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BREAKDOWN OF NORMAL THRESHOLD DISPERSION RELATION
FOR THE VACUUM POLARIZATION

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A B S T R A C T

Stating a concrete example, it is shown that the vacuum polarization develops anomalous singularities. The analytic structure of the vacuum polarization in these circumstances is discussed in some detail. It appears that its large q^2 behaviour is not linked anymore to the asymptotic form of the e^+e^- total cross-section. Since it is the large q^2 behaviour of the vacuum polarization which solely can be calculated in asymptotically free gauge theories (giving in leading order the naive quark model result), this is a warning to using the naive quark model result for the e^+e^- total cross-section as well in fitting the data.

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In order to explain the unexpectedly large e^+e^- annihilation cross-section, essentially two pictures have emerged.

i) - It appears that the inclusive e^+e^- cross-section is at least one order of magnitude larger, for pions, than what one expects via (naive) continuation from the electroproduction analogue ¹⁾. This hints at the fact that the deep Euclidean and time-like regions are disconnected by some anomalous singularities ²⁾ which, for their part, almost completely govern the annihilation process. The quantitative analysis of the continuation procedure ¹⁾ as well as the direct investigation of exclusive hadronic channels ³⁾ (taking proper account of anomalous singularities) indeed give strong support to this observation.

ii) - Disregarding this observation, several authors have converted lowest order perturbation calculations ⁴⁾ of asymptotically free quark theories via a normal threshold dispersion relation for the vacuum polarization into direct information about the e^+e^- total cross-section ⁵⁾. The assumption of a normal threshold dispersion relation is vital here since quark perturbation theory is clearly not justifiable in the physical region. In leading order this approach agrees with the naive quark model result. The relatively large e^+e^- cross-section then usually is described at the expense of introducing higher degrees of freedom (6×3 quarks at SPEAR II energies).

Both pictures are hard to reconcile. It is expected that the anomalous singularities appearing in the inclusive and exclusive e^+e^- cross-section also reproduce in the vacuum polarization so that the normal threshold dispersion relation breaks down. In these circumstances the naive quark model result should be regarded with great reservations.

It is the purpose of this note to state a concrete case showing that the vacuum polarization in fact develops anomalous singularities and, for the second time, giving an idea of how the analytic structure of the vacuum polarization looks like in general.

Let us consider the vacuum polarization diagram shown in Fig. 1a. The (final state) interaction between particles 1 and 2 (labelled by their masses m_1 and m_2) is assumed to be purely elastic which, for our aims, implies no restriction. The discontinuity of diagram 1a across cut (I) can then be traced back to the form factor-like diagram shown in Fig. 1b. Cut (II) does not give rise to anomalous singularities ^{*)} and will be ignored here.

*) For the photon coupling point-like.

Diagram 1b has been extensively studied in Ref. 3) (to be referred to as KSS) which we shall heavily lean on. To begin with, we assume $q^2 < m_R^2 + m_1^2$ and $m_3^2 < m_R^2 + m_2^2$ which guarantees that diagram 1b has no anomalous singularities *) in the variable $s = (p_1 + p_2)^2$ (p_1 and p_2 being the momenta associated with particles 1 and 2, respectively). For simplicity we set $m_1 = m_2 = m$ and only retain s waves **) in channel (12). Furthermore, we restrict ourselves to scalar photons and all particles being scalar. Following the notation of KSS, the imaginary part of the vacuum polarization diagram 1a can then be written

$$\text{Im } \Pi(q^2) = \frac{g_{R23}^2 m_3^2}{2(16\pi)^2 q^2} \int_{s_-}^{s_+} ds \left(\frac{(s - 4m^2)(s - s_-)(s - s_+)}{s} \right)^{\frac{1}{2}} A^0(q^2 + i\epsilon, s + i\epsilon) A^{0*}(q^2 + i\epsilon, s + i\epsilon) \quad (1)$$

where g_{R23} is the coupling constant of particles R, 2 and 3 and $s_- = (\sqrt{q^2} - m_3)^2$.

