Curvaton Potential Terms, Scale-Dependent Perturbation Spectra and Chaotic Initial Conditions

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Abstract

The curvaton scenario predicts an almost scale-invariant spectrum of perturbations in most inflation models. We consider the possibility that renormalisable $\phi^4$ or Planck scale-suppressed non-renormalisable curvaton potential terms may result in an observable deviation from scale-invariance. We show that if the curvaton initially has a large amplitude and if the total number of e-foldings of inflation is less than about 300 then a running blue perturbation spectrum with an observable deviation from scale-invariance is likely. D-term inflation is considered as an example with a potentially low total number of e-foldings of inflation. A secondary role for the curvaton, in which it drives a period of chaotic inflation leading to D-term or other flat potential inflation from an initially chaotic state, is suggested.

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1 Introduction

The curvaton scenario is an alternative model of the origin of cosmological density perturbations [1, 2, 3]. A scalar field, the curvaton, is assumed to be effectively massless during inflation. De Sitter fluctuations of the curvaton are transferred into energy density perturbations when coherent oscillations of the curvaton field come to dominate the energy density of the Universe and subsequently decay. A number of particle physics candidates have been proposed and models analysed [4, 5]. In particular, it has been noted that supersymmetric (SUSY) D-term hybrid inflation [6] with natural values for the dimensionless couplings must have a curvaton in order to be consistent with cosmic microwave background (CMB) constraints [7].

A prediction of the massless curvaton model is that the spectrum of density perturbations is almost exactly scale-invariant in most inflation models [3]. An observable scale-dependent curvaton perturbation ($|\Delta n| \gtrsim 10^{-2}$, where $n$ is the spectral index [8]) could be obtained if the curvaton mass was close to $0.1H$ during inflation [3, 5], but there is no strong reason to expect this.

Another possibility is that the curvaton could have renormalisable or Planck scale-suppressed non-renormalisable terms in its potential. It is therefore important to ask whether such terms could naturally result in an observable deviation from scale-invariance and to establish the nature of the deviation. We will consider this question in the following. We will show that an observable deviation from scale-invariance is possible if the total number of e-foldings of inflation is relatively small, less than around 300, and if the curvaton makes the transition to slow-rolling during inflation, which is likely if the initial curvaton amplitude is large.

D-term inflation provides an example of an inflation model which may have a small number of e-foldings of inflation. This is the case if the initial value of the inflaton field is less than the reduced Planck scale $M = M_{Pl}/\sqrt{8\pi} = 2.4 \times 10^{18}$ GeV, implying a small number of e-foldings of inflation ($<1000$) if the $U(1)_{FI}$ Fayet-Iliopoulos gauge coupling is larger than 0.2. An initial value for the inflaton field less than the Planck scale is expected if dangerous non-renormalisable terms in the inflaton potential are
suppressed by a discrete symmetry, leaving only high-order Planck-scale suppressed
terms. Alternatively, in the case where inflation originates from chaotic initial condi-
tions, the curvaton may play a secondary role, providing an initial period of chaotic
inflation [9] which allows the transition from an initially chaotic state characterized
by the Planck scale (‘space-time foam’ due to quantum fluctuations of the metric [9])
to D-term inflation (or any other inflation with a nearly flat potential) on a much
lower energy scale. In this scenario the initial value of the inflaton field is naturally
of the order of the Planck scale, even in the case where the inflaton potential has no
non-renormalisable corrections. Thus a small total number of e-foldings of D-term
inflation is also possible in this case.

The paper is organised as folllows. In Section 2 we discuss the spectral index and
deviation from scale-invariance due to a curvaton potential. In Section 3 we consider
the specific case of the D-term inflation/curvaton scenario. In Section 4 we present
our conclusions.

2 Spectral Index from Curvaton Potential Terms

In this section we will consider a simple model consisting of a constant inflaton poten-
tial, $V_o$, and a curvaton potential consisting of terms of the form,

$$V(\phi) = \frac{\lambda \phi^d}{d! M_{Pl}^{d-4}}, \quad (1)$$

where $\lambda$ is taken to be positive and of the order of 1, $d!$ is a symmetry factor and
$M_{Pl} = 1.2 \times 10^{19}$ GeV. This includes a renormalisable $\phi^4$ interaction when $d = 4$
and Planck scale-suppressed non-renormalisable terms, associated with the natural
quantum gravity cut-off, for $d \geq 5$. The curvaton equation of motion is then,

$$\ddot{\phi} + 3H \dot{\phi} = -\alpha_d \phi^{d-1} ; \quad \alpha_d = \frac{\lambda}{(d-1)! M_{Pl}^{d-4}}, \quad (2)$$

where the expansion rate $H$ is assumed to be approximately constant during inflation.

