

Information-Theoretic Determination of Ponderomotive Forces

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Abstract

From the equilibrium condition $\delta S = 0$ applied to an isolated thermodynamic system of electrically charged particles and the fundamental equation of thermodynamics $(dU = TdS - (\mathbf{f} \cdot d\mathbf{r}))$ subject to a new procedure, it is obtained the Lorentz’s force together with non-inertial terms of mechanical nature. Other well known ponderomotive forces, like the Stern-Gerlach’s force and a force term related to the Einstein-de Haas’s effect are also obtained. In addition, a new force term appears, possibly related to a change in weight when a system of charged particles is accelerated.

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I. INTRODUCTION

There has been an effort to understand macroscopic phenomena, characterized by thermal equilibrium and irreversible approaches to it, based on the crucially important property: the entropy. E. T. Jaynes [1] was a precursor of this line of thought. In fact, the application of information-theoretic techniques to a number of fields has been very successful, specially in studying colloidal systems. Interestingly, it was discovered entropy depletion effects in a binary mixture consisting in the increase of entropy of one component and forcing the other component to a greater order, leading through this process to the possibility of entropy control of the directed motion of colloidal particles [2, 3]. Direct measurements of entropic forces were measured for a single colloidal sphere close to a wall in a suspension of rigid colloidal rods [4]. Also, stochastic resonance-like phenomena in non-linear systems can be described by a maximization of the information-theoretic distance measures between probability distributions of the output variable [5, 6].

In this paper we follow through this line of research of maximum entropy methods, but proposing a special procedure to analyze systems imparted with general kind of motion, both translational and rotational. One advantage of this method is that a simple method of calculation is devised from energy and entropy considerations. It gives known results through the use of entropy of a system dynamics and in specific problems could be a possible alternative to Hamilton theory. We don’t analyze the eventual deep meaning of the present procedure, but only claim that the results obtained mean the entropy, as it appears in the fundamental equations of thermodynamics, could be a mathematical entity useful for the calculation of non equilibrium processes.

Based on the maximization of entropy principle and the fundamental equation of thermodynamics \(dU = TdS - (\mathbf{f} \cdot d\mathbf{r})\), it was previously suggested a new procedure [8] to obtain the dynamical equation of motion of a particle in a rotating frame.

In this paper we extend the same procedure to classical electrodynamics, obtaining what we would like to call information-theoretic ponderomotive forces. Besides the classical force in a rotating frame, it is additionally obtained the Lorentz’s force acting over an electrically charged particle, the Stern-Gerlach’s force, a term related to the Einstein-de Haas’s effect, as well a new term, possibly related to a change in weight in an accelerated frame.
II. EXTREMUM PRINCIPLE

Our initial theoretical framework is the one advanced by Landau and Lifshitz [11]. Let us consider an isolated macroscopic system $S$ composed by $N$ infinitesimal macroscopic subsystems $S'$ (with an internal structure possessing a great number of degrees of freedom to ensure the validity of the entropy concept) with $E_i$, $p_i$ and $J_i$, all constituted of classical charged particles with charge $q_i$ and mass $m_i$. The system is assumed to be isolated and so there exist seven independent additives integrals: the energy, 3 components of the linear momentum $p$ and 3 components of the angular momentum $L$.

The energetics of the charged particles is modelled on the Maxwell-Lorentz theory of the electromagnetic field. Thus, the internal energy $U_i$ of each subsystem moving with momentum $p_i$ in a Galilean frame of reference is given by

$$E_i = U_i + \frac{p_i^2}{2m_i} + \frac{J_i^2}{2I_i} - q_i V_i + q_i (A_i \cdot v_i),$$

(1)

where, besides the mechanical energy, the electrostatic and magnetic energy ($U_{m,i} \rightarrow -q_i (A_i \cdot v_i)$) terms are also duly inserted.

