New realizations of observables in dynamical systems with second class constraints.

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Abstract

In the Dirac bracket approach to dynamical systems with second class constraints observables are represented by elements of a quotient Dirac bracket algebra. We describe families of new realizations of this algebra through quotients of the original Poisson algebra. Explicit expressions for generators and brackets of the algebras under consideration are found.
1 Introduction

In a dynamical system with first class constraints physical functions are elements of a
Poisson bracket algebra of first class functions (see e.g. [1]). Observables are classes of the
physical functions modulo the functions vanishing on constraint surface.

In the Dirac bracket approach to a system with second class constraints [2] the original
Poisson bracket is replaced by the Dirac one and constraints become first class. In this case
all the functions on phase space are first class and observables are elements of the Dirac
bracket algebra of all the functions modulo the functions vanishing on constraint surface.

The latter quotient algebra can be realized as a Poisson bracket algebra of the functions
on constraint surface [3]. Another useful realization can be obtained by using the Abelian
conversion of second class constraints [4]. The algebra of observables is also realized as a
quotient of the original Poisson algebra of first class functions [5].

The aim of this article is to present the new Poisson algebras with respect to the
original bracket which are isomorphic to the algebra of observables in a dynamical system
with second class constraints.

The construction uses a family of nested subalgebras of the original Poisson algebra
of first class functions and their ideals which are generated by the functions vanishing on
constraint surface. Existence of such subalgebras imposes some restrictions on possible
constraints. Solving the defining equations we find explicit expressions for generators of
the algebras under consideration. This enables us to construct families of new isomorphic
images of the algebra of observables. The new algebras are Poisson ones with respect
to the original bracket. Using these realizations of a constrained system one can avoid
quantization of the Dirac bracket.

The paper is organized as follows. In Section 2 we review a description of a system
with second class constraints through the original Poisson bracket. In Section 3 we find
explicit expressions for the functions on phase space which serve as generators of new
Poisson algebras. These algebras are constructed and studied in Section 4. In Section 5
we describe new realizations of the algebra of observables in a system with second class
constraints.
2 A realization of observables through the original Poisson bracket

Let $M$ be a phase space with the phase variables $\eta_n, n = 1...2N$, and the Poisson bracket $[\eta_m, \eta_n] = \omega_{mn}(\eta)$. Let $H(\eta)$ be the original hamiltonian and $\varphi_j(\eta), j = 1...2J$, the second class constraints $\det [\varphi_j, \varphi_k]_{\varphi=0} \neq 0$. We shall assume [2] that all the quantities vanishing on constraint surface are linear functions of $(\varphi_i)$.

Hamilton equations of the system under consideration are read

$$\frac{d}{dt}\eta_n = [\eta_n, H_T], \quad \varphi_j = 0.$$  \hspace{1cm} (1)

Here

$$H_T = H + \lambda_j \varphi_j$$  \hspace{1cm} (2)

and $\lambda_j = \lambda_j(\eta)$ are defined by the equations

$$[H_T, \varphi_j]_{\varphi=0} = 0.$$  \hspace{1cm} (3)

From (1) it follows

$$\frac{d}{dt}f = [f, H_T], \quad \varphi_j = 0$$  \hspace{1cm} (4)

for all $f = f(\eta)$.

Using (3) one can write equations (4) as

$$\frac{d}{dt}f = [f, H_T]_D, \quad \varphi_j = 0.$$  \hspace{1cm} (5)

Here the Dirac bracket was introduced

$$[g, h]_D = [g, h] - [g, \varphi_j] c_{jk}[\varphi_k, h], \quad c_{jk}[\varphi_k, \varphi_l] = \delta_{jl}.$$  

The constraints $(\varphi_j)$ are first class with respect to the Dirac bracket: $[\varphi_j, \varphi_k]_D = 0$ and the physical functions are defined by the equations

$$[f, \varphi_j]_D|_{\varphi=0} = 0.$$

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which are satisfied identically. Let $A$ be the space of functions on $M$ and $\Phi \subseteq A$ be the subspace of the functions which vanish on constraint surface. Then the algebra of observables is the Dirac bracket algebra $A/\Phi$. Note that $A/\Phi$ is also an algebra with respect to the pointwise multiplication and hence $A/\Phi$ is a Poisson algebra.