A natural possibility is that the curvaton has a large initial amplitude. For example,
if all values of the initial curvaton amplitude are equally probable (at least for a small
effective mass, \( V''(\phi) < H^2 \), then on average we would expect to find a large initial amplitude. Alternatively, it has been suggested that the initial value of the potential energy density in quantum gravity is naturally of the order of \( M_{Pl}^4 \), corresponding to a chaotic initial state with quantum fluctuations of energy density \( \sim M_{Pl}^4 \) and size \( \sim M_{Pl}^{-1} \) [9]. In this case a large initial value of the curvaton amplitude would be expected. The curvaton could then play a secondary role, driving a period of chaotic inflation with energy density initially of the order of \( M_{Pl}^4 \), which allows the transition from the chaotic initial state to inflation at a much lower energy density.

For a large initial amplitude the curvaton evolution during inflation has two distinct phases, coherent oscillation and slow-rolling. A slow-rolling curvaton satisfies \( \ddot{\phi} \ll 3H\dot{\phi} \). In this limit Eq. (2) has the solution
\[
\phi^{d-2} = \frac{\phi_o^{d-2}}{1 + \frac{3(d-2)}{(d-1)} \left( \frac{\phi_o}{\phi_*} \right)^{d-2} \Delta N},
\]
where
\[
\phi_*^{d-2} = \frac{9H^2}{\alpha_d (d-1)}
\]
and \( \Delta N \) is the number of e-foldings of inflation that have passed since \( \phi \) was equal to \( \phi_o \). \( \dddot{\phi} \) can be obtained in terms of \( \dot{\phi} \) by taking the time derivative of Eq. (2) in the slow-roll limit,
\[
\dddot{\phi} = -\frac{\alpha_d (d-1)}{3H} \phi^{d-2} \dot{\phi}.
\]
The condition for a slow-rolling curvaton, \( \ddot{\phi} \ll 3H\dot{\phi} \), then becomes \( \phi_*^{d-2} \ll \phi_*^{d-2} \), such \( \phi_* \) is approximately the value of \( \phi \) at which curvaton slow-rolling begins.

The spectral index of a slowly rolling curvaton is given by [3]
\[
n = 1 - 2\epsilon + \frac{2 V''(\phi)}{3H^2},
\]
where \( \epsilon = -\dot{H}/H^2 \) and all terms are evaluated when the scale of present wavenumber \( k \) leaves the horizon. The \( \epsilon \) term is negligible in most inflation models [10]. The second term is due to the evolution of the curvaton during inflation. Applying Eq. (6) to Eq. (3) (with \( \epsilon = 0 \)) gives
\[
\Delta n \equiv n - 1 = \frac{6\gamma^{d-2}}{\left(1 + \frac{3(d-2)}{(d-1)} \gamma^{d-2} \Delta N \right)}; \quad \gamma = \frac{\phi_o}{\phi_*}.
\]

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Thus a curvaton slowly rolling in the potential Eq. (1) produces a blue perturbation spectrum \((n > 1)\). In addition, the running of the spectral index with present wavenumber \(k\) is given by

\[
\frac{dn}{d\ln k} = \frac{dn}{d\Delta N} = -\frac{(d - 2)}{2(d - 1)}\Delta n^2 ,
\]

where \(N\) is the number of e-foldings until the end of inflation and we have used \(d\Delta N/dk = -dN/dk = 1/k\). (The present wavenumber \(k\) is proportional to \(e^{-N}\), where \(N\) is the number of e-foldings before the end of inflation at which the perturbation crosses the horizon. Therefore \(k \propto e^{-N}\) and \(dN/dk = -1/k\).)

We next consider the number of e-foldings of inflation over which a slow-rolling curvaton can produce a potentially observable deviation from scale-invariance. We will consider as examples the case \(d = 4\), corresponding to a renormalisable \(\phi^4\) interaction, and \(d = 6\) for the non-renormalisable curvaton potential. (The \(d = 6\) case is the natural lowest order term in supersymmetry, corresponding to a non-renormalisable superpotential term \(\propto \phi^4\).) We will consider the slow-rolling curvaton approximation to be reasonably well satisfied once \(3H\dot{\phi} \geq 4\ddot{\phi}\), which requires that \(\gamma^{d-2} \leq 0.25\). Thus the initial value of \(\phi\) we will consider corresponds to \(\phi_o\) such that \(\gamma^{d-2} = 0.25\). For the case \(d = 4\) this implies that \(\gamma = 0.5\) (i.e. \(\phi_o = 0.5\phi_*\)) \(^1\). Then

\[
\Delta n = \frac{1.5}{(1 + 0.5\Delta N)} .
\]

