Electromagnetic Pion Form Factor and Neutral Pion Decay Width

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Abstract

The electromagnetic pion form factor, \( F_\pi(q^2) \), is calculated for spacelike \( q^2 \) in generalised impulse approximation using a confining quark propagator, \( S \), and a dressed quark-photon vertex, \( \Gamma_\mu \), obtained from realistic, nonperturbative Dyson-Schwinger equation studies. Excellent agreement with the available data is obtained for \( F_\pi(q^2) \) and many other pion observables, including the decay \( \pi^0 \rightarrow \gamma \gamma \). The implementation of confinement eliminates endpoint and pinch singularities in the calculation of \( F_\pi(q^2) \). With asymptotic freedom manifest in \( S \) this calculation yields \( q^2 F_\pi(q^2) = \text{constant} \), up to \( \ln[q^2]\)-corrections, for spacelike \( q^2 \gtrsim 35 \text{ GeV}^2 \), which indicates that soft, nonperturbative contributions dominate the form factor at presently accessible \( q^2 \).

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I. INTRODUCTION

The electromagnetic pion form factor, \( F_\pi(q^2) \), has been much studied but the applicability of perturbative QCD to the associated exclusive process remains controversial. All perturbative analyses of its behaviour at large spacelike-\( q^2 \) rely on the separation of the amplitude into a product of soft and hard contributions using the “factorisation theorem” [1–3]. In that approach the net result is the product of a soft contribution that depends on the bound-state Bethe-Salpeter amplitude, which provides only the overall normalisation, and a hard contribution that is independent of the bound-state Bethe-Salpeter amplitude and is taken to be given by the Born amplitude for a collinear quark-antiquark pair, each massless, to scatter coherently from a virtual photon. This analysis yields \( q^2 F_\pi(q^2) = 16 \alpha(q^2) f_\pi^2 \) as spacelike-\( q^2 \to \infty \), where \( f_\pi \approx 93 \text{ MeV} \) and \( \alpha(q^2) \) is the running coupling constant in QCD.

A controversy exists in whether this result is useful; i.e., are presently accessible values of \( q^2 \) really large enough to be testing predictions based on perturbative analyses in QCD? The conclusion drawn in Refs. [4, 5] is no; i.e., that the “factorisation theorem” is invalid at presently accessible values of \( q^2 \) and hence that the \( q^2 \) dependence of the quark momentum distribution in the pion provides an important contribution to \( F_\pi(q^2) \). This conclusion is supported by Ref. [6] and by the fact that a good fit to the experimental data, over the entire range of available \( q^2 \), is possible using the light-front formulation of a relativistic constituent quark model [7], which has no obvious connection with perturbative QCD. Nevertheless, this conclusion is denied in Ref. [8], which claims that “Sudakov” suppression [QCD radiative corrections to the quark-photon vertex] extends the range of applicability of QCD based perturbative analyses down to spacelike-\( q^2 \sim 4 \text{ GeV}^2 \).

Herein the generalised impulse approximation to \( F_\pi(q^2) \), illustrated in Fig. 1, is calculated for spacelike-\( q^2 \) using the nonperturbative Dyson-Schwinger equation [DSE] approach to QCD, which is reviewed in Ref. [9]. The essential ingredients of this calculation are: 1) the dressed quark propagator, \( S(p) \), which is confining in the sense that it has no singularities that can lead to free-quark production thresholds in Fig. 1; i.e., there is no quark mass-shell; 2) the pion Bethe-Salpeter amplitude, \( \Gamma_\pi(p, P) \), which is regular for spacelike values of \( p \), the relative \( q = \vec{7} \) momentum; and 3) the dressed quark-photon vertex, \( \Gamma_\mu(p_1, p_2) \), which is regular in the spacelike region; i.e., away from resonances such as the \( \rho \)-meson. These properties, which together ensure confinement, entail the important consequence that the generalised impulse approximation is free of both the “endpoint” and “pinch” singularities that arise in perturbative analyses.

It is important to note that Fig. 1 reduces to the the sum of diagrams considered in perturbative studies of \( F_\pi(q^2) \). This can be seen by replacing, sequentially, each of the dressed vertices by its lowest-order perturbative contribution. It therefore follows that with \( S \), \( \Gamma_\pi \) and \( \Gamma_\mu \) behaving for large spacelike-\( q^2 \) as prescribed by the renormalisation group in QCD, which is the case in realistic DSE studies, the generalised impulse approximation should yield a model-independent prediction for the behaviour of \( F_\pi(q^2) \) at large spacelike-\( q^2 \).

In the phenomenological application of the DSE approach the model dependence is restricted to the small spacelike-\( q^2 \) domain; i.e., spacelike-\( q^2 \lesssim 2 \text{ GeV}^2 \), and is realised in a modelling of the form of the quark-quark interaction in the infrared. This not only incorporates the information obtained about, for example, the gluon condensate in the QCD sum rules approach [10] but also extends it. The calculation of experimental observables in
this approach therefore allows one to place constraints on the qualitative and quantitative features of the effective quark-quark interaction at small spacelike-$q^2$ in QCD and to infer the $q^2$ scale where perturbative, model-independent, effects begin to dominate.

In Sec. II the generalised impulse approximation and its essential ingredients $[S, \Gamma_\pi$ and $\Gamma_\mu]$ are discussed in detail. The width $\Gamma_{\mu\Gamma_{\pi\pi}}$ is calculated in generalised impulse approximation in Sec. III and shown to be independent of the details of $S$, $\Gamma_\pi$ and $\Gamma_\mu$ in the chiral limit. The calculation of $F_\pi(q^2)$ for spacelike-$q^2$, and other pion observables, is described in Sec. IV and the results compared with experiment. The behaviour of $F_\pi(q^2)$ at large spacelike-$q^2$ in generalised impulse approximation is determined analytically in Sec. V, with the result: $F_\pi(q^2) \propto 1/q^4$, up to $ln[q^2]$-corrections. This prediction is verified numerically and found to become evident only for spacelike-$q^2 \gtrsim 35$ GeV$^2$, which is presently inaccessible experimentally. The results are summarised and conclusions presented in Sec. VI.

II. GENERALISED IMPULSE APPROXIMATION

Herein all calculations are carried out in Euclidean space, with $\gamma_\mu$ hermitian and metric $\delta_{\mu\nu} = diag(1,1,1,1)$.