In principle, we now can write down a dispersion relation for $\Pi(q^2)$:

$$\Pi(q^2) = \frac{q^2}{\pi} \int_{(2m+m_3)^2}^{\infty} dq'^2 \frac{\text{Im } \Pi(q'^2)}{q'^2(q'^2 - q^2)} \quad (2)$$

with $\text{Im } \Pi(q^2)$ given by Eq. (1) provided we give m_R^2 a q'^2 dependence such that $q'^2 < 2m_R^2 + 2m^2 - m_3^2$ (i.e., absence of anomalous singularities). To obtain the physically interesting case of m_R^2 having some fixed value, Eqs. (1) and (2) have to be continued into the region where $A^0(q^2, s)$ develops anomalous singularities.

Before doing that, let us briefly recapitulate the analytic structure of $A^0(q^2, s)$. Below the critical point $q^2 = 2m_R^2 + 2m^2 - m_3^2$ we have, according to KSS,

*) Actually, it would be sufficient to assume $q^2 + m_3^2 < 2m_R^2 + m_1^2 + m_2^2$.

**) One could argue that this might lead to the wrong answer since individual anomalous singularities could well cancel doing the partial wave summation. However, as the individual phase shifts can be chosen quite independently, this is not to be expected.

$$\begin{aligned}
 -\frac{1}{2} A^0(q^2 + i\epsilon, s) = & \frac{1}{\mathcal{J}(s)} \int_{-\infty}^0 ds' \frac{1}{s' - s} \frac{\sqrt{s'} \mathcal{D}(s')}{\sqrt{(s' - 4m^2)(s' - s_-)(s' - s_+)}} \\
 & + \frac{1}{\mathcal{J}(s)} \int_{s_2}^{s_1} ds' \frac{1}{s' - s} \frac{\sqrt{s'} \mathcal{D}(s')}{\sqrt{(s' - 4m^2)(s' - s_-)(s' - s_+)}}
 \end{aligned} \tag{3}$$

with s_1, s_2 and the intermediate path of integration ^{*)} as shown in Fig. 2a. Beyond the critical point, s_1 migrates around $4m^2$ as indicated in Fig. 2b, giving

$$\begin{aligned}
 -\frac{1}{2} A^0(q^2 + i\epsilon, s) = & \frac{1}{\mathcal{J}(s)} \int_{-\infty}^0 ds' \frac{1}{s' - s} \frac{\sqrt{s'} \mathcal{D}(s')}{\sqrt{(s' - 4m^2)(s' - s_-)(s' - s_+)}} \\
 & + \frac{1}{\mathcal{J}(s)} \int_{s_2}^{4m^2} ds' \frac{1}{s' - s} \frac{\sqrt{s'} \mathcal{D}(s')}{\sqrt{(s' - 4m^2)(s' - s_-)(s' - s_+)}} + \frac{1}{\mathcal{J}(s)} \int_{4m^2}^{s_1} ds' \frac{1}{s' - s} \frac{\sqrt{s'} \mathcal{D}(s') e^{2i\delta(s')}}{\sqrt{(s' - 4m^2)(s' - s_-)(s' - s_+)}}
 \end{aligned} \tag{4}$$

where the path of integration follows the dashed curve in Fig. 2b.

The analytic continuation of Eqs. (1) and (2) is now easily done. Let us first concentrate on Eq. (1). According to Eq. (4), A^0 develops an anomalous cut in s moving through the unitarity cut into the lower half plane. The analogue situation occurs for A^{0*} (branch point s_1^*) as can be inferred from Fig. 3a. In order not to cross any branch cuts, this means that the integral in Eq. (1) (which was along the real axis) has to be deformed as indicated in Fig. 3a, passing around the branch points ^{**)} s_1 and s_1^* . The

^{*)} The contour around s_- is due to the fact that s_- , being a branch point of the integrand of Eq. (3), crosses the path of integration connecting s_1 and s_2 .