Requiring that \(\Delta n \geq 10^{-2}\) then requires that \(\Delta N \leq 298\). Thus for \(d = 4\) an observable deviation from a scale-invariant perturbation spectrum is obtained during the first 298 e-foldings of inflation following the onset of curvaton slow-rolling. For the case \(d = 6\) we find \(\gamma = 0.71\) and

\[
\Delta n = \frac{1.5}{(1 + 0.6\Delta N)} .
\]

Requiring \(\Delta n \geq 10^{-2}\) then requires that \(\Delta N \leq 248\).

From this we see that if the length scales of cosmological interest (those corresponding to CMB temperature fluctuations and large-scale structure) leave the horizon less

\(^1\)For \(d = 4\) we have \(\phi_* \approx H\). Therefore a large curvaton amplitude at the end of inflation is possible even with a renormalisable \(\phi^4\) potential.
than about 250 e-foldings after the onset of curvaton slow-rolling, and if the curvaton makes the transition to slow-rolling during inflation, then we would expect to be able to observe a running blue spectrum of density perturbations, assuming a leading order curvaton potential term with $d \leq 6$. Any positive value of $\Delta n$ can be accommodated, depending on how many e-foldings of curvaton slow-rolling have occurred when the scales of cosmological interest exit the horizon.

In the simple model we are considering here inflation begins once the curvaton energy density becomes less than $V_o$. We can then estimate the number of e-foldings from the onset of inflation until the onset of curvaton slow-rolling. For $\phi > \phi_*$ the curvaton will have a large effective mass, $V''(\phi) \gg H^2$. Therefore we expect the curvaton to undergo damped coherent oscillations in a $\phi^d$ potential. We expect the transition from coherent damped oscillations to slow-rolling to occur rapidly as a function of $\phi$ in a $\phi^d$ ($d \geq 4$) potential. Therefore we will consider the curvaton to be coherently oscillating for $\phi \gtrsim \phi_*$ and slow-rolling for $\phi \lesssim \phi_*$. The amplitude of coherent oscillations in a $\phi^d$ potential evolve with scale factor $a$ as $\phi \propto a^{-\frac{6}{d-2}}$ [11], thus the energy density evolves as $V(\phi) \propto a^{-\frac{6d}{d-2}}$. Therefore the number of e-foldings from the beginning of inflation ($V(\phi) = V_o$) until the onset of curvaton slow-rolling, $N_S$, is

$$N_S = \frac{(d + 2)}{6d} \ln \left( \frac{V_o}{V(\phi_*)} \right),$$

where the ratio of the energy density of the inflaton to the energy density of the curvaton at $\phi_*$ is given by

$$\frac{V_o}{V(\phi_*)} \approx \left( \frac{M_{Pl}}{V_o} \right)^{\frac{2}{d-2}}.$$

(This also shows that at the onset of curvaton slow-rolling the energy density is typically inflaton dominated.) Therefore, using as an example $V_o^{1/4} = 10^{15}$ GeV, we find for $d = 4$ ($d = 6$) that $N_S \approx 9.4$ ($N_S \approx 4.2$).

Thus the total number of e-foldings from the onset of inflation during which an observable deviation from scale-invariance is obtained is $N_S + \Delta N \approx 310$ for $d = 4$ and $\approx 250$ for $d = 6$. Therefore, since the scales of cosmological interest leave the horizon at around 50 e-foldings before the end of inflation, if the total number of e-foldings of inflation is less than about 360 ($d = 4$) or 300 ($d = 6$), and if the initial value of
the curvaton is large such that the transition to curvaton slow-rolling occurs during inflation, then we would expect to find a running blue perturbation spectrum with an observable deviation from scale-invariance.

3 D-term Inflation/Curvaton Scenario

A possible 'application' of the curvaton scenario is to SUSY hybrid inflation models. Hybrid inflation models [12] are a favoured class of inflation model, due to their ability to account both for a sufficiently flat potential during inflation and a sufficiently large inflaton mass for reheating after inflation, without requiring small couplings. In the context of SUSY, a particularly interesting class of hybrid inflation model is D-term inflation [6]. This is because D-term inflation naturally evades the \( \eta \)-problem of supergravity (SUGRA) inflation models [13] i.e. the generation of order \( H \) SUSY-breaking mass terms due to non-zero F-terms during inflation. Thus D-term inflation models are favoured, at least as low-energy effective theories.

