Summary

The Luminosity and luminous region for an accelerator for bunched beams and a crossing angle are considered. Time averaged values for both are calculated based on the assumptions that the bunch length increases linearly and the intensity decreases exponentially. Specific calculations are made for the forthcoming Large Hadron Collider (LHC).

1 Introduction

A brief summary of the definitions of luminosity and luminous region is given, followed by explicit calculations using LHC parameters. The time average of the luminosity and hence the luminous region is also considered, this is of interest to future experiments at the LHC as it may change the best position of the detectors with time. The changes of the parameters of luminosity with time are based on previous experimental evidence and mathematical models.

Some mathematical preliminaries are discussed, followed by the standard derivation of the luminosity given a crossing angle. In the next section, the derived formulae are applied to the LHC parameters and the hourglass effect is studied numerically and shown not to be important. Subsequently, the reduction factor, due to the crossing angle, is examined and plots are given of the effect different parameters such as bunch length or $\beta^*$ have on the luminous region.

In the next section, the average luminosity and luminous region are defined and investigated using varying basic parameters such as bunch length and/or intensity. Typical variations are: an increase in bunch length by 30% and an exponential decrease in intensity over a period of 10 hours. The results are shown as plots for the nominal starting parameters of the LHC only, however links are given to computer programs where all parameters may be varied if necessary.

This is an internal CERN publication and does not necessarily reflect the views of the LHC project management.
2 Mathematical Preliminaries

Particles, in accelerators, travel in bunches or form a continuous beam. They either have head-on collisions at the interaction points or collide at a small crossing angle $\phi$. In the case of the LHC, they travel in bunches and collide at a total angle of $\phi = 300 \mu\text{rad}$. A bunch can be described by considering the number of particles it contains, $N$, and the normalised density, $\rho(x, y, s, ct)$ [1]. In what follows it is assumed that there is no coupling between the three spatial coordinates and that the beams move in the $s$ direction with $x$ and $y$ being the two transverse directions. The beams are also assumed to have Gaussian profiles in all three directions, namely

$$\rho^{(i)}(x, y, s, ct) = \rho_{x}^{(i)}(x)\rho_{y}^{(i)}(y)\rho_{s}^{(i)}(s - ct)$$

where $i = 1, 2$ (depending on the beam) and

$$\rho_{z}^{(i)}(z) = \frac{1}{\sigma_z\sqrt{2\pi}} \exp\left(-\frac{z^2}{2\sigma_z^2}\right), \quad i = 1, 2, \ z = x, y,$$

$$\rho_{s}^{(i)}(s \pm ct) = \frac{1}{\sigma_s\sqrt{2\pi}} \exp\left(-\frac{(s \pm ct)^2}{2\sigma_s^2}\right),$$

and the $\sigma_k’s$ ($k = x, y, s$) represent the standard deviation or beam sizes in the three coordinates.

The crossing angle $\phi$ is assumed to lie in the $(x, s)$ plane and the beams can be represented by two rotations as follows

$$x_1 = x \cos \frac{\phi}{2} - s \sin \frac{\phi}{2}, \quad s_1 = s \cos \frac{\phi}{2} + x \sin \frac{\phi}{2},$$

$$x_2 = x \cos \frac{\phi}{2} + s \sin \frac{\phi}{2}, \quad s_2 = s \cos \frac{\phi}{2} - x \sin \frac{\phi}{2}$$

as shown in Fig.1 below. Because the transverse beam size is orders of magnitude smaller than the longitudinal bunch length and the crossing angle is small, the simplification

$$s_1 = s_2 = s \cos \frac{\phi}{2}$$

is introduced and the reader is referred to [2] for the full calculation.