One may define the generalised impulse approximation to the connected $\pi-\pi-A_\mu$ vertex in QCD as, with $m_u = m_d$,

$$\Lambda_\mu(P + q, -P) = \frac{N_c}{f_\pi^2} \int \frac{d^4k}{(2\pi)^4} \text{tr}_D \left[ \Gamma_\pi(k; P + q) S(k_{++}) i\Gamma_\mu(k_{++}, k_{--}) S(k_{--}) \Gamma_\pi(k - \frac{1}{2}q; -P) S(k_{--}) \right],$$

(1)

where $q$ is the photon momentum and $P$ is the initial momentum of the pion. Here the trace over colour and flavour indices has been evaluated leaving only the trace over Dirac indices and

$$k_{\alpha\beta} = k + \frac{\alpha}{2}q + \frac{\beta}{2}P.$$  

(2)

In Eq. (1): $\Gamma_\mu(p_1, p_2)$ denotes the dressed quark-photon vertex; $\Gamma_\pi(p; P)$ the pion Bethe-Salpeter amplitude, with $p$ the relative momentum and $P$ the centre-of-mass momentum; and $S(p)$ the dressed quark propagator. This vertex is illustrated in Fig. 1.

A. Quark Propagator

The dressed quark propagator in Eq. (1) can be obtained by solving the following Dyson-Schwinger equation [DSE]:

$$S^{-1}(p) = i\gamma \cdot p + m + \frac{4}{3}g^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\nu S(k) \Gamma^\nu_\mu(k, p) D_{\mu\nu}((p - k)^2),$$

(3)

where $m = (m_u + m_d)/2$ is the current-quark mass. Here $D_{\mu\nu}(k)$ is the dressed gluon propagator and $\Gamma^\nu_\mu(p_1, p_2)$ is the dressed quark-gluon vertex; each of which satisfies its own DSE. The general form of the solution of Eq. (3) is

$$S(p) = -i\gamma \cdot p\sigma V(p^2) + \sigma S(p^2),$$

(4)
which can also be written as

\[
S(p) = \frac{1}{i\gamma \cdot p A(p^2) + m + B(p^2)} .
\] (5)

This equation has been much studied and the general properties of \(\sigma_S\) and \(\sigma_V\) in QCD are well known [9]. [The renormalisation of this equation has no effect on the general features of the solution that are used herein.]

In Ref. [11] Eq. (3) was solved with

\[
g^2 D_{\mu\nu}(k) = \left( \delta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2} \right) 8\pi^4 D \delta^4(k) \quad \text{and} \quad \Gamma_{\mu}(p, p) = -i\partial_{\mu}S^{-1}(p) ,
\] (6)

where \(D\) is a mass-scale parameter. This Ansatz for the dressed gluon propagator models the infrared behaviour of the quark-quark interaction in QCD via an integrable infrared singularity and ensures confinement, as described below, and the Ansatz for the dressed-quark gluon vertex is the result of extensive analysis of its general form [9,12-14]. The solution of Eq. (3) in Ref. [11] is

\[
\bar{\sigma}_S(y) = \frac{C}{2m y} \exp(-2y^2)J_1(4\sqrt{m} y) + \frac{\overline{m}^2}{y} \int_0^\infty d\xi \xi K_1(\overline{m} \xi) J_1(y \xi) \exp \left( -\frac{\xi^2}{8} \right) ,
\] (7)

with \(\overline{m} = m/\sqrt{2D}\), \(y^2 = p^2/(2D)\) and where \(J_1\) and \(K_1\) are Bessel functions, and

\[
\bar{\sigma}_V(y) = \frac{1}{\bar{m}} \left( \frac{\bar{\sigma}_S(y)}{y} + \frac{1}{4y} \frac{d}{dy} \bar{\sigma}_S(y) \right) ,
\] (8)

with \(\bar{\sigma}_S(y^2) = \sqrt{2D} \sigma_S(k^2)\) and \(\bar{\sigma}_V(y^2) = 2D \sigma_V(k^2)\). In Eq. (7), \(C\) is a parameter associated with dynamical chiral symmetry breaking and it is not determined by Eq. (3) with Eq. (6), while the integral, which is associated with explicit chiral symmetry breaking, cannot be evaluated in terms of known functions.

At large spacelike-\(p^2\) one finds from Eqs. (7) and (8) that

\[
\sigma_S(p^2) \approx \frac{m}{p^2} - \frac{m^3}{p^4} + \ldots \quad \text{and} \quad \sigma_V(p^2) \approx \frac{1}{p^2} - \frac{D + m^2}{p^4} + \ldots .
\] (9)

At large spacelike-\(p^2\) in QCD one has at leading order

\[
\sigma_S(p^2) \approx \frac{\hat{m}}{p^2} \left[ \frac{1}{2} \ln \left( \frac{p^2}{\Lambda^2_{QCD}} \right) \right]^d ,
\] (10)

with \(\hat{m}\) a renormalisation point invariant and \(d = 12/[33 - 2N_f]; N_f\) is the number of quark flavours. So, neglecting \(\ln[p^2]\) terms, one sees that the model defined by Eqs. (3) and (6) incorporates asymptotic freedom.

Another feature of this model is that both \(\sigma_S\) and \(\sigma_V\) are entire functions in the complex-\(p^2\) plane but for an essential singularity. As a consequence the quark propagator does not have a Lehmann representation and can be interpreted as describing a confined particle. This is because, when used in Eq. (1), for example, this property ensures the absence of
free-quark production thresholds, under the reasonable assumptions that $\Gamma_\pi$ is regular for spacelike-$p^2$ and $\Gamma_\rho$ is regular for spacelike-$q^2$. It follows from this that Eq. (1) is free of “endpoint” and “pinch” singularities. (This is a particular, sufficient manner in which to realise the requirement that Fig. 1 have no free-quark production thresholds. There are other, more complicated, means but the effect is the same [9]. Some of the phenomenological implications of a model with a simple realisation of this confinement mechanism have been discussed in Ref. [15].)

The solution described by Eqs. (7) and (8) has a defect. To see this one sets $\bar{m} = 0$ in Eq. (7), which yields

$$\sigma_S(y) = Ce^{-2y^2}. \quad (11)$$

This poorly represents $\sigma_S$ away from $y = 0$ since, in the absence of a bare mass and when chiral symmetry is dynamically broken, it is well known [16] that

$$\sigma_S(y)|_{y \to -\infty} \to \frac{4\pi^2d}{3} \frac{\kappa}{y^2(\ln y^2)^{1-d}} \quad (12)$$

with $\kappa = -(\ln[\mu^2/\Lambda^2_{\text{CD}}])^{-d}(\bar{q}q)_{ij}^2$, a renormalisation point invariant. This defect results from the fact that although the form of $D_{\mu\nu}$ in Eq. (6) generates confinement, it underestimates the strength of the coupling in QCD away from $k^2 = 0$. In the numerical studies of Eq. (3) that have used a better approximation to $D_{\mu\nu}(k)$ [9,17] there is no such defect.