However, the high precision observations of CMB temperature fluctuations made by the Wilkinson Microwave Anisotropy Probe (WMAP) [14] have introduced a difficulty for D-term inflation models. The contribution of \( U(1)_{FI} \) cosmic strings, formed at the end of inflation, to the CMB is too large unless the superpotential coupling satisfies \( \lambda \gtrsim 10^{-4} \) [7]. This is an unattractive possibility if the motivation for hybrid inflation models is that inflation can proceed with natural values of the dimensionless coupling constants of the order of 1.

The cosmic string problem and constraint on \( \lambda \) arises if the energy density perturbations responsible for structure formation are due to conventional inflaton quantum fluctuations. However, if the energy density perturbations were generated by a curvaton, it might be possible to reduce the energy density during inflation such that the cosmic string contribution to the CMB is acceptably small [7].

We first review the relevant features of D-term inflation [6]. For inflaton field \( s \) large compared with \( s_c \) (where \( s_c \) is the critical value of \( s \) at which the \( U(1)_{FI} \) symmetry breaking transition occurs), the 1-loop effective potential of D-term inflation is given
by $V = V_o + \Delta V$ ($\Delta V \ll V_o$), where \[6\]

$$V_o = \frac{g^2 \xi^4}{2}; \quad \Delta V = \frac{g^4 \xi^4}{32\pi^2} \ln \left(\frac{s^2}{\Lambda^2}\right),$$

and where $\Lambda$ is a renormalisation scale, $g$ is the $U(1)_{FI}$ gauge coupling and $\xi$ is the Fayet-Illiopoulos term. During slow-rolling the inflaton evolves according to

$$3H s \approx -\frac{dV}{ds} = -\frac{g^4 \xi^4}{16\pi^2 s}. \tag{14}$$

This has the solution

$$s^2 = s_c^2 + \frac{g^4 \xi^4 N}{24\pi^2 H^2}, \tag{15}$$

where $s$ is the value of the curvaton at $N$ e-foldings before the end of inflation. For $s^2 \gg s_c^2$ the number of e-foldings until the end of inflation is related to $s$ by

$$s^2 = \frac{g^2 M^2 N}{4\pi^2}, \tag{16}$$

where we have used $V \approx V_o$ ($\Delta V \ll V_o$) in $H$. Thus $s \propto \sqrt{N}$.

In the absence of curvaton evolution during inflation, the spectral index is given by

$$n - 1 = 1 - 2\epsilon \equiv 1 - \frac{1}{V} \frac{dV}{dN} = 1 - \frac{g^2}{16\pi^2 N}. \tag{17}$$

Thus with $N \approx 50$, corresponding to scales of cosmological interest, we find $\Delta n = n - 1 = -1.3 \times 10^{-4}g^2$. Therefore the deviation from scale-invariance is unobservably small in the absence of curvaton evolution.

For natural values of the gauge coupling $g$, the value of inflaton at $N \approx 50$ is close to $M$. This leads to a problem associated with non-renormalisable corections to the superpotential of the form $S^m/M^{m-3}$ and $U(1)_{FI}$ gauge superfield terms of the form $S^k W^a W^a / M^k$ [15]. These would result in an unacceptable deviation of the potential from flatness at $N \approx 50$ (so preventing sufficient inflation) unless $m > 9$ and $k > 6$ [15]. The possible solutions of this problem are to impose either an R-symmetry or a discrete symmetry on the superpotential to eliminate the dangerous terms [15]. Discrete gauge symmetries are preferred if quantum gravity effects violate global symmetries such as R-symmetry [15].
A high-order discrete symmetry would eliminate the dangerous terms up to a large value of $m$. In this case, the inflaton potential would rapidly increase as $s$ approaches $M$. Thus we would expect that inflation starts at $s < M$ in the case of a discrete symmetry. In this case the total number of e-foldings of inflation is less than $4\pi^2/g^2$. For $g > 0.2$ the total number of e-foldings is less than $1000^2$.