^{**)} Note that the integrand of Eq. (1) is analytic in s in the relevant region so that we are permitted to deform the contour.

discontinuity of the integrand of Eq. (1) across the anomalous cuts can be read off from Eq. (4), so that in the anomalous case ($q^2 > 2m_R^2 + 2m^2 - m_3^2$), Eq. (1) is replaced by

$$\begin{aligned} \text{Im } \Pi(q^2) = & \frac{g_{\pi\pi 3}^2 m_3^2}{2(16\pi)^2 q^2} \int_{4m^2}^{s_-} ds \left(\frac{(s-4m^2)(s-s_-)(s-s_+)}{s} \right)^{\frac{1}{2}} A^0(q^2+i\epsilon, s+i\epsilon) A^{0*}(q^2+i\epsilon, s+i\epsilon) \\ & - \frac{g_{\pi\pi 3}^2 m_3^2}{64\pi q^2} \text{Re } i \int_{4m^2}^{s_1^*} ds e^{-2i\delta\alpha} A^0(q^2+i\epsilon, s) \end{aligned} \quad (5)$$

We recall that

$$\begin{aligned} s_1 = & \frac{1}{2m_R^2} \left\{ -m_R^4 + m_R^2 (q^2 + m_3^2 + 2m^2) - (q^2 - m^2)(m_3^2 - m^2) \right. \\ & \left. + \sqrt{((m_R+m)^2 - q^2)(q^2 - (m_R-m)^2)((m_R+m)^2 - m_3^2)(m_3^2 - (m_R-m)^2)} \right\} \end{aligned} \quad (6)$$

Let us next consider Eq. (2). Here, two cases may be distinguished. If $(2m+m_2)^2 < 2m_R^2 + 2m^2 - m_3^2$, corresponding to m_R being unstable with respect to m_3 and m (e.g., $m_R = m_\rho$, $m_3 = m = m_\pi$), A^0 has no anomalous singularities anymore at q^2 close (or equal) to threshold ($q^2 = (2m+m_3)^2$). In this case the vacuum polarization takes on the form (2) with Eq. (1) replaced ^{*} by Eq. (5). The first integral in Eq. (5) (and only the first) is to be identified with the total cross-section, so that we may write

^{*}) It is self-understood that once s_1 migrates back into the upper half plane, the second integral in Eq. (5) is to be set to zero.

$$\begin{aligned} \pi(q^2) = & \frac{q^2}{32\pi^2\alpha^2} \int_{(2m+m_3)^2}^{\infty} d q'^2 \frac{\sigma_{tot}(q'^2)}{q'^2(q'^2 - q^2)} \\ & - \frac{g_{\pi\pi 3}^2 m_3^2}{64\pi} \frac{q^2}{\pi} \int_{(2m+m_3)^2}^{\infty} d q'^2 \frac{1}{(q'^2)^2(q'^2 - q^2)} \operatorname{Re} i \int_{4m^2}^{s_1^*} ds e^{-2i\delta(s)} A^0(q'^2 + i\epsilon, s) \end{aligned} \quad (7)$$

This does not conflict with unitarity since, in Eq. (7), we always have $s_1 < 4m^2$ or s_1 being complex and lying in the lower half plane. Note that for $q'^2 \rightarrow \infty$ also $|s_1| \rightarrow \infty$.

If, on the other hand, $(2m+m_3)^2 > 2m_R^2 + 2m^2 - m_3^2$, corresponding to m_R being stable with respect to m_3 and m , anomalous singularities do contribute down to threshold. In this case, the integral in Eq. (2) has to be deformed such that $s_- (= (\sqrt{q'^2} - m_3)^2)$ never crosses the anomalous cut inherent in A^0 appearing under the second integral in Eq. (5) which constitutes a branch cut in s_- along this cut. In order to economise the discussion of this situation ^{*)} let us consider the case [where Eq. (5) has been continued to] $m_R = m = m_\pi$ and $m_3 = m_\rho$ which has been amply studied in KSS. It appears that the contour of integration in Eq. (2) is to be deformed along the real axis such as to surround $q'^2 = 0$ where $s_1 = s_-$. As q'^2 moves along, s_1 and s_- describe the paths as shown in Fig. 3b. The result can be read off from Eq. (5) and Fig. 3b, giving

$$\pi(q^2) = \frac{q^2}{32\pi^2\alpha^2} \int_{(2m+m_3)^2}^{\infty} d q'^2 \frac{\sigma_{tot}(q'^2)}{q'^2(q'^2 - q^2)}$$