1. Approximate, algebraic quark propagator

Herein, to avoid the need for a numerical solution of Eq. (3), Eqs. (7) and (8) are simply modified so as to restore the missing strength at intermediate-$x$ ($= y^2$) and thereby provide a better approximation to the numerical solutions, while retaining the confining characteristics present in Eqs. (7) and (8). The following approximating algebraic forms are used:

$$\overline{\sigma}_S(x) = Ce^{-2x} + \frac{1 - e^{-bx}}{b_1x} \frac{1 - e^{-bx}}{b_3x} \left( b_0 + b_2 \frac{1 - e^{-\Lambda x}}{\Lambda x} + \frac{\bar{m}}{x + \bar{m}^2} \left( 1 - e^{-2(x+\bar{m}^2)} \right) \right) \quad (13)$$

$$\overline{\sigma}_V(x) = \frac{2(x + \bar{m}^2) - 1 + e^{-2(x+\bar{m}^2)}}{2(x + \bar{m}^2)^2} - \bar{m}Ce^{-2x}, \quad (14)$$

which allow the propagator to be consistent with realistic numerical solutions of Eq. (3).

When $b_0 = 0 = b_2$, Eqs. (13) and (14) provide an excellent approximation to Eqs. (7) and (8) [18], while for nonzero values of these parameters it is clear that the behaviour given in Eq. (12) is recovered, up to $\ln[p^2]$-corrections. These model forms are also entire functions in the complex $p^2$ plane but for an essential singularity.

The expressions in Eqs. (13) and (14) provide a six parameter model of the quark propagator in QCD: $C$, $\bar{m}$, $b_0$, $\ldots$, $b_3$. $[\Lambda = 10^{-4}]$ is introduced simply to decouple $b_2$ from the quark condensate, as will be shown below.] These parameters can easily be fitted to experimental observables and, in principle, Eq. (3) then inverted to extract $D_{\mu\nu}(k)$, the effective gluon propagator. So, in $S$ one has an implicit parametrisation of $D_{\mu\nu}(k)$ and hence a connection between experimental observables and the nature of the effective quark-quark interaction in the infrared. Precise experimental data can therefore be used to determine the effective quark-quark interaction in the infrared.
B. Pion Bethe-Salpeter Amplitude

The Bethe-Salpeter amplitude in Eq. (1) is the solution of the homogeneous Bethe-Salpeter equation [BSE]:

\[
\Gamma^\alpha_\pi(p; P) = \int \frac{d^4 k}{(2\pi)^4} K^\alpha_{\pi}(k; p; P) \left(S(k - \frac{1}{2} P)\Gamma^\alpha_\pi(k; P)S(k + \frac{1}{2} P)\right)^{tu} \tag{15}
\]

where \(P\) is the centre-of-mass momentum of the bound state, \(p\) is the relative momentum between the quarks in the bound state and the superscripts are associated with the Dirac structure of the amplitude. In the isospin symmetric case, \(m_u = m_d\), \(K^\alpha_{\pi}(p, k; P) \propto I_F\), the identity matrix in flavour-space. Further, since \(\Gamma_\pi\) and \(S(p)\) are \(\propto I_C\), the identity matrix in colour-space, then \(K^\alpha_{\pi}(p, k; P) \propto I_C\) also.

The generalised-ladder approximation is defined by the choice

\[
K^\alpha_{\pi}(p, k; P) = \frac{4}{3} g^2 D_{\mu\nu}(p - k)(\gamma_{\mu})^{rt}(\gamma_{\nu})^{us} \tag{16}
\]

in Eq. (15), with \(S(p)\) obtained from Eq. (3), and has been much studied [19]. The amplitude obtained as a solution in this case defines the nonperturbative “dressed-quark core” of the meson and provides the dominant contribution to physical observables, as will be seen herein.

The most general form of \(\Gamma_\pi\) allowed by Lorentz covariance, which is odd under parity transformations, is [20]

\[
\Gamma_\pi(p; P) = i\gamma_5 \left\{ E(p; P) + i\gamma \cdot p \cdot P F(p; P) + i\gamma \cdot P G(p; P) + [\gamma \cdot p, \gamma \cdot P] H(p; P) \right\} \tag{17}
\]

where, since \(\pi^0\) is even under charge-conjugation, \(E, F, G\) and \(H\) are even functions of \((p \cdot P)\). However, the many studies of Eq. (15) using (16) [19] suggest strongly that the dominant amplitude in Eq. (17) is \(E(p; P)\), with the other functions providing less than a 10% contribution to physical observables; i.e., that it is a good approximation to write

\[
\Gamma_\pi(p, P) = i\gamma_5 E(p; P). \tag{18}
\]

In generalised-ladder approximation \(K\) is independent of \(P\) and the standard normalisation condition for \(\Gamma_\pi\) reduces to the requirement that, for \(P^2 = -m_\pi^2\):

\[
2P_\mu = N_c \int \frac{d^4 k}{(2\pi)^4} t r D \left[ \Gamma_\pi(k; P)S(k_0)\Gamma_\pi(k; -P)\frac{\partial S(k_0)}{\partial P_\mu} + \Gamma_\pi(k; P)\frac{\partial S(k_0)}{\partial P_\mu}\Gamma_\pi(k; -P)S(k_0) \right] \tag{19}
\]

In the chiral limit; i.e., when the current quark mass, \(m\), is zero, the pseudoscalar generalised-ladder-approximation BSE and quark rainbow-DSE [which has \(\Gamma^\alpha_\pi = \gamma_{\alpha}\) in Eq. (3)] are identical [21] and one has a massless excitation in the pseudoscalar channel with

\[
E(p; P^2 = 0) = \frac{1}{f_\pi} B_{m=0}(p^2) \tag{20}
\]

where \(B(p^2)\) is given in Eq. (5) and \(f_\pi\) the calculated normalisation constant. This is an illustration of the manner in which Goldstone’s theorem is realised in the Dyson-Schwinger equation framework. In this case, Eqs. (13) and (14) completely determine \(\Gamma_\pi\).
For $m \neq 0$, the Bethe-Salpeter amplitude must still vanish as the relative momentum $p^2 \to \infty$ [22]. A first, simple approximation in this case is

$$E(p; P^2 = -m^2) = \frac{1}{f_\pi} B_{m \neq 0}(p^2),$$

which is very good for small current-quark mass. One notes that using Eqs. (13) and (14) in Eq. (21) entails

$$\Gamma_\pi(p, P) \propto \frac{1}{p^2},$$

which, up to the $\ln[p^2]$-corrections associated with the anomalous dimension, reproduces the ultraviolet behaviour of the Bethe-Salpeter amplitude given by QCD [22].

