillator initially (at $t = 0$) is in the vacuum state $|0_t\rangle$ is annihilated by the operator $a_0$ and the initial conditions for the functions $\alpha(t)$ and $\beta(t)$ are

$$|\alpha(0)| = 1, \quad \beta(0) = 0.$$  (B10)

At the moment $t$ it will not be in the vacuum state $|0_t\rangle$, annihilated by the operator $a(t)$. Rather, there is the following relation between the states considered

$$|0_t\rangle = \frac{1}{\sqrt{\alpha(t)}} \exp \left( \frac{\beta^*(t)}{2\alpha^*(t)} \left( a \dagger(t) \right)^2 \right) |0_t\rangle,$$  (B11)

which follows from the equations (B8). At the moment $t$ the average number of the oscillator excitation level (number of quanta) is

$$N(t) = \langle 0_t | a \dagger(t) a(t) |0_t\rangle = |\beta(t)|^2.$$  (B12)

In the Heisenberg representation, the equation of motion for the operator $Q(t)$ is

$$\ddot{Q} + \Omega^2 Q = 0.$$  (B13)

The solution to this equation can be expressed in terms of the operators $a_0$ and $a_0^\dagger$ so that

$$Q(t) = Q^{(-)}(t) a_0 + Q^{(+)}(t) a_0^\dagger.$$  (B14)

Comparing this expression with the equations (B3) and (B8) we can express the coefficient $\beta(t)$ in terms of $Q^{(-)}(t)$ at the moment of time at which the first time derivative of the frequency $\Omega$ vanishes as follows

$$\beta = \frac{1}{\sqrt{2\Omega}} e^{-i\int \Omega dt} \left( \Omega Q^{(-)} - i \dot{Q}^{(-)} \right).$$  (B15)

b) Resonance particle production.
Now let us turn to the issue of scalar particle creation in the external spatially homogeneous periodic field $\epsilon g(\omega t)$ whose properties were described in Appendix A (see the equations (A2)-(A4)). Remember that $\epsilon$ is a small parameter of the perturbation theory. The scalar field operator $\chi$ which describes particles with mass $m_\chi$ can be decomposed into spatial Fourier modes

$$\chi = \int \frac{d^3k}{(2\pi)^3} Q_k e^{ikx}, \quad (B16)$$

so that every mode corresponds to a quantum oscillator with Hamiltonian (B1), complex coordinate $Q = Q_k$ and frequency $\Omega = \Omega_k$ given by

$$\Omega_k^2 = \omega_k^2 + \epsilon g(\omega t), \quad (B17)$$

where $\omega_k^2 = m_\chi^2 + k^2$. Because the frequency $\Omega_k$ depends on time, the oscillator with label $k$ will be excited and this means particle production. We can use all the above expressions obtained for the generic oscillator by simply adding a subscript $k$. Thus, the theory of scalar particle production in the case under consideration is only a slight modification of the oscillator formalism described above. One only has to take into account correctly the presence of infinite number of oscillators labelled by wavenumbers $k$ and interrelations between them. The expression analogous to (B11) relating the initial scalar field vacuum to the vacuum at the moment of time $t$ will look like follows

$$|0_0 > = \prod_k \frac{1}{\sqrt{[\alpha_k(t)]}} \exp \left( \frac{\beta_k^*(t)}{2\alpha_k^*(t)} a_k(t) a_{-k}^*(t) \right) |0_t >, \quad (B18)$$

where $a_k^*(t)$ and $a_k(t)$ are the time-dependent creation and annihilation operators which diagonalize the scalar field Hamiltonian at the moment $t$. From the expression (B18) it can be seen that scalar particles are created in pairs with opposite wavenumbers $k$. The average number of particles produced at the moment $t$ in the $k$-th mode will be given by the expression similar to (B12) with the only modification being the addition of subscript $k$. To determine the average number $N_k$
of particles produced we can use the formulas (B12) and (B15). We only need to know the solution for \( Q_k^{(-)}(t) \). This function obeys the equation
\[
\ddot{Q}_k^{(-)} + (\omega_k^2 + \epsilon g(\omega t)) Q_k^{(-)} = 0,
\]
which is just the equation (A1) which we considered in the Appendix A. Initial conditions for \( Q_k^{(-)} \) stem from the initial vacuum conditions (B10) and are
\[
| Q_k^{(-)}(0) | = \frac{1}{\sqrt{2\Omega_k(0)}}, \quad \left( \Omega_k Q_k^{(-)} - i\dot{Q}_k^{(-)} \right) \bigg|_{t=0} = 0.
\]