^{*)} As far as discussing the structure of the various branch cuts is concerned.

$$\begin{aligned}
 & - \frac{g_{223}^2 m_3^2}{64\pi} \frac{q^2}{\pi} \int_0^\infty dq'^2 \frac{1}{(2m+m_3)^2 (q'^2)^2 (q'^2 - q^2)} \operatorname{Re} i \int_{4m^2}^{s_1^*} ds e^{-2i\delta(s)} A^0(q'^2, i\epsilon, s) \\
 & - \frac{g_{223}^2 m_3^2}{16\pi} \frac{q^2}{\pi} \int_0^{(2m+m_3)^2} dq'^2 \frac{1}{(q'^2)^2 (q'^2 - q^2)} \operatorname{Re} i \int_{4m^2}^{s_1^*} ds e^{-2i\delta(s)} D^{-1}(s) \\
 & \times \int_{s_-}^{s_1} ds' \frac{1}{s' - s} \frac{\sqrt{s'} D(s') e^{2i\delta(s')}}{\sqrt{(s' - 4m^2)(s' - s_-)(s' - s_+)}} \quad (8)
 \end{aligned}$$

It should be noted that Eq. (8) is not singular at $q^2 = 0$ since $\operatorname{Re} i \int_{4m^2}^{s_1^*} \dots$ vanishes quadratically ^{*)} in q'^2 as $q'^2 \rightarrow 0$.

Equation (8) is not bound to the special mass configuration considered in its derivation. It can be shown, either by continuation or directly, that Eq. (8) is valid quite generally. In particular, it matches with Eq. (7) as the discontinuity of the cut [Eq. (8)] from $q^2 = 0$ to $q^2 = (2m+m_3)^2$ vanishes in the limit $(2m+m_3)^2 < 2m_R^2 + 2m^2 - m_3^2$.

The message of our investigations is that the vacuum polarization develops for a wide range of internal masses, including the equal mass case, an anomalous cut from $q^2 = 0$ to $q^2 = (2m+m_3)^2$. Furthermore, it says that the discontinuity of $\Pi(q^2)$ for physical q^2 consists, in all cases, of a normal threshold part, to be associated with the total cross-section, and an anomalous threshold contribution having no physical counterpart. Without knowing (or being able to estimate) the anomalous threshold part, it thus appears impossible to draw any conclusions from the asymptotic behaviour of $\Pi(q^2)$ for the total cross-section, contrary to the customary belief. We hope to come back elsewhere to estimating the effect of anomalous thresholds on the vacuum polarization.

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^{*)} Firstly, $s_- \rightarrow s_1$ as $q'^2 \rightarrow 0$. Secondly, the integrand becomes purely real.

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FIGURE CAPTIONS

- Figure 1 (a) Vacuum polarization diagram.
 (b) Form factor-like diagram resulting from cut (I) of (a).
- Figure 2 Position of branch points s_1 and s_2 ,
 (a) below, and
 (b) beyond the critical point.
- Figure 3 (a) Displaced contour of s integration in the case of s_1 (s_1^*)
 lying in the lower (upper) half plane.
 (b) Path of s_- surrounding s_1 . The numbers mark the corresponding values of q'^2 :
 ① $q'^2 = (2m+m_3)^2$;
 ② $q'^2 = m_3^2$;
 ③ $q'^2 = 4m^2$, and
 ④ $q'^2 = 0$.
 The analogue picture occurs for s_1^* .

(a)

(b)

FIG. 1

(a)

(b)

FIG. 2

FIG.3