C. Quark-photon Vertex

The quark-photon vertex, $\Gamma_\mu(p_1, p_2)$, satisfies a DSE that describes both strong and electromagnetic dressing of the interaction. Solving this equation is a difficult problem that has only recently begun to be addressed [23]. However, much progress has been made in constraining the form of $\Gamma_\mu(p_1, p_2)$ and developing a realistic Ansatz [12-14].

It is obvious that the bare vertex: $\Gamma_\mu(p_1, p_2) = \gamma_\mu$, is inadequate when the fermion propagator has momentum dependent dressing because it violates the Ward-Takahashi identity:

$$(p_1 - p_2)_\mu i\gamma_\mu \neq S^{-1}(p_1) - S^{-1}(p_2);$$

and hence leads to an electromagnetic current for the pion that is not conserved.

An Ansatz [13] satisfying the criteria of Ref. [14]; i.e., it a) satisfies the Ward-Takahashi identity; b) is free of kinematic singularities; c) reduces to the bare vertex in the free field limit as prescribed by perturbation theory; and d) has the same transformation properties as the bare vertex under charge conjugation and Lorentz transformations, is

$$\Gamma_\mu(p, k) = \Gamma^{BC}_\mu(p, k) + \Gamma^T_\mu(p, k)$$

where [12], with $S^{-1}(p) = i\gamma \cdot p A(p^2) + m + B(p^2)$,

$$\Gamma^{BC}_\mu(p, k) = \frac{[A(p^2) + A(k^2)]}{2} \gamma_\mu + \frac{(p + k)_\mu}{p^2 - k^2} \left\{ [A(p^2) - A(k^2)] \frac{\gamma \cdot p + \gamma \cdot k}{2} - i \left[ B(p^2) - B(k^2) \right] \right\}$$

and $(p - k)_\mu \Gamma^T_\mu(p, k) = 0$ with $\Gamma^T_\mu(p, p) = 0$. Using the bare quark propagator, which has $A = 1$ and $B = \text{constant}$, $\Gamma^T_\mu = 0$. In this Ansatz the $\Gamma^{BC}_\mu$ piece is completely determined by the dressed quark propagator but $\Gamma^T_\mu$ is undetermined.

Vector-meson-photon mixing contributions to $F_\pi(q^2)$ enter through the dressed quark-photon vertex and can only appear in $\Gamma^T_\mu(p, k)$. This is especially clear in the Nambu–Jona-Lasinio model where $\Gamma^{BC}_\mu(p, k) = \gamma_\mu$ [24]. The generalised impulse approximation is defined as Eq. (1) with
\[ \Gamma_\mu(p, k) = \Gamma_{\mu}^{BC}(p, k) , \]  

(26)

which is completely determined by the quark propagator and introduces no new parameters. This entails that vector meson contributions are explicitly excluded in this approximation.

A feature of Eq. (26), which follows from criterion c), is that with a quark propagator that incorporates asymptotic freedom the quark-photon vertex reduces to \( \gamma_\mu \) at large spacelike-\( q^2 \), in the manner prescribed by perturbation theory in QCD. This Ansatz therefore provides a realistically constrained extrapolation of the quark-photon vertex to small spacelike-\( q^2 \).

1. \( \rho-\gamma \) mixing

The \( \rho \)-meson contributes little to \( F_\pi(q^2) \) in the spacelike region, even though it is important for timelike-\( q^2 \approx m_\rho^2 \). This follows from a consideration of vector-meson–photon \([\rho-\gamma]\) mixing in QCD. Firstly one notes that electromagnetic gauge invariance requires the \( \rho-\gamma \) mixing amplitude, \( \Pi_{\rho\gamma}^{(\pi)}(q^2) \), to vanish at \( q^2 = 0 \) and that asymptotic freedom ensures it vanishes at large spacelike-\( q^2 \). Secondly, the \( \rho \)-meson is well described in the phenomenological DSE approach, with the dressed-quark core giving \( > 90\% \) of its mass [25], and a calculation based on this shows that \( \Pi_{\rho\gamma}^{(\pi)}(q^2) \) is a slowly varying function for spacelike-\( q^2 \). From these results it is clear that the \( \rho \)-meson is unimportant in the spacelike region. This conclusion is supported by the calculations reported herein and elsewhere [26].

A form of the \( \rho \)-dominance Ansatz can nevertheless be understood in quark-gluon models. For example, in the Nambu–Jona-Lasinio model one can eliminate the electromagnetic field, \( A_\mu \), from the fermion determinant that results from bosonising the theory, by redefining the \( \rho \)-field, which is local in this model [27]. The redefinition of the \( \rho \)-field, \( \rho \rightarrow \rho' \), induces an interaction term of the form \( \rho'_\mu A_\mu \) outside of the determinant but destroys the connection between the field and the dressed-quark bound state that is the solution of the BSE, Eq. (15). This modified-\( \rho \) field, \( \rho' \), then mediates the electromagnetic interaction of all hadrons. In the approach employed herein, where the mesons are not pointlike, a similar procedure can be followed [28], however, it is more complicated and destroying the connection with the BSE is a significant qualitative loss.

D. Current Conservation

Using Eq. (26) and the identities: \( S(-k)^T = C^\dagger S(k) C \); \( \Gamma^T_{\mu}(k; p) = C^\dagger \Gamma_\mu(k; p) C \); and \( \Gamma^T_\mu(-k; p) = C^\dagger \Gamma_\mu(k; p) C \); and \( \Gamma^T_\mu(-k,-p) = -C^\dagger \Gamma_\mu(p, k) C \), where \( C = \gamma_2 \gamma_4 \) is the charge conjugation matrix, one finds easily that the \( \pi \)-current is conserved:

\[ q_\mu A_\mu(P + q, -P) = 0 . \]  

(27)

For elastic scattering, with \([2q \cdot P = q^2]\), one can therefore write

\[ A_\mu(P + q, -P) = (2P + q)_\mu F_\pi(q^2, P^2) . \]  

(28)