The entropy of the system is the sum of the entropy of each subsystems (and function of their internal energy $U$, $S = S(U)$):

$$S = \sum_i S_i \left( E_i - \frac{p_i^2}{2m} - \frac{J_i^2}{2I_i} - q_i V_i + q_i (A_i \cdot v_i) \right).$$

(2)

Energy, momentum and angular momentum conservation laws must be verified for a totally isolated system:

$$E = \sum_{i=1}^N E_i,$$

(3)

$$P = \sum_{i=1}^N p_i,$$

(4)

and

$$L = \sum_{i=1}^N (r_i \times p_i) + J_i.$$  

(5)

In the above equations, $r_i$ is the position vector relatively to a fixed frame of reference $R$, $p_i$ is the total momentum (particle + field) and $J_i$ is the total intrinsic angular momentum of the particle, comprising a vector sum of the electron orbital angular momentum and the angular momentum contributed by electron spin and nuclear spin (since the electromagnetic
momentum is already included in the preceding term through $\mathbf{p}_i$). The exploitation of the entropy principle introduces Lagrange multipliers from which, as we will see, ponderomotive forces are obtained.

It is necessary to find the conditional extremum; they are set up not for the function $S$ itself but rather for the changed function $\bar{S}$:

$$\bar{S} = \sum_{i=1}^{N} \left\{ S_i \left[ E_i - \frac{p_i^2}{2m} - \frac{J_i^2}{2I_i} - q_i V_i + q_i (A_i \cdot \mathbf{v}_i) \right] + (a \cdot \mathbf{p}_i) + b \cdot ([\mathbf{r}_i \times \mathbf{p}_i] + \mathbf{J}_i) \right\},$$  \hspace{0.5cm} (6)

where $a$ and $b$ are Lagrange multipliers. The conditional extremum points are obtained for

$$\frac{\partial \bar{S}}{\partial \mathbf{r}} = 0; \ \frac{\partial \bar{S}}{\partial \mathbf{p}} = 0. \hspace{0.5cm} (7)$$

At thermodynamic equilibrium the total entropy of the body has a maximum constrained to supplementary conditions 3, 4 and 5.

III. ELECTRODYNAMIC PONDEROMOTIVE FORCES

At this point we intend to go a step further than Landau and Lifschitz presentation [11]. As we are interested in situations in which separate parts of a system (discrete sub-systems) are internally in equilibrium (this should apply to continuous systems as well) small changes in extensive coordinates $\delta X_\mu$ will produce small deviations $\delta \overline{S}$ in the total entropy of the system. As is known, we can in general define a set of intensive parameters $F_\mu$ in terms of the local parameter $S(X_\mu)$ by the equation

$$F_\mu = \frac{\partial S(X_\mu)}{\partial X_\mu} \hspace{0.5cm} (8)$$

with $F_\mu$ being a local position- and time-dependent quantity called the affinity of the sub-system. After multiplying the affinity by the "temperature" $T$ it is obtained the entropic forces. This is what we seek right now and we will show that with an appropriate procedure electrodynamic ponderomotive (body) forces are obtained. Of course we are aware of the implications the use of entropy in nonequilibrium can have, but in this paper we don’t analyze the concept of probability in nonequilibrium statistical mechanics. Quoting Guttmann [10], it is sure that this matter is still

ill understood...and don’t have a secure foundation.
Retrospectively, the developments hereby exposed can be an illustration of the potential use of entropy for others applications than the conventional role played in the determination of equilibrium states. From the partial derivative, $\frac{\partial S}{\partial p_i}$, it is obtained the total (particle’s + field) canonical momentum (when putting $\mathbf{v}_{rel} = Ta$ and $\omega = Tb$, and thus giving a clear meaning to the Lagrangian multipliers):

$$p_i = \mathbf{p}_{rel} + m[\omega \times \mathbf{r}_i] + q_i \mathbf{A}_i.$$  \hspace{1cm} (9)

