Analytical Bounce Solution in a Dissipative Quantum Tunneling

D. K. Park*

Department of Physics, Kyungnam University, Masan, 631-701, Korea

Abstract

The analytical bounce solution is derived in terms of the polygamma function in the Caldeira-Leggett’s dissipative quantum tunneling model. The classical action for the bounce solution lies between the upper and lower bounds in the full range of $\alpha$, where $\alpha$ is a dissipation coefficient. The bounce peak point increases from 1 to $4/3$ with increase of $\alpha$. In spite of various nice features we have shown that the solution we have derived is not exact one by making use of the zero mode argument in the linearized fluctuation equation. However, our solution can be a starting point for approximate computation of the prefactor in this model.

*Email: dkpark@hep.kyungnam.ac.kr
How to describe a dissipation at the level of quantum mechanics is a long-standing puzzle in physics. Upon our knowledge Feynman and Vernon (FV) [1] firstly described it as an interaction between a system of interest and its environments. Especially, they developed a formalism to investigate the quantum dissipation systematically by introducing an influence functional. The influence functional is an extremely important quantity in the sense that it contains all quantum effects of the environments and, thus makes it possible to describe the quantum dissipation in terms of only system’s coordinates.

Based on FV formalism Caldeira and Leggett (CL) considered in Ref. [2] a quantum tunneling model interacting with the harmonic oscillator environments. Introducing a potential which does not have a true vacuum, CL examined the effect of dissipation on quantum tunneling within a semi-classical approximation which was developed several decades ago [3–5]. The full application of the semi-classical application, however, is very difficult in this setup because the exact bounce solution is still unknown. In fact, it seems to be extremely hard (or may be impossible) to derive a bounce solution in an analytic form due to the non-local term the CL model involves. Without the analytic bounce it is impossible to exploit the power of the semi-classical approximation maximally.

In Ref. [2] CL obtained the upper and lower bounds of the classical action in terms of the dissipation coefficient \( \alpha \) without the explicit bounce solution. These two bounds are monotonically increasing function with respect to \( \alpha \), which indicates that the presence of the dissipation causes the decrease of the tunneling probability within an exponential approximation. The similar physical setup with a double-well potential was examined by a canonical method in Ref. [6,7]. The authors in Ref. [6,7], however, claimed that the dissipation may enhance the tunneling probability. In order to reconcile these two apparently discrepancy results we think the prefactor should be examined in the semi-classical side. However, it is very difficult to compute the prefactor without the analytic bounce solution. Thus we may need a bounce solution in the analytic form for the computation of the prefactor although it is not exact. It is main purpose of this letter to derive an analytic bounce which interpolates between the exact no-damping solution and the strong-damping solution.
We start with a dimensionless action [2]
\[
\sigma[z] = \int_{-\infty}^{\infty} du \left[ \left( \frac{dz}{du} \right)^2 + (z^2 - z^3) \right] + \frac{\alpha}{\pi} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} du' \left( \frac{z(u) - z(u')}{u - u'} \right)^2
\]
(1)
which yields an equation of motion as
\[
\ddot{z} = z - \frac{3}{2} z^2 + \frac{2\alpha}{\pi} \int_{-\infty}^{\infty} du \frac{z(u) - z(u')}{(u - u')^2}.
\]
(2)
We know the exact no-damping \((\alpha \to 0)\) solution
\[
z_0(u) = \text{sech}^2 \frac{u}{2}
\]
(3)
and strong-damping \((\alpha \to \infty)\) solution
\[
z_\infty(u) = \frac{4}{3[1 + \left(\frac{u}{2\alpha}\right)^2]}
\]
(4)
Thus, the real bounce solution should interpolate between (3) and (4) with increasing the dissipation coefficient \(\alpha\).