One obtains similarly that
\[ \Lambda_{\mu}(P, -P) = 2P_\mu F_{\mu}(0, P^2) = \]
\[ N_c \int \frac{d^4k}{(2\pi)^4} tr_D \left[ \Gamma_{\pi}(k; P) S(k_{0^-}) \Gamma_{\pi}(k; -P) \frac{\partial S(k_{0^+})}{\partial P_\mu} + \Gamma_{\pi}(k; P) \frac{\partial S(k_{0^-})}{\partial P_\mu} \Gamma_{\pi}(k; -P) S(k_{0^+}) \right]. \]

Comparing this with Eq. (19) it is clear that in generalised impulse approximation \( F_{\pi}(0, P^2) = 1 \) only if the Bethe-Salpeter kernel is independent of \( P \); i.e., generalised impulse approximation combined with a \( P \)-independent Bethe-Salpeter kernel provides a consistent approximation scheme. In this case one has, in the chiral limit \( P^2 = -m_{\pi}^2 = 0 \):

\[ f_\pi^2 = \]
\[ \frac{N_c}{8\pi^2} \int_{0}^{\infty} ds s B(s)^2 \left( \sigma_V^2 - 2[s_\sigma s'_\sigma + s_\sigma V_\sigma V_\sigma] - s[s_\sigma s'_\sigma - (\sigma_\sigma)^2] - s^2[\sigma_\sigma V_\sigma V_\sigma - (\sigma_\sigma)^2] \right), \]

with \( s = p^2 \).

This shows that the generalised impulse approximation to the form factor, Eq. (1), is regular in the chiral limit. In fact, the calculated results are only weakly dependent on \( m_{\pi} \).

The study of Ref. [26] indicates that, at \( m_{\pi} = 0.14 \text{ GeV} \), Eq. (1) provides the dominant contribution to \( F_{\pi}(q^2) \) for spacelike-\( q^2 \) and that that pion-loops are unimportant, a result supported by the lattice QCD studies of Ref. [29].

### III. NEUTRAL PION DECAY

In Euclidean space the matrix element for the decay \( \pi^0 \to \gamma \gamma \) can be written

\[ \mathcal{M}(k_1, k_2) = -2 i \frac{\alpha_{em}}{f_{\pi}} \epsilon_{\mu}(k_1) \epsilon_{\nu}(k_2) k_1\mu k_2\nu G(k_1 \cdot k_2), \]

where \( k_\nu \) are the photon momenta and \( \epsilon(k_\nu) \) are their polarisation vectors. Here, the \( \pi^0 \) momentum is \( P = (k_1 + k_2) \) and \( P^2 = 2 k_1 \cdot k_2 \).

Using Eq. (31) one finds easily that

\[ \Gamma_{\pi^0\to\gamma\gamma} = \frac{m_{\pi}^2}{16\pi} \left( \frac{\alpha_{em}}{\pi f_{\pi}} \right)^2 G(-m_{\pi}^2)^2. \]

Experimentally one has \( \Gamma_{\pi^0\to\gamma\gamma} = (7.74 \pm 0.56) \text{ eV} \), which corresponds to

\[ g_{\pi^0\gamma\gamma} \equiv G(-m_{\pi}^2) = 0.504 \pm 0.019, \]

using \( m_{\pi} = 135 \text{ MeV} \) and \( f_{\pi} = 93.1 \text{ MeV} \).

In generalised impulse approximation one has

\[ \frac{1}{4\pi} \int d^4k tr_D \left[ S(k) i\Gamma_{\mu}(k, k - k_1) S(k - k_1) \Gamma_{\pi}(k + k_2/2, P) S(k + k_2) i\Gamma_{\pi}(k + k_2, k) \right]. \]

In the chiral limit, \( P^2 = 0 \), and using Eq. (18), one easily obtains
\[
g_{\phi_{\gamma\gamma}}^{0} \equiv G(0) = 
\int_{0}^{\infty} ds \frac{f_{\pi} E(s; 0)}{A(s)} A \left[ \sigma_{V} \sigma_{S} + s \left( \sigma_{V}' \sigma_{S} - \sigma_{V} \sigma_{S}' \right) \right] + s \sigma_{V} \left( A' \sigma_{S} - B' \sigma_{V} \right). 
\]

Now, defining
\[
C(s) = \frac{B(s)}{s A(s)} = \frac{\sigma_{S}(s)}{s \sigma_{V}(s)}
\]

one obtains a dramatic simplification and Eq. (35) becomes
\[
g_{\phi_{\gamma\gamma}}^{0} = - \int_{0}^{\infty} ds \frac{f_{\pi} E(s; 0)}{B(s)} \frac{C'(s)}{1 + C(s)^{3}},
\]

It therefore follows from the manner in which dynamical chiral symmetry breaking is realised in the Dyson-Schwinger equation framework; i.e., from Eq. (20), that in the chiral limit
\[
g_{\phi_{\gamma\gamma}}^{0} = \int_{0}^{\infty} dC \frac{1}{(1 + C)^{3}} = \frac{1}{2} \,,
\]

since \(C(s = 0) = \infty\) and \(C(s = \infty) = 0\). Hence, the experimental value is reproduced independent of the details of \(S(p)\).

This illustrates the manner in which the Abelian anomaly is incorporated in the Dyson-Schwinger equation framework. A similar result emerges [30] in connection with the Wess-Zumino five-pseudoscalar term [31].

In order to obtain the result in Eq. (38) it is essential that, in addition to Eq. (20), the photon-quark vertex satisfy the Ward identity. This is not surprising. However, the fact that one must dress all of the elements in a calculation consistently is often overlooked.

The subtle cancellations that are required to obtain this result also make it clear that it cannot be obtained in model calculations where an arbitrary cutoff function (or “form-factor”) is introduced into each integral. The fact that \(E(p, P^2 = 0)\) is the pion Bethe-Salpeter amplitude and \(f_{\pi} E(p; P^2 = 0) = B(p^2)\) in the chiral limit, is crucial.

A good approximation to \(g_{\phi_{\gamma\gamma}}\) for nonzero pion mass is given by Eq. (34) with Eq. (21).

**IV. CALCULATED SPACELIKE FORM FACTOR**

The generalised impulse approximation to \(F_{\pi}(q^2)\) is defined by Eq. (1) with the dressed quark propagator obtained from Eq. (3), the Bethe-Salpeter amplitude obtained using Eqs. (15), (16) and the dressed quark-photon vertex in Eq. (26).