Here, $\mathbf{p}_{rel} = m\mathbf{v}_{rel}$ is the apparent momentum as seen in a rotating frame $\mathcal{R}'$. As it should be, besides the mechanical terms related to translation and rotation, it appears the electromagnetic momentum term $q_i \mathbf{A}_i$. By the other hand, a further refinement in the formulation of the maximum entropy method by postulating the existence of a multiplicative factor $T/2$ associated to each degree of freedom, allowing us from the partial derivative $\frac{\partial S}{\partial r_i}$ to obtain the expression

$$\frac{T}{2} \frac{\partial S}{\partial r_i} = \nabla_{r_i} U_i + m_i \frac{d\mathbf{v}_i}{dt} + \frac{1}{2} \nabla_{r_i} (\omega \cdot \mathbf{J}_i).$$  \hspace{1cm} (10)

In nonequilibrium, as $T$ does not have the current meaning of thermodynamic temperature, it is merely a parameter useful to come through with calculations ($T$ could be a kind of kinetic temperature).

When there is no flux of entropy ($\frac{T}{2} \frac{\partial S}{\partial r_i} = 0$) and consequently the following relation prevails

$$- \nabla_{r_i} U_i^{eq} = m_i \frac{d\mathbf{v}_i}{dt} + \frac{1}{2} \nabla_{r_i} (\omega \cdot \mathbf{J}_i).$$  \hspace{1cm} (11)

Here, we denote by $U_i^{eq}$ the $ith$ sub-system internal energy at equilibrium (since it was obtained through condition $\frac{\partial S}{\partial r_i} = 0$) and the first term on the right-hand side (rhs) is the mechanical force acting on the $ith$ particle on $\mathcal{R}$ frame. The physical meaning of Eq. 11 is that, at equilibrium, there is no production of entropy and the force acting on over the $ith$ particle may be found by taking the gradient of a conservative potential (which includes the rotational energy) - in fact, an effective potential energy, useful in dynamics to find turning points of orbits in a central field of force. Remark that the electromagnetic field is affected to $U_i^{eq}$ and that’s why does not appear explicitly here.

Going back to Eq. $\ref{eq:2}$, when considering the flux of the internal energy, through Eq. 2 it is easy to obtain the following full expression

$$\frac{T}{2} \frac{\partial S}{\partial r_i} = -q_i \nabla_{r_i} V_i + q_i \nabla_{r_i} (\mathbf{v}_i \cdot \mathbf{A}_i)$$
\[ + m_i \frac{dv_i}{dt} - \frac{1}{2} \nabla_{r_i} \left( \frac{J_i^2}{I_i} \right) + \frac{1}{2} \nabla_{r_i} (\omega \cdot J_i). \]  

(12)

It is worth to rearrange some terms in order to give more clear physical meaning to the all expression. So, using the definition of \( B \) in terms of vector potential, we have the mathematical identity

\[ \nabla_{r_i} (A_i \cdot v_i) = (v_i \cdot \nabla_{r_i}) A_i + (A_i \cdot \nabla_{r_i}) v_i + [v_i \times B_i] + [A_i \times [\nabla_{r_i} \times v_i]]. \]  

(13)

The cross product \((v_i \cdot \nabla) A_i\) requires an interpretation in physical grounds. In fact, consider a charge \( q \) moving with velocity \( v \) along a given direction \( x \), such that it is displaced by \( x = vt \). The Maxwell equations determine the electromagnetic field in terms of the position of field sources relative to the position of the charge, \( A(x - vt, y, z) \). It follows, for a planar wave, a time dependence of the form \( \frac{\partial A}{\partial t} = -v \frac{\partial A}{\partial x} \). Taking this into account, the extremum condition \( \frac{\partial S}{\partial r_i} = 0 \), for example, lead us to the relationship

\[ E_i + [v_i \times B_i] = \frac{1}{2} q_i \nabla_{r_i} \left( \frac{J_i^2}{I_i} - \omega \cdot J_i \right) - [A_i \times \omega], \]  

(14)

which is a condition of charges equilibrium in a rotating frame. The fields are defined as usually by \( E = -\nabla V - \frac{\partial A}{\partial t} \) and \( B = [\nabla \times A] \), with the usual meaning. In the last expression, the electromagnetic field was computed explicitly and so the equilibrium condition acquired a new form. In particular, we see that a gyroscopic term plays an important role in equilibrating the plasma.