For a typical accelerator with bunched beams the expression for the luminosity is given by

$$L = 2cN_1N_2fB \cos \frac{\phi}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho_{x}^{(1)}(x)\rho_{y}^{(1)}(y)\rho_{s}^{(1)}(s - ct)$$

$$\times \rho_{x}^{(2)}(x)\rho_{y}^{(2)}(y)\rho_{s}^{(2)}(s + ct) dx dy ds dt$$

where $f$ is the revolution frequency, $B$ is the number of colliding bunches per beam and $\phi$ is the crossing angle.

Using $\int_{-\infty}^{+\infty} \exp(-x^2/A)dx = \sqrt{\pi A}$, the $y$ and $t$ integrations can be done immediately and, after applying the rotations (1), the following is obtained
\begin{equation}
\mathcal{L} = \frac{N_1 N_2 f B}{4 \pi^2 \sigma_s} \cos^2 \phi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sigma_x^2 \sigma_y^2} e^{-\frac{x^2 \cos^2(\phi/2)}{\sigma_x^2}} e^{-\frac{x^2 \sin^2(\phi/2) + y^2 \cos^2(\phi/2)}{\sigma_y^2}} dx ds.
\end{equation}

Similarly, the x integration may be done to yield

\begin{equation}
\mathcal{L} = \frac{N_1 N_2 f B}{4 \pi \sigma_s} \cos \phi \int_{-\infty}^{\infty} e^{-\frac{y^2 \sin^2 \phi}{\sigma_y^2}} e^{-\frac{x^2 \cos^2 \phi}{\sigma_x^2}} ds. \tag{2}
\end{equation}

Note that $\sigma_x$ and $\sigma_y$ are left inside the integral sign for generality as these may still depend on $s$.

If it is assumed that there is no hourglass effect, so any dependence of either of the transverse beam sizes on the longitudinal coordinate $s$ may be neglected, a further integration of (2) (in $s$) gives

\begin{equation}
\mathcal{L} = \frac{N_1 N_2 f B}{4 \pi \sigma_s \sigma_y} \frac{1}{\sqrt{1 + \left(\frac{\sigma_x}{\sigma_s} \tan \frac{\phi}{2}\right)^2}}. \tag{3}
\end{equation}

From (3), one can recover the approximate expression for the luminosity of a bunched beam with a crossing angle. In particular, $\sigma_s \gg \sigma_x$ and so the first square root in (3) can be approximated to 1 and
\[ \mathcal{L} = \frac{N_1 N_2 f B}{4\pi \sigma_x \sigma_y} S, \]  
\[ S = \frac{1}{\sqrt{1 + \left(\frac{\phi}{\sigma_x}\right)^2}}, \]

where \( S \) is known as the luminosity reduction factor and is given by

3 Evaluation of Luminosity for the LHC

The evaluation of (3) for the LHC requires the following parameters:

\( N_1 = N_2 = 1.1 \times 10^{11} \), the number of particles per bunch, with 2808 bunches per beam,
\( f = 11.2455 \text{ kHz} \), \( \gamma = 7461 \), \( \phi = 300 \mu\text{rad} \), \( \beta_x^* = \beta_y^* = 0.5 \text{ m} \), \( \sigma_s = 7.7 \text{ cm} \), and
\( \epsilon_n = 3.75 \mu\text{m} \), \( \sigma^*_x = \sigma^*_y = \sqrt{\frac{\epsilon_n}{\beta}} \), with \( \beta \) the relativistic factor, so

\[ \mathcal{L} = 1.21 \times 10^{34} \times 0.808 \text{ cm}^{-2}\text{s}^{-1} = 9.78 \times 10^{33} \text{ cm}^{-2}\text{s}^{-1} \]  

where the first two numbers represent the maximum luminosity (without crossing angle) and the reduction factor, respectively.