The solutions to the equation (B19) in the resonance band were studied in the Appendix A. According to (A5) we introduce the value \( \Delta \) by
\[
\omega_k^2 = \left( \frac{s}{2} \omega \right)^2 + \epsilon \Delta,
\]
where we replaced \( p/q \) in advance by the half-integer \( s/2 \) according to the results of the Appendix A. Now the solution to \( Q_k^{(-)} \) will be given by a complex linear combination of the solutions of type (A7), namely
\[
Q_k^{(-)} = \frac{1}{\sqrt{2\omega_k}} (a_1 \cos \psi_1 + ia_2 \cos \psi_2) + O(\epsilon),
\]
with
\[
\psi_{1,2} = \frac{s}{2} \omega t + \theta_{1,2}.
\]
The initial conditions for \( a_{1,2} \) and \( \theta_{1,2} \) stem from (B20)
\[
a_1 = a_2 = 1, \quad \theta_2 - \theta_1 = \frac{\pi}{2}.
\]

If in analogy to (A15) we introduce the variables \( x_{1,2} \) and \( y_{1,2} \) as
\[
x_1 = a_1 \cos \theta_1, \quad x_2 = a_2 \cos \theta_2, \quad y_1 = a_1 \sin \theta_1, \quad y_2 = a_2 \sin \theta_2,
\]
then to first order in the small parameter \( \epsilon \) the equations for \( x_{1,2} \) and \( y_{1,2} \) will be

1) The constant value of the phase \( \alpha_n \) defined in (A4) can be eliminated without loss of generality by a shift of the values of \( \theta_{1,2} \).
identical to (A16). The initial conditions for these functions follow from (B24)
\[ x_1(0) = y_2(0), \quad x_2(0) = -y_1(0), \]
\[ |x_1|^2 + |y_1|^2 = |x_2|^2 + |y_2|^2 = 1. \]  
(B26)

To the lowest order in \( \epsilon \), the expression for the value \( \beta_k \) is given by
\[ \beta_k = \frac{1}{2} (x_1 + ix_2 + iy_1 + iy_2). \]  
(B27)

It obeys the equation
\[ \ddot{\beta}_k = \mu_+^2 \beta_k, \]  
(B28)

which follows from the system (A16), with \( \mu_+ \) given by (A17). The initial conditions for the value \( \beta_k \) stem from (B26) and are
\[ \beta_k(0) = 0, \quad |\dot{\beta}_k(0)| = \mu_+ |\Delta_{-0}|. \]  
(B29)

Solving the linear system (B28) for the function \( \beta_k \) with the initial conditions (B29) we obtain the expression for the mean number of the particles produced
\[ N_k \equiv |\beta_k|^2 = \frac{1}{1 - \Delta^2/|g_4|^2} \sinh^2 (\mu_+ t), \]  
(B30)

where as we recall, the value \( \mu_+ \) is given by (A17).

The analysis developed in this Appendix can be also used in the case when the value of \( \mu_+ \) is not constant in time but changes adiabatically. Namely, if
\[ |\dot{\mu_+}| \ll \mu_+^2, \]  
(B31)

then the mean number of the particles created can be approximated by
\[ N_k \simeq \sinh^2 \left( \int \mu_+ \, dt \right), \]  
(B32)

where the integral in the argument of the hyperbolic sine is taken over the time during which resonance condition (A18) is satisfied for the mode with wavenumber \( k \), and the prefactor similar to that of (B30) has been set to unity which is
reasonable because at the edge of the resonance waveband (when $\Delta \approx |g_4|$) the adiabaticity condition (B31) ceases to be valid.
REFERENCES