In a closed system, the total differential of \( U \) in the variables \( r \) and \( S \) is given by \( dU = T dS - (f \cdot r) \). During the interval of time \( dt \) and at a given point \((r, S)\) together with the introduction of our postulate, we have the fundamental equation of thermodynamics in a differential form:

\[ \nabla_{r_i} U_i = \frac{T}{2} \nabla_{r_i} S - f_i. \]  

(15)

This is the well known fundamental equation of thermodynamics, only differing by the introduction of a factor \( \frac{T}{2} \) by each degree of freedom. This is a necessary condition in order to fully exploit all the theoretical predictions which are possible to achieve through the hereby proposed procedure. In our current perspective, the entropy provides a useful guide for the likely outcome of a plasma. Through entropy we have access to end-points but not to the intermediate physical processes that give birth to a final state. Thus, applying
this equation (same procedure as presented in [8] for Newton’s second law) and after some vectorial algebra, it is easily obtained the force acting on electrically charged particles

\[ f_i = q_i E_i + q_i [v_i \times B_i] + q_i (A_i \cdot \nabla r_i) v_i + 2q_i [A_i \times \omega] + \frac{1}{2} \nabla r_i (\omega \cdot J_i) \]

\[ - \frac{1}{2} \frac{\nabla r_i}{r_i} \left( \frac{J_i^2}{I_i} \right) - \nabla r_i U_i. \]  

(16)

But at this stage we need to take care of the physical meaning of the mathematical entities involved, in particular the last term - the gradient of energy in space. It is understandable that, not far away from equilibrium, we can always assert that

\[ - \nabla r_i U_i = - \nabla r_i U^\text{eq}_i + f^S_i. \]

(17)

Here, we denote by \( f^S_i \) a force term representing a small disturbance of a constant equilibrium state characterizing \( \text{ith} \) system. Hence, the above expression can be rewritten in the form

\[ f_i = f^M_i + f^L_i + q_i (A_i \cdot \nabla r_i) v_i - 2q_i [\omega \times A_i] \]

\[ - \frac{1}{2} \frac{\nabla r_i}{r_i} \left( \frac{J_i^2}{I_i} \right) + \nabla r_i (\omega \cdot J_i) + f^S_i. \]

(18)

Here, \( f^M_i \) is the mechanical force acting over the \( \text{ith} \) particle, \( f^L_i = q_i E_i + q_i [v \times B_i] \) is the respective Lorentz’s force. The third term on the right-hand side of the above equation is the rate of variation of the particle’s velocity along the potential vector acting on it. Now, notice that \( A_i = \sum_j \frac{q_j v_j}{cr_{ij}} \) is the potential vector (here, in Gaussian units) at the point where the charge \( q_i \) is, as obtained by a development to first order in \( v_i \) of the Liénard-Wiechert potentials. This solution corresponds to a retarded potential and is correct only for \( \frac{v}{c} \ll 1 \), otherwise we should add a relativistic correction. It can be easily shown that the following relation holds

\[ q_i (A_i \cdot \nabla r_i) v_i = -q_i [\omega \times A_i]. \]

(19)

To disclose its full physical meaning after a better arrangement, it is easily seen to correspond to a kind of gyroscopic force

\[ q_i (A_i \cdot \nabla r_i) v_i = \sum_j \frac{q_j q_i}{cr_{ij}} [\omega \times v_j]. \]