The bounce solution we obtained in this letter is
\[
z(u) = \sqrt{\frac{2}{\pi}} \frac{A_\alpha}{4 B_\alpha^2} \left[ \psi' \left( \frac{C_\alpha + B_\alpha + iu}{2B_\alpha} \right) + \psi' \left( \frac{C_\alpha + B_\alpha - iu}{2B_\alpha} \right) \right]
\]
(5)
where \(\psi'\) is an usual polygamma function. The \(A_\alpha, B_\alpha,\) and \(C_\alpha\) are \(\alpha\)-dependent but \(u\)-independent constants which obey the following equations:
\[
\begin{align*}
\mu \psi''(\mu) + 2\psi'(\mu) &= \frac{2\sqrt{2\pi}}{3} \frac{B_\alpha^2}{A_\alpha} \\
\mu^2 \psi''(\mu) + 2\mu \psi'(\mu) &= \frac{4\sqrt{2\pi \alpha}}{3} \frac{B_\alpha}{A_\alpha} \\
(3 - 8\alpha^2)\mu^3 \psi''(\mu) + 6(1 - 4\alpha^2)\mu^2 \psi'(\mu) + 16\alpha^2 \mu + 4\alpha^2 &= 0
\end{align*}
\]
(6)
where \(\psi''\) is polygamma function and \(\mu = C_\alpha/B_\alpha\). From the first and second equations of Eq.(6) one can show easily \(C_\alpha = 2\alpha\). One can show also numerically that \(A_\alpha\) and \(B_\alpha\) are monotonically increasing functions with respect to \(\alpha\).

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\(^1\)In this letter we will follow the same conventions with Ref. [2]
Before we explain how Eq.(5) is derived, we would like to show its nice features. Fig. 1 shows the $\alpha$-dependence of its classical solution which is represented by the red line. The green and blue lines represent the classical actions for the strong-damping and no-damping solutions respectively. Two black lines are upper and lower bounds of the classical action which were derived explicitly in Ref. [2]. Fig. 1 indicates that the classical action for our bounce solution (5) lies between the upper and lower bounds in the full range of $\alpha$. Fig. 1 also implies that our solution interpolates between the no-damping and the strong-damping solutions. The small difference between the red and green lines at the large $\alpha$ region indicates that our solution is an approximate analytical solution. This fact will be proven later by making use of the zero mode argument in the fluctuation equation level. Fig. 2 shows the $\alpha$-dependence of $z(0)$, which means the peak point of the bounce. As CL predicted in Ref. [2], the peak point increases from 1 to $4/3$ with increase of $\alpha$.

Now, let me explain how the bounce solution (5) is derived. Taking a Fourier transform from $z(u)$ to $\tilde{z}(\omega)$, one can change the equation of motion (2) in terms of $\tilde{z}(\omega)$ as following

$$(\omega^2 + 2\alpha|\omega| + 1)\tilde{z}(\omega) = \frac{3}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} d\Omega \tilde{z}(\omega - \Omega)\tilde{z}(\Omega).$$

(7)

The explicit form of $\tilde{z}_0(\omega)$ and $\tilde{z}_\infty(\omega)$ which are Fourier transform of $z_0(u)$ and $z_\infty(u)$ become

$$\tilde{z}_0(\omega) = 2\sqrt{2\pi} \frac{\omega}{\sinh \pi \omega},$$
$$\tilde{z}_\infty(u) = \frac{4\sqrt{2\pi \alpha}}{3} e^{-2\alpha|\omega|}.$$

(8)

As expected $\tilde{z}_0(\omega)$ and $\tilde{z}_\infty(\omega)$ are solutions of Eq.(7) without second term and first term in the left-handed side of Eq.(7) respectively. These are properties of no-damping and strong-damping in $\omega$-space. It is worthwhile noting the $\omega$-dependence of $\tilde{z}_0(\omega)$ and $\tilde{z}_\infty(\omega)$. Although $\omega$ is a dimensionless quantity, it should have a dimension if we go back to the CL’s original model which has a dimension. Thus, from the dimensional consideration in Eq.(7) we can conjecture that $\tilde{z}(\omega)$ should have a same dimension with $\omega$ in the original theory except the strong-damping solution, which should be dimensionless. From this point of view we can understand the $\omega$-dependence of $\tilde{z}_0(\omega)$ and $\tilde{z}_\infty(\omega)$. Thus one can take an ansatz

4
\[ \tilde{z}(\omega) = A_\alpha \frac{\omega}{\sinh B_\alpha \omega} e^{-C_\alpha |\omega|}. \]  

(9)

Then the \( \alpha \)-dependent constants \( A_\alpha, B_\alpha, \) and \( C_\alpha \) should satisfy \( A_0 = 2\sqrt{2\pi}, B_0 = \pi, C_0 = 0, \)
\( A_\infty/B_\infty = 4\sqrt{2\pi}\alpha/3 \) and \( C_\infty = 2\alpha \) in order for \( \tilde{z}(\omega) \) to interpolate between the no-damping and the strong-damping solutions.