3.1 Hourglass effect

In the above calculation, the so-called hourglass effect was not included, which is usually a good approximation. However, this may not always be the case (e.g. small \( \beta^* \)) and the reduction factor is calculated again numerically in the more general case below. Returning to (2), and putting

\[ \sigma_z = \sigma^*_z \sqrt{1 + \left(\frac{s}{\beta^*}\right)^2}, \]

where \( z = x, y \), and \( \sigma^*_z, \beta^* \) are the values of the transverse beam sizes and beta function at the interaction point, (2) may be reduced to

\[ \mathcal{L}_{HG} = \left( \frac{N_1 N_2 f B}{4\pi \sigma^*_x \sigma^*_y} \right) \frac{\cos^2 \frac{\phi}{2}}{\sqrt{\pi} \sigma_s} \int_{-\infty}^{+\infty} \frac{e^{-s^2 A}}{1 + \left(\frac{s}{\beta^*}\right)^2} ds, \]  

where

\[ A = \frac{\sin^2 \frac{\phi}{2}}{(\sigma_x)^2} + \frac{\cos^2 \frac{\phi}{2}}{\sigma^2_s} = \frac{\sigma^2_s \sin^2 \frac{\phi}{2} + (\sigma^*_x)^2 [1 + (\frac{s}{\beta^*})^2] \cos^2 \frac{\phi}{2}}{(\sigma^*_x)^2 [1 + (\frac{s}{\beta^*})^2] \sigma^2_s}. \]

The last integral has to be done numerically and gives

\[ \mathcal{L}_{HG} = 1.21 \times 10^{34} \times 0.806 \text{ cm}^{-2}\text{s}^{-1} = 9.75 \times 10^{33} \text{ cm}^{-2}\text{s}^{-1}, \]
where, as before, the notation used is that the first term is the maximum luminosity and the second is the new reduction factor due to the combined effects of a crossing angle and the hourglass. So the inclusion of the hourglass effect ($L \rightarrow L_{HG}$) amounts to a very small reduction in luminosity and it can be concluded that, in the case of a bunch length of 7.7 cm and a $\beta^*$ of 50 cm this effect may be neglected and a plot comparing calculations both with and without the hourglass effect will be shown in the next section. However, the hourglass effect can become important and this is illustrated in Fig. 2 where the bunch length is taken as 10 cm and the values of $\beta^*$ considered are 5, 25 and 50 cm, i.e for $\sigma_s \approx \beta^*$.

![Figure 2: Hourglass effect for different $\beta^*$ and a bunch length of 10 cm, $\phi = 0.0 \mu$rad](image)

## 4 Detailed analysis of the reduction factor and Luminous region

The density of the interaction vertices (where events are located) is a function of the beam sizes ($\sigma_x, \sigma_y$), the bunch length ($\sigma_s$) and the overall geometry. Therefore, a careful analysis of the luminous region or

$$L = \int_{-s}^{+s} L(s') ds',$$

is required. Definition (9) is a modification of

$$L_0 = \int_{-\infty}^{+\infty} L(s') ds',$$
and the ratio of (9) to (10) refers to the percentage of luminosity lying between $-s$ and $+s$ only. Using a program written for this purpose [3], several cases were considered and are listed below:

Firstly, the luminous region was considered for several crossing angles, ranging from 0.0 $\mu$rad to 500 $\mu$rad and increasing in steps of 100 $\mu$rad at a time. The results are shown in (3), from which one can see that the total luminosity available is reduced as the crossing angle is increased. If the results of (3) are normalised as shown in (4), it becomes clear that increasing the crossing angle is a key factor in reducing both the luminosity and the luminous region, much more so than a small change in bunch length as will be explained below.

Subsequently five bunch lengths are considered ranging from 5.0 cm to 15.0 cm and increasing in steps of 2.5 cm at a time. At first, a zero crossing angle (Fig.5 is considered for comparison. The results shown are with the vertical axis in units of $10^{-34}$ cm$^2$s$^{-1}$ and the horizontal one representing the distance $s$ either side of the IP (given in cm). In figure 6, the same bunch lengths are considered, but the crossing angle is 300 $\mu$rad (LHC). All integrals were shown to converge (in the case of figure 5 to 1 and for the others lower) but only the region from 0 cm to 30 cm is shown as this is the one of interest. Figure 7 shows the same results as Fig.6, but without the hourglass effect included, in this case the expression for the luminous region may be given analytically as shown below (11). From this it is concluded that the hourglass effect may indeed be neglected for the LHC parameters.