It is clear that if, at large spacelike-\(q^2\), each of the elements in Eq. (1) behaves as prescribed by the renormalisation group in QCD then this amplitude reduces to that studied in perturbative analyses of the elastic form factor [3]. [To see that the “hard-scattering” contribution is included one need only once-iterate the \(\pi-q\bar{q}\) vertices using Eq. (15).] Therefore, a model such as the one described by Eqs. (13), (14) and (21), constructed so as to preserve this asymptotic behaviour, will provide a model-independent prediction for the large spacelike-\(q^2\) behaviour of \(F_{\pi}(q^2)\).
Such a model will also provide an extrapolation to small spacelike-$q^2$ that is sensitive to the nature of the effective quark-quark interaction in the infrared. In this feature the phenomenological DSE approach both complements and extends the QCD sum rule approach [5]. The DSE framework therefore provides an interpretation of the model parameters in terms of qualitative and quantitative features of the quark-quark interaction in the infrared.

To evaluate Eq. (1) it is convenient to work in the Breit frame with

$$q = (0, 0, -q, 0) \quad \text{and} \quad P = (0, 0, \frac{1}{2}q, i\sqrt{m^2 + \frac{1}{4}q^2})$$

(39)

in which case, with $k = k(\sin \beta \sin \theta \cos \phi, \sin \beta \sin \theta \sin \phi, \sin \beta \cos \theta, \cos \beta)$,

$$k \cdot q = -k q \sin \beta \cos \theta$$

$$k \cdot P = \frac{1}{2}k q \sin \beta \cos \theta + ik\sqrt{m^2 + \frac{1}{4}q^2} \cos \beta$$

(40)

and one is left with three integrals to evaluate, one radial and two angular.

Here the feature of confinement is crucial since it ensures that, for all spacelike-$q^2$, the integrand is regular and hence that the integrals can be evaluated using straightforward Gaussian quadrature techniques; i.e., there are no “endpoint” or “pinch” singularities. In any model that does not have this feature there will be some spacelike $q^2$ for which the calculation becomes nonsensical because the integration begins to sample the unphysical singularities associated with free-quark production.

A. Fitting the Parameters

Equations (13), (14) and (21) provide a six parameter model $[C, m, b_0 \ldots b_3]$ of the nonperturbative dressed-quark substructure of the pion based on DSE studies. These parameters are fixed herein by requiring that the model reproduce, as well as possible, the following experimental values of the dimensionless quantities:

$$\frac{f_\pi}{\langle \bar{q}q \rangle^{1/2}} = 0.423 \quad f_\pi r_\pi = 0.318 \quad \frac{m_\pi^2}{\langle \bar{q}q \rangle^{1/2}} = 0.396$$

(42)

the dimensionless $\pi$-$\pi$ scattering lengths (see Refs. [32,33] for a discussion):

$$a_0^0 = 0.21 \pm 0.02 \quad a_0^1 = -0.040 \pm 0.003 \quad a_1^1 = 0.038 \pm 0.002 \quad a_2^0 = 0.0017 \pm 0.0003$$

(43)

and a least-squares fit to $F_\pi(q^2)$ on the spacelike-$q^2$ domain: $[0,4]$ GeV$^2$. The fitting procedure was performed using the expression for the pion decay constant, $f_\pi$, given in Eq. (30),

$$\langle \bar{q}q \rangle_\mu^2 = - \left( \frac{\mu^2}{\Lambda_{QCD}^2} \right)^{a} \lim_{N\to\infty} \left( \frac{\Lambda_{QCD}^2}{\Lambda_{QCD}^2} \right)^{-a} \frac{3}{4\pi^2} \int_{0}^{\Lambda_{QCD}^2} \left[ \sigma_s(s) - \frac{m^2}{s + m^2} \right] ds s$$

(44)

$s = p^2$ and $\Lambda_{QCD} = 0.20$ GeV, following Ref. [17] but with $\alpha = 1$ rather than the anomalous dimension, $d$, because the associated $\ln[p^2]$-corrections have not been included in Eqs. (13)
and (14), and the expressions for $a^0_0$, $a^2_0$, $a^1_1$, $a^0_2$ and $r_\pi$ given in Ref. [32]. It should be noted that using Eq. (13) in Eq. (44) one obtains

$$\langle \bar{q}q \rangle_{\mu^2} = -(2D)^{1/2} \left( \ln \frac{\mu^2}{\Lambda_{QCD}^2} \right) \frac{3}{4\pi^2} \frac{b_0}{b_1 b_3},$$

which is independent of $b_2$ for arbitrarily small but nonzero $\Lambda$.

Following this procedure one obtains

$$C = 0.0406 , \ m = 0.0119 ,$$
$$b_0 = 0.118 , \ b_1 = 2.51 , \ b_2 = 0.525 , \ b_3 = 0.169 .$$

(46)

The mass scale is set by requiring equality between the percentage error in $f_\pi$ and $r_\pi$, which yields $D = 0.133$.

B. Results for $F_\pi(q^2)$ and Other Observables.

In Table I the low-energy physical observables calculated with the parameter set of Eq. (46) are compared with their experimental values. The agreement is excellent.

In Fig. 2 the five $\pi$-$\pi$ partial wave amplitudes associated with the scattering lengths in the table, calculated using the formulae in Ref. [32], are plotted and can be seen to be in excellent agreement with the data up to $x \approx 3$, which corresponds to $E \approx 4m_\pi$.

The form factor, $F_\pi(q^2)$, at small spacelike-$q^2$ is shown in Fig. 3 and for larger spacelike-$q^2$ in Fig. 4. Given that the “experimental” point at $q^2 = 6.3$ GeV$^2$, measured in pion electroproduction [38], depends strongly on the model used to separate strong and electromagnetic effects, the agreement with the experimental data is again excellent.

This demonstrates that the generalised impulse approximation to $F_\pi(q^2)$, calculated with a confining quark propagator and pion Bethe-Salpeter amplitude, both following from realistic, nonperturbative DSE studies and thereby incorporating asymptotic freedom, embodies the essential physics of the electromagnetic structure of the pion at presently accessible values of spacelike-$q^2$. Taken together with the other calculated results it is clear that the phenomenological DSE approach employed herein, which describes the pion as a bound state of dressed quarks interacting via nonperturbative gluon exchange, provides a concise, uniformly good description of the properties of the pion. This can be rephrased in the statement that, away from resonances, the nonperturbative dressed-quark core of the pion is its dominant determining characteristic, as argued in Refs. [26,32]. [The small contributions from the “tail” of the $\rho$ resonance and $\pi$-loops can easily be accommodated by small changes in the parameter values in Eq. (46).]