An R-symmetry would, in most cases, eliminate all non-renormalisable terms from the inflaton potential. This would result in a nearly flat inflaton potential for all values of the inflaton. In this case the question of the initial value of the inflaton field is related to the dynamics of the onset of inflation. One possibility is that the Universe begins in a chaotic state, characterised by quantum fluctuations of energy density of the order of $M^4$ and length scales $M^{-1}$ [9] \(^3\). The energy density will consist of gradient, kinetic and potential terms. For inflation to begin, it is essential that one of the fluctuations can become dominated by the potential term. This requires that inflation can occur with a potential energy density of the order of $M^4$, as in chaotic inflation [9]. D-term inflation with a flat potential has an energy density characterised by the scale $\xi \lesssim 10^{16}$ GeV $\ll M$. Therefore, in order to enter D-term inflation (or any other inflation with a nearly flat potential), an initial period of chaotic inflation due to another field is essential. The curvaton field with a potential could play exactly this role. If the curvaton starts with a large energy density then we may have an initial period of curvaton-driven chaotic inflation followed by D-term inflation. In this case we also expect the transition to curvaton slow-rolling to occur during D-term inflation.

The initial energy density of the inflaton will be due gradient terms characterised by the length scale $M^{-1}$, $\rho \sim M^2 s^2$, assuming that the initial value of the inflaton field, $s$, is due to a chaotic fluctuation of length scale $M^{-1}$ which subsequently inflates. Since the initial gradient energy density is expected to be of the order of $M^4$, we expect that $s \sim M$. Therefore, with chaotic initial conditions, it is plausible that a small

\[^2\]A concern with high-order discrete symmetries is that they may suppress the reheating temperature by suppressing couplings of the inflaton to the Standard Model sector fields. This would make it difficult for the curvaton to dominate the energy density before it decays.

\[^3\]We follow [17] and assume that the scale of chaotic fluctuations in SUGRA is set by the reduced Planck scale.
number of e-foldings of D-term inflation will occur, depending on the random initial fluctuation which inflates become to the observed Universe 4.

Thus D-term inflation with a discrete symmetry solution of the Planck-scale inflaton problem naturally results in a small total number of e-foldings of inflation. In addition, with chaotic initial conditions it is also plausible that a small number of e-foldings of D-term inflation will occur. Therefore an observable deviation of the curvaton perturbation spectrum from scale-invariance is possible in these cases.

4 Conclusions

We have shown that a curvaton scenario with a natural curvaton potential, large initial curvaton amplitude and less than around 300 total e-foldings of inflation can result in a blue perturbation spectrum with an observable deviation from scale-invariance and running spectral index.

Of the two conditions required to obtain an observable deviation from scale-invariance, the requirement of a large initial curvaton amplitude (such that the curvaton makes the transition to slow-rolling during inflation) is not manifestly unnatural. Indeed, it may be the most natural initial condition to consider if the Universe evolves from a chaotic initial state.

The question is then whether it is natural to have a total number of e-foldings of inflation not much larger than 300. In general, there is no obvious reason to expect the total number of e-foldings to be as low as 300. However, in some models, such as D-term inflation with a discrete symmetry suppression of dangerous non-renormalisable inflaton potential terms, a low number of e-foldings ($\lesssim 1000$ for $g \gtrsim 0.2$) is likely. Alternatively, if the Universe starts in an initial chaotic state characterised by energy density fluctuations $\sim M^4$ on length scales $M^{-1}$, then the curvaton may play a secondary role by providing an initial period of chaotic inflation which allows the tran-

\[ ^{4\text{In order to drive chaotic inflation, it is necessary for the curvaton to take a value initially larger than } M, \text{ which requires a non-trivial structure for the SUGRA K"ahler potential [16, 17]. In our case the initial value need not be too large, since we only require a small number of e-foldings of chaotic inflation in order to make the transition from the chaotic initial state to potential-driven evolution.}}\]
situation from the initial chaotic state to D-term inflation at a much lower energy scale. In this case we expect the initial value of inflaton field to be of the order of $M$, which can plausibly result in a low total number of e-foldings of D-term inflation. It would be interesting to develop a complete D-term inflation/curvaton scenario, compatible with chaotic initial conditions as well as CMB constraints.

The idea that the curvaton may provide an initial period of chaotic inflation is more general than the specific application to D-term inflation, and may be applied to other inflation models based on a nearly flat potential.

The simplest interpretation of the results presented here is that they support the idea that the curvaton is likely to produce an effectively scale-invariant perturbation spectrum as far as observations are concerned, given that most inflation models will have a total number of e-foldings much larger than 300. However, if a running blue perturbation spectrum is observed, it could be interpreted as the effect of a curvaton potential combined with a relatively small number of e-foldings of inflation, requiring a particular inflation model such as D-term inflation with a high-order discrete symmetry or chaotic initial conditions and an initial period of curvaton-driven chaotic inflation.

References