Now, we will insert the ansatz (9) into Eq.(7) to extract an information on \( A_\alpha, B_\alpha, \) and \( C_\alpha \). The most difficult term we need to compute is the following convolution term

\[ \tilde{I}_\alpha(\omega) = \int_{-\infty}^{\infty} d\Omega \tilde{z}(\omega - \Omega) \tilde{z}(\Omega) = A^2_\alpha \frac{A_\alpha}{2} \left[ \tilde{I}_{1,\alpha}(\omega) - \omega^2 \tilde{I}_{2,\alpha}(\omega) \right] \]

(10)

where

\[ \tilde{I}_{1,\alpha}(\omega) = \int_0^{\infty} dy \frac{y^2}{\cosh B_\alpha y - \cosh B_\alpha \omega} e^{-\frac{\epsilon_\alpha}{2}(|\omega + y| + |\omega - y|)} \]  

(11)

\[ \tilde{I}_{2,\alpha}(\omega) = \int_0^{\infty} dy \frac{1}{\cosh B_\alpha y - \cosh B_\alpha \omega} e^{-\frac{\epsilon_\alpha}{2}(|\omega + y| + |\omega - y|)}. \]

Note that \( \tilde{I}_{1,\alpha}(\omega) \) and \( \tilde{I}_{2,\alpha}(\omega) \) are even function with respect to \( \omega \). Thus we can assume \( \omega > 0 \) without loss of generality. With this assumption \( \tilde{I}_{1,\alpha}(\omega) \) and \( \tilde{I}_{2,\alpha}(\omega) \) are expressed as following

\[ \tilde{I}_{1,\alpha}(\omega) = \frac{2}{3} \left[ \left( \frac{\pi}{B_\alpha} \right)^2 - \frac{\omega^2}{2} \right] \frac{\omega}{\sinh B_\alpha \omega} + \frac{1}{B_\alpha^3} \left( \frac{\partial^2 K - e^{-\mu z_0} \partial^2 K}{\mu = 0} \right) \]

(12)

\[ \tilde{I}_{2,\alpha}(\omega) = -\frac{\omega}{\sinh B_\alpha \omega} e^{-C_\alpha \omega} + \frac{1}{B_\alpha} \left( K - e^{-\mu z_0} K \right) \]

where

\[ K \equiv \int_{z_0}^{\infty} dz \frac{e^{-\mu z}}{\cosh z - \cosh z_0} \]  

(13)

and \( z_0 \equiv B_\alpha \omega \) and \( \mu \equiv C_\alpha/B_\alpha \). When we compute the first term of \( \tilde{I}_{1,\alpha}(\omega) \) we used the property of the Lerch function in Ref. [8].

Now, the remaining problem for the computation of \( \tilde{I}_\alpha(\omega) \) is to compute \( K \) which have an infrared-like infinity as a field theory terminology. In order to take into account the infinity carefully we take a change of variable \( x = e^z \), which makes \( K \) to be

\[ K = 2(x_0 - x_0^{-1})^{-1} \int_{x_0 + \epsilon}^{\infty} dx \left[ \frac{x^{-\mu}}{x - x_0} - \frac{x^{-\mu}}{x - x_0^{-1}} \right] \]

(14)
where \( x_0 = e^{z_0} \). In Eq.(14) we introduced an infinitesimal parameter \( \epsilon \) explicitly for the regularization of the infrared-like infinity. Performing the integration in Eq.(14) one can express \( K \) as a difference of two hypergeometric functions. Making use of the relation between the hypergeometric and digamma function \([9]\) the final expression of \( K \) becomes

\[
K = \frac{e^{-\mu z_0}}{\sinh z_0} \left[ (z_0 - \ln \epsilon) + e^{2\mu z_0} \ln(e^{2z_0} - 1) + (e^{2\mu z_0} - 1)\psi(\mu) - \sum_{n=1}^{\infty} \frac{(\mu)_n}{n!} \psi(n + 1)(1 - e^{-2z_0})^n \right]
\]

(15)

where \((\mu)_n = \mu(\mu + 1) \cdots (\mu + n - 1)\) and \( \psi \) is a digamma function. Note that \( K \) has a logarithmic divergence as expected. Using Eq.(15) it is straightforward to compute \( \tilde{I}_{2,\alpha}(\omega) \) which reduces to

\[
\tilde{I}_{2,\alpha}(\omega) = \frac{e^{-C_{\alpha}\omega}}{B_{\alpha} \sinh B_{\alpha}\omega} \left[ z_0 + \left( e^{2\mu z_0} - 1 \right) \left\{ \psi(\mu) + \ln \left( e^{2z_0} - 1 \right) \right\} - \sum_{n=1}^{\infty} \frac{(\mu)_n}{n!} \psi(n + 1) \left( 1 - e^{-2z_0} \right)^n \right].
\]