If the hourglass effect is neglected, equation (7) may be calculated analytically to give

$$L = \int_{-s}^{+s} L(s')ds' = \left( \frac{N_1N_2fB}{4\pi\sigma_x\sigma_y}\right) \frac{\cos \frac{\phi}{2}}{\sqrt{\pi\sigma_s}} \sqrt{\frac{\pi}{A}} \text{erf} \left( \sqrt{A} s \right)$$

where all the symbols used are the same as those defined for (7).

5 Average Luminosity

In this section, the average luminosity and the average of the luminous region over a certain time span are considered. During a coast, it is expected that the bunch length increases and also that the intensity decreases with time, the models assumed for the two decays are based on [4] and [6], respectively. The bunch length is expected to increase by 30 % over a period of 10 hours and the intensity is expected to decrease exponentially. The worst possible exponential decrease which has been observed is [5]

$$N \rightarrow N_0 \exp \left( -\frac{t}{10} \right)$$

where $t$ is the time given in hours and, therefore, this is the one used in the simulations.

The integrated averaged luminosity is given by:

$$L_{av}(s) = \frac{1}{T} \int_0^T \int_{-s}^{+s} L(s',t)ds'dt.$$  

Using the purpose built program [3] for the LHC nominal values, the following results are found and are listed in Figs.8 - 12. Both decays, bunch length increase and intensity
Figure 3: Luminous region for a bunch length of 7.5 cm, $\beta^* = 50$ cm and different crossing angles

decrease were considered separately. Figure 8 shows the initial nominal case, before any time evolution at all. Figures 9 and 10 show the average luminosity over a time of 10 hours with the bunch length increasing linearly, until it has reached a 30 % increase with respect to the initial 7.7 cm. Finally, figures 11 and 12 show the same as the previous two, in the case of an exponential decrease in intensity as well as the increase in bunch length. From all the figures it may be concluded that the luminous region does not change significantly with time regardless of the decays considered above.

The table below compares the percentages of luminosity desired with the size of the luminous regions required, both before and after a 10 hour coast.
Figure 4: Luminous region for a bunch length of 7.5 cm, $\beta^* = 50$ cm and different crossing angles (normalised)
6 Conclusions

The luminosity for the LHC was evaluated and, in particular, the luminous region lying 30 cm either side of the interaction point was looked at for a sample of crossing angles and bunch lengths near the LHC parameters. It was also shown that, for the parameters considered, the hourglass effect may be neglected. A 10 hour coast was considered, together with two decays in the factors of luminosity, one a 30 % increase in bunch length and the other an exponential decay of the intensity due to interactions and other effects. The results show that there is no significant increase in the required luminous region to capture a certain percentage of the available luminosity.

7 Acknowledgements

The author thanks M. Hayes, W. Herr, M. Placidi, F. Ruggiero, A. Verdier and F. Zimmerman for very useful comments and a careful reading of the manuscript.

References

Figure 6: Luminous region for $\phi = 300 \, \mu rad$, $\beta^* = 50 \, cm$


Figure 7: Luminous region for $\phi = 300 \, \mu$rad, $\beta^* = 50 \, $cm, no hourglass

Figure 8: Luminous region for $\phi = 300 \, \mu$rad, $\beta^* = 50 \, $cm, bunch l. 7.7 cm
Figure 9: Same as Fig.8 after 10 hour coast with bunch length increase only

Figure 10: Same as Fig.9 normalised with respect to the nominal case
Figure 11: Same as Fig.8 after 10 hour coast with bunch length increase and intensity decay

Figure 12: Same as Fig.11 normalised with respect to the nominal case