V. ASYMPTOTIC BEHAVIOUR

A simple way to analyse the behaviour of $F_\pi(q^2)$ at large spacelike-$q^2$ is to rewrite Eq. (1) in terms of the Bethe-Salpeter wave-function:

$$\chi_\pi(p, P) = S(k - \frac{1}{2}P) \Gamma_\pi(p, P) S(k + \frac{1}{2}P) ,$$

(47)
in which case

\[ \Lambda_\mu(P + q, -P) = \frac{N_c}{f_\pi^2} \int \frac{d^4k}{(2\pi)^4} i r_D \left[ \chi_\pi(k; P + q) i \Gamma_\mu(k^{++}, k_{--}) \chi_\pi(k - \frac{1}{2}q; -P) S^{-1}(k_{--}) \right]. \]  \hspace{1cm} (48)

Asymptotic freedom, which is represented up to \( \ln[p^2] \)-corrections by Eqs. (9) and (22), entails that the behaviour of \( F_\pi(q^2) \) at large spacelike-\( q^2 \) can be obtained from Eq. (48) with

\[ S(p) \approx \frac{1}{i\gamma \cdot p + M_c} \quad \text{and} \quad \chi_\pi(p; P) \approx \frac{\Lambda_1^3}{(p^2 + \Lambda_2^2)(p^2 + M_c^2)}, \] \hspace{1cm} (49)

where \( \Lambda_1, \Lambda_2 \) and \( M_c \) are characteristic parameters \( \text{[the value of which is unimportant but typically [26] } \Lambda_1 \sim 500 \text{ MeV} \sim \Lambda_2 \text{ and } M_c \sim 220 \text{ MeV} \] \( \text{]} \) and \( \Gamma_\mu(p_1, p_2) \approx \gamma_\mu \). It therefore follows from Eqs. (28), (48) and (49) that, at large spacelike-\( q^2 \),

\[ F_\pi(q^2) \propto \int \frac{d^4k}{(2\pi)^4} \chi_\pi(k) \chi_\pi(k - \frac{1}{2}q). \] \hspace{1cm} (50)

Making use of the approximation

\[ \frac{2}{\pi} \int_0^\pi d\theta \sin^2 \theta \frac{f(k^2 + p^2 - 2kp\cos \theta)}{k^2 + p^2 - 2kp\cos \theta} \approx \frac{f(k^2)}{k^2} \theta(k^2 - p^2) + \frac{f(p^2)}{p^2} \theta(p^2 - k^2), \] \hspace{1cm} (51)

which is often used in the analysis of the asymptotic behaviour of Eq. (3) and is very good for large spacelike-\( q^2 \) [40], one obtains

\[ \left( \frac{F_\pi(x)}{\chi_\pi(x)} \right)^1 \propto x \chi_\pi(x), \] \hspace{1cm} (52)

where \( x = q^2/4 \). The solution of this equation, which satisfies the boundary condition \( F_\pi(q^2 = \infty) = 0 \), is

\[ F_\pi(x) \propto C_1 \chi_\pi(x) + \int_0^x dy \chi_\pi(y) \int_y^x dz z \chi_\pi(z), \] \hspace{1cm} (53)

where \( C_1 \) is an undetermined constant.

One observes immediately from Eq. (53) that in generalised impulse approximation the asymptotic form of the elastic form factor depends on the Bethe-Salpeter amplitude of the bound state. This result signals the failure of the “factorisation theorem” in this exclusive process. It is not complicated by considerations associated with “endpoint” or “pinch” singularities since these are absent in this calculation, as required by confinement.

Using the form of \( \chi_\pi \) given in Eq. (49), which is a consequence only of asymptotic freedom, one finds

\[ F_\pi(q^2) \underset{q^2 \to \infty}{\sim} \frac{\ln q^2}{q^4}. \] \hspace{1cm} (54)

Taking into account the \( \ln[p^2] \)-corrections to Eqs. (9) and (22), which arise because of the anomalous dimension of the propagator and Bethe-Salpeter amplitude in QCD, would only lead to the modification \( \ln[q^2] \to \ln[q^2] \gamma \), where \( |\gamma| \) is \( O(1) \), in Eq. (54).
The numerical methods used herein to calculate $F_x(q^2)$ have been constructed so as to ensure that the result is independent of the details of the numerical procedure for $0 < q^2 < 20$ GeV$^2$. Therefore, in order to verify the prediction of Eq. (54) and to estimate the spacelike-$q^2$ at which the asymptotic regime is reached, a least squares fit of the calculated results for $1/F_x(X)$ to

$$a_0 + a_1 \frac{X}{\ln X} + a_2 \frac{X^2}{\ln X},$$

with $X = Q^2/(1 \text{ GeV}^2)$, was performed on the domain $5 < X < 20$. This procedure yielded

$$a_0 = 41.1, \quad a_1 = -15.4, \quad a_2 = 1.59$$

and the comparison between the fitted curve and the calculated results is presented in the upper panel of Fig. 5. In the lower panel the comparison between $F_x(X)$ and the reciprocal of the fitting function is presented. The prediction of Eq. (54) receives strong support from this analysis. Further, this analysis suggests that, with the parameter values of Eq. (46), which are fixed by physics at spacelike-$q^2 \leq 4$ GeV$^2$, the asymptotic term; i.e., the $\ln[X]/X^2$ term, only becomes dominant for spacelike-$q^2 \geq 35$ GeV$^2$. This result provides further support for the arguments of Ref. [4]; i.e., that soft, nonperturbative physics dominates at presently accessible spacelike-$q^2$ in exclusive processes.

Equation (54) is an interesting, general result that follows directly from the constraints that 1) asymptotic freedom and 2) the realisation of confinement used herein, place on the fermion propagator and pion Bethe-Salpeter amplitude at large spacelike-$q^2$. Given that Eq. (1) is the amplitude used in the perturbative analysis of the asymptotic form of the elastic pion form factor in QCD, then Eq. (54) contradicts the assertion that $q^2F_x(q^2) = \text{constant}$ [1,2], up to $\ln[q^2]$-corrections. The implication of Eq. (54) is that the factorisation approach fails for this exclusive process because the $q^2$ dependence of the pion Bethe-Salpeter amplitude makes an important contribution to the asymptotic form of $F_x(q^2)$.

A dependence of the asymptotic fall-off of $F_x(q^2)$ on the pion's Bethe-Salpeter amplitude is also found using the light-front formulation of relativistic quantum mechanics [41]. In this approach the asymptotic form of $F_x(q^2)$ is only independent of $\chi_\pi$ if the constituent-mass of the quark is zero, when the mass-shell singularity dominates the integral that arises. In such an approach, however, a constituent-quark mass of zero is a phenomenologically untenable assumption [7], with a value of $\sim 210$ MeV being required to fit the available data.