(16)

Note that the infinity term in Eq.(15) disappears in Eq.(16) because of the exact cancellation. This exact cancellation also takes place in \( \tilde{I}_{1,\alpha}(\omega) \). After tedious calculation the final form of \( \tilde{I}_{\alpha}(\omega) \) reduces to

\[
\tilde{I}_{\alpha}(\omega) \tilde{z}^{-1}(\omega) = \frac{1}{3} A_{\alpha} \left[ \omega^2 + \left( \frac{\pi}{B_{\alpha}} \right)^2 \right] + \frac{A_{\alpha}}{2B_{\alpha}^2} \left\{ \frac{e^{2\mu z_0} - 1}{z_0} \psi''(\mu) + 2 \left( e^{2\mu z_0} + 1 \right) \psi'(\mu) \right\} - \left( \frac{4}{3} \frac{\omega^2}{z_0} + \frac{2\pi^2}{3} \right)
\]

\[
+ 2 \sum_{n=1}^{\infty} \frac{(\mu)_n}{n!} \left[ \psi(n + \mu) - \psi(\mu) \right] \psi(n + 1) \left( 1 - e^{-2z_0} \right)^n
\]

\[
- 2 \sum_{n=1}^{\infty} \frac{\psi(n + 1)}{n} \left( 1 - e^{-2z_0} \right)^n
\]

\[
- \sum_{n=2}^{\infty} \frac{(\mu)_n}{n!} \left( \psi''(n + \mu) + \psi'(\mu) \right) \psi(n + 1) \left( 1 - e^{-2z_0} \right)^n
\]

\[
+ 2 \sum_{n=2}^{\infty} \frac{\gamma + \psi(n)}{n} \psi(n + 1) \left( 1 - e^{-2z_0} \right)^n \left( \frac{1}{z_0} \right)
\]

where \( \gamma \) is an Euler’s constant. In order for \( \tilde{z}(\omega) \) to be an exact solution the right-handed side of Eq.(17) should be equal to \( 2\sqrt{2\pi}(\omega^2 + 2\alpha\omega + 1)/3 \). To extract an information on \( A_{\alpha} \),

6
and $C_{\alpha}$ we assume this equality. Repeating to take a $\omega \to 0$ limit and subsequently to differentiate the right-handed side of Eq.(17) three times, we can derive Eq.(6). Taking an inverse-Fourier transform to $\tilde{z}(\omega)$, we can derive Eq.(5).

Although our bounce solution has many nice features as discussed before, it is not an exact solution unfortunately except $\alpha = 0$. For $\alpha = 0$ case we can show analytically $A_0 = 2\sqrt{2}\pi$, $B_0 = \pi$ and $C_0 = 0$ by solving Eq.(6) in the $\mu \to 0$ limit. This means our solution exactly coincides with no-damping solution at $\alpha \to 0$ limit. However, for the nonzero $\alpha$ we can show that our bounce solution is an approximate one by using the zero mode of the linearized fluctuation equation as following.

Note that the CL model has a time-translational symmetry in spite of the presence of the non-local term. Although one can show this fact simply from the equation of motion (2), we would like to show it at the fluctuation level for a later use. Inserting $z(u) = z_{cl}(u) + \eta(u)$ into Eq.(2) one can construct easily the linearized fluctuation equation

\[-c_1\ddot{\eta} + c_2(1 - 3z_{cl})\eta + c_3\frac{2\alpha}{\pi} \int_{-\infty}^{\infty} du' \frac{\eta(u) - \eta(u')}{(u - u')^2} = \lambda \eta \] (18)

where $\lambda$ is an eigenvalue of the fluctuation equation and the constants $c_1$, $c_2$, and $c_3$ are introduced for convenience. If $c_1 = c_2 = 1$ and $c_3 = 0$, Eq.(18) is a fluctuation equation around the no-damping solution. In this case it is trivial to show that $dz_0(u)/du$ is a zero mode. If $c_1 = 0$ and $c_2 = c_3 = 1$, Eq.(18) corresponds to a fluctuation around the strong-damping solution. In this case also one can show directly that $dz_\infty(u)/du$ is a zero mode. This means that the CL model has a time-translational symmetry in spite of the presence of the dissipation.