(20)

Inserting Eq. 20 into Eq. 18 the total force acting over the \( \text{ith} \) particle can be written in the final form

\[ f_i = f^M_i + f^L_i - 2 \sum_j m^{em}_{ij} [\omega \times v_j]. \]
\[-\frac{1}{2} \nabla r_i \left( \frac{j^2_i}{I_i} \right) + \nabla r_i (\omega \cdot J_i) + f_i^S. \] (21)

We have introduced in the last equation the electromagnetic mass of the system resulting from the interaction between charges \(q_i\) and \(q_j\) distant \(r_{ij}\) apart

\[m_{ij}^{em} = \frac{1}{2} \frac{q_i q_j}{c r_{ij}}. \] (22)

The third term on the rhs - a kind of gyroscopic force - is unexpected and requires further study. It can lead to striking new phenomena, such as levitation, because it means that the interaction between a system of charges with different sign leads to a mass increase, whereas equal sign charged particles when interacting lead to a mass decrease. An effect similar to some extent to this one was referred by Boyer [9] when investigating the change in weight associated with the electrostatic potential energy for a system of two point charges supported side by side against a weak downwards gravitation field.

Finally, the fourth term in the rhs is closely related to the Einstein-de Haas effect. In fact, there is a natural relationship between \(J_i\) and the Ampèrian current constituting part of the material medium and the magnetization vector (in particular, \(\mu = \frac{g e J_{2mc}}{2mc}\)), and from this result a body force [12]. The fifth term in the rhs is the Stern-Gerlach’s force when applied to a magnetic moment.

The last term in the rhs, \(f_i^S\), was introduced \textit{mutatis mutandis} to represent non-equilibrium processes, such as bremsstrahlung radiation, line radiation, turbulence, \textit{inter alia}. Altogether, in order to represent fully non equilibrium processes, other terms should be included, such as the pressure gradients \(\nabla p\) as well the dissipative terms related to the momentum gain of the ion fluid caused by collisions with electrons - as admitted in [13], they are of the form \(P_{ei} = \eta c^2 n^2 (v_e - v_i)\), where \(\eta\) is the specific resistivity (in general, a tensor), but in particular for an isotropic plasma it is a scalar. When searching for a more complete description, and transposing the equations from the discrete to the continuum level, the magnetohydrodynamics fluid equations of motion should be retrieved.

We don’t attempt to derive Maxwell equations from the actual framework since they are essentially relativistic equations obeying Lorentz transformations and a very careful definition of time and space should be addressed.
IV. CONCLUSION

This letter has extended the application of a new procedure to electromagnetic fields. We believe that the development hereby presented can show up new features related to the entropy concept itself. It was shown that the inertial force terms appearing in accelerated frames and the electromagnetic force acting over an electrically charged particle are a manifestation of an entropic force. But if the known forces are obtained with a given kind of system energy, therefore, working backward we can conclude that a given arrangement of the system in terms of microstates (and hence through entropy) are generating those forces. In this sense, as the application of the entropy concept is not well funded in non-equilibrium situations, we have here possible example of how far we can go. Besides the already known force terms a kind of new electromagnetic gyroscopic force was shown to appear in a rotating frame implying a mass variation of a system of charges.

Research of phenomena involving electromagnetic fields could take advantage of this tool.

[7] This equation has its roots in an attempt to provide a mechanical foundation to the second law and it was established mathematically by Gibbs and Helmholtz. It was established by Hermann von Helmholtz when he worked on the mechanical interpretation of thermodynamics, see
Jrmn91, pp.111-183 (1884). John Williard Gibbs, almost a decade earlier, gave grounds to this fundamental equation, see: J. W. Gibbs, Transactions of the Connecticut Academy, II. pp. 309-342 (1873); J. W. Gibbs, Transactions of the Connecticut Academy, II. pp. 382-404 (1873)