VI. SUMMARY AND CONCLUSIONS

The generalised impulse approximation, illustrated in Fig. 1, has been used to calculate $F_x(q^2)$ at spacelike-$q^2$; i.e., away from resonance contributions. The crucial elements of this calculation are: the dressed quark propagator; the pion Bethe-Salpeter amplitude; and the dressed quark-photon vertex, all of which follow from realistic, nonperturbative Dyson-Schwinger equation studies. This means that in this calculation each of these elements behaves at large spacelike-$p^2$ in the manner prescribed by the renormalisation group in QCD, up to $\ln[p^2]$-corrections, and at small spacelike-$p^2$ in such a way as to ensure confinement. In addition, the same approach has been used to simultaneously calculate: $f_\pi$; $m_\pi$; $\langle \bar{q}q \rangle$; $v_\pi$;
the $\pi^0 \rightarrow \gamma\gamma$ decay width; the $\pi-\pi$ scattering lengths: $a_{0}^0$, $a_{0}^1$, $a_{2}^0$, $a_{2}^2$; and the associated partial wave amplitudes.

The calculated results for all quantities agree extremely well with the data, as shown for $F_\pi(q^2)$ in Figs. 3 and 4. This supports the contention that the confined, nonperturbative “dressed-quark core” is the dominant determining characteristic of the pion away from resonance contributions. Particularly interesting in this context is the calculation of the $\pi^0 \rightarrow \gamma\gamma$ decay width, which illustrates the manner in which the Abelian anomaly manifests itself in the Dyson-Schwinger equation approach. The successful application of this approach to pion observables suggests, and provides the foundation for, a study of the nucleon as a bound state of three dressed-quarks using the covariant Faddeev equation. This is currently underway.

The Dyson-Schwinger equation approach provides a phenomenological framework in which to relate experimental observables to the qualitative and quantitative features of the effective quark-quark interaction in the infrared, which is a crucial but presently unknown element of QCD. Although there have been attempts to calculate the gluon propagator directly, many by solving the Dyson-Schwinger equation for the gluon vacuum polarisation [9] and recently via lattice simulations [12], which, unfortunately, are unreliable for spacelike-$q^2\lesssim2$ GeV$^2$, the extraction from experimental observables advocated herein presently appears to be the most practically efficacious approach. This analysis and the reliability of the extracted form of the interaction would benefit greatly from more precise measurements of $\pi-\pi$ scattering, at and near threshold, and $F_\pi(q^2)$ for spacelike-$q^2\gtrsim1$ GeV$^2$.

The fact that the dressed quarks are described by a confining propagator means that there is no quark mass-shell and eliminates “endpoint” and “pinch” singularities in the calculation of $F_\pi(q^2)$. This, coupled only with asymptotic freedom, entails that

$$F_\pi(q^2) \propto 1/q^4 \quad (57)$$

for large spacelike-$q^2$, up to ln$[q^2]$-corrections. This prediction is independent of the form of the quark-quark interaction at small and intermediate spacelike-$q^2$, the modelling of which is characteristic of the phenomenological application of the Dyson-Schwinger equation approach.

Equation (57) signals the breakdown of the “factorisation theorem” in the study of $F_\pi(q^2)$ since its asymptotic form is sensitive to the momentum dependence of the pion Bethe-Salpeter amplitude, which follows from asymptotic freedom in QCD.

Importantly, the present calculation indicates that the asymptotic form given by Eq. (57) only becomes evident for spacelike-$q^2\gtrsim35$ GeV$^2$; i.e., that soft, nonperturbative contributions dominate this exclusive process in the domain that is presently accessible experimentally.

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REFERENCES

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**TABLE I.** This table allows a comparison between the low-energy $\pi$ observables calculated using the parameters of Eq. (46) and their experimental values. Here: $m_{av,c} = \frac{1}{2}(m_u + m_d)$ and the pion mass is obtained from: $m_{\pi}^2 f_\pi^2 = -2 m_{\pi}^{av,c} \langle \bar{q}q \rangle_{\mu^2}$, which is tabulated simply for convenience. Each of the calculated quantities tabulated here was evaluated at the listed value of $m_\pi$; i.e., the chiral limit expressions for $f_\pi$, Eq. (30), $r_\pi$, Eq. (A21) of Ref. [32], and $g_{\pi^0\gamma\gamma}$, Eq. (35), were not used, however, the finite pion mass corrections are less than 1% for each of these. The experimental values of $f_\pi$, $m_{av,c}$, $m_\pi$ and $g_{\pi^0\gamma\gamma}$ are extracted from Ref. [34]; $r_\pi$ from Ref. [39]; and the scattering lengths are discussed in Ref. [32]. The value of $\langle \bar{q}q \rangle_{\tau}$ is that typically used in QCD sum rules analysis.
FIGURES

FIG. 1. This figure is a pictorial representation of the amplitude identified with the $\pi\pi A_\mu$ vertex in the generalised impulse approximation. The thick, straight external lines represent the incoming and outgoing $\pi$, the filled circles at the $\pi$ legs represent the $\langle \pi | \bar{q} q \rangle$ Bethe-Salpeter amplitudes, the wiggly line represents the photon, $\gamma$, the shaded circle at the $\gamma$ leg represents the regular part of the dressed quark-photon vertex [which satisfies the Ward-Takahashi Identity, Eq. (23)] and the thin internal lines represent the dressed quark propagator.

FIG. 2. This figure allows a comparison between the $\pi\pi$ partial wave amplitudes calculated using Eq. (46) (solid line) and experiment (the data are taken from Fig. 4 in Ref. [32]). The dashed line is the current algebra prediction of Ref. [35]. This is identically zero for $T_0^2(x)$ and $T_2^2(x)$. The dimensionless variable $x = E^2/(4m_\pi^2) - 1$ is defined so that threshold is at $x = 0$. Note that $x = 3$ corresponds to $E = 4m_\pi$.

FIG. 3. This plot shows the generalised impulse approximation to $F_\pi(q^2)$ calculated using the parameters in Eq. (46). The experimental data are from Refs. [37] (circles) and [39] (crosses).

FIG. 4. This plot shows the generalised impulse approximation to $q^2 F_\pi(q^2)$ calculated using the parameters in Eq. (46). The experimental data are from Refs. [36] (crosses), [37] (diamonds) and [38] (circles).

FIG. 5. The upper panel compares $1/F_\pi(X)$ (plotted points) with the fitting function of Eq. (55), $X = Q^2/(1\text{GeV}^2)$. The lower panel compares $F_\pi(X)$ with the reciprocal of the fitting function. The $a_2$ term provides more than $75\%$ of the value of $F_\pi(X)$ only for $X > 35$. 

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