Now, let us consider the full fluctuation equation, i.e. $c_1 = c_2 = c_3 = 1$. If our bounce solution (5) is an exact one, by same reason

\[\eta \equiv \frac{dz(u)}{du} = \sqrt{\frac{2}{\pi}} \frac{iA_{\alpha}}{8B_{\alpha}^3} \left[ \psi'' \left( \frac{C_{\alpha} + B_{\alpha} + iu}{2B_{\alpha}} \right) - \psi'' \left( \frac{C_{\alpha} + B_{\alpha} - iu}{2B_{\alpha}} \right) \right] \] (19)

should be a zero mode. Inserting Eq.(19) into the non-local term in Eq.(18) one can show

\[\frac{2\alpha}{\pi} \int_{-\infty}^{\infty} du' \frac{\eta(u) - \eta(u')}{(u - u')^2} = \sqrt{\frac{2}{\pi}} \frac{3i\alpha A_{\alpha}}{4B_{\alpha}^2} \left[ \zeta \left( 4, \frac{C_{\alpha} + B_{\alpha} - iu}{2B_{\alpha}} \right) - \zeta \left( 4, \frac{C_{\alpha} + B_{\alpha} + iu}{2B_{\alpha}} \right) \right] \] (20)
where $\zeta(p, q)$ is a Riemann Zeta function defined as $\zeta(p, q) = \sum_{k=0}^{\infty} 1/(q + k)^p$.

Using Eq.(20) the left-handed side of the fluctuation can be plotted numerically. When $\alpha$ is small, the left-handed side of the fluctuation equation is plotted in Fig. 3 in terms of $u$ for various $\alpha$, which indicates that $\eta$ is not exact zero mode although approximately it is. Fig. 3 also shows how $\eta$ can be a zero mode in the $\alpha \to 0$ limit. Fig. 4 is a plot of the left-handed side of Eq.(18) when $\alpha$ is large. This figure also shows how our solution goes to the strong-damping solution in the $\alpha \to \infty$ limit. But we should comment that our bounce solution does not seem to exactly coincide with the strong-damping solution in the $\alpha \to \infty$ limit. That is why there is a small difference in Fig.1 between the classical action for our solution and its lower bound at the large $\alpha$ regime.

In this letter we have derived the analytic bounce solution in the CL model. The classical action for our solution lies between the upper and lower bounds in the full range of $\alpha$. Although it has many nice features, we have shown that it is not an exact solution except $\alpha = 0$ case. However, using this approximate solution, one may be able to compute the prefactor approximately. We guess this prefactor may be important factor to reconcile the discrepancy between the semi-classical method and the canonical method. We hope to visit this issue in the near future. Our analytic bounce solution might be extended to the non-Ohmic dissipation case. The explicit result will be discussed elsewhere in detail.

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REFERENCES


FIGURES

FIG. 1. Plot of $\alpha$-dependence of classical actions for our bounce solution (5) (red line), no-damping solution (3) (blue line) and strong-damping solution (4) (green line). Two black lines are upper and lower bounds of the classical action which were derived by CL in Ref.[2] without an explicit bounce solution. This figure indicates that the classical action for our solution lies between the upper and lower limits in the full range of $\alpha$.

FIG. 2. Plot of $\alpha$-dependence of the bounce peak point. As CL have argued in Ref.[2], the bounce peak point increases from 1 to $4/3$ with increase of $\alpha$.

FIG. 3. The plot of the left-handed side of Eq.(18) for various small $\alpha$. The difference from zero indicates that our solution is not an exact one. The increase of the peak point and the wide-spreading shape of peak with increase of $\alpha$ denotes that our bounce solution (5) goes away from the exact one when $\alpha$ becomes larger in the small $\alpha$ regime.

FIG. 4. Plot of the left-handed side of Eq.(18) for various large $\alpha$. The decrease of the peak point with increase of $\alpha$ means that our solution approaches to the exact one when $\alpha$ becomes larger in the large $\alpha$ regime.
Fig. 1
Fig. 2

$Z(0)$ vs $\alpha$

$Z(0)$ vs $\alpha$

$4/3$ vs $\alpha$

$4/3$ vs $\alpha$
Fig. 3
Fig. 4
Fig. 1
Fig. 2
Fig. 3

\( \alpha = 0.5 \)
Fig. 4

$\alpha = 1$

$u$