Let $\{f\} \in A/\Phi$ be the coset represented by $f \in A$. Then, using (5) and (3) one can obtain the Hamilton equations for observables

$$\frac{d}{dt}\{f\} = ([\{f\}, \{H\}])_D.$$

(6)

In a recent article [5] a new approach to quantization of the system (1) was proposed. Let $\Upsilon$ be the algebra of the functions which are quadratic in $\varphi_j$ and let $\Omega$ be the algebra of first class functions:

$$\Omega = \{f \in A | [f, \varphi_j]|_{\varphi=0} = 0 \}.$$  

(7)

For $u \in \Upsilon$ we have $u|_{\varphi=0} = 0$. The set $\Upsilon$ includes the element $\varphi_j \varphi_j$ and hence from the equations $u = 0$, $u \in \Upsilon$, it follows $\varphi_j = 0$. Thus, the constraints ($\varphi_j$) and $\Upsilon$ are equivalent and we can replace equations (1) by

$$\frac{d}{dt}f = [f, H_T], \quad u_{ij}\varphi_i\varphi_j = 0.$$  

(8)

Here $u_{ij} = u_{ij}(\eta)$ are arbitrary functions. In contrast with the original ones the new constraints $\Upsilon$ are first class.

In this approach the algebra of physical functions consists of all the functions which satisfy the equations

$$[f, u] \in \Upsilon$$

(9)

for all $u \in \Upsilon$. One can show that these equations are equivalent to the definition of first class functions (7) in the original second class system. Due to equations (9) and (7) the algebra of observables is the Poisson algebra $\Omega/\Upsilon$.

Let $\{f\}^\bullet \in \Omega/\Upsilon$ be the coset represented by $f \in \Omega$. Then the equations for observables read

$$\frac{d}{dt}\{f\}^\bullet = ([\{f\}, \{H_T\}]^\bullet).$$

(10)

The present approach and the Dirac bracket one are related by the isomorphism of the algebras of observables $\Omega/\Upsilon$ and $A/\Phi$ [5]:

$$T(\{g\}^\bullet) = \{g\}.$$
Below we shall obtain new realizations of the algebra $A/\Phi$ through quotients of the original Poisson algebra.

3 Generators of $\Omega_{s+1}$

Let $\Omega_{s+1}$, $s \in N$, be the space of the functions on $M$ which are defined by the equations

$$[\varphi_j, \tilde{g}] \in \Upsilon_s. \quad (11)$$

Here

$$\Upsilon_s = \{ u \in A | u = u_{j_1 \ldots j_s} \varphi_{j_1} \ldots \varphi_{j_s}, u_{j_1 \ldots j_s}(\eta) \in A \}.$$

It is seen that $\Upsilon_{s+1} \subset \Omega_{s+1} \subset \Omega_s$, $\Omega_2 = \Omega$, $\Upsilon_{s+1} \subset \Upsilon_s$, $\Upsilon_2 = \Upsilon$ and $\Upsilon_1 = \Phi$. We shall denote $\Upsilon_0 = \Omega_1 = A$.

To describe elements of $\Omega_{s+1}$ explicitly let us consider equations (11) with the initial condition

$$\tilde{g}(\eta) \in \{ g(\eta) \} \quad (12)$$

for some $g \in A$. A solution to these equations can be represented in the form

$$\tilde{g} = g + \sum_{r=1}^{s} \frac{1}{r!} \nu_{i_1 \ldots i_r}(\eta) \varphi_{i_1} \ldots \varphi_{i_r} + \nu_{i_1 \ldots i_{s+1}}(\eta) \varphi_{i_1} \ldots \varphi_{i_{s+1}}. \quad (13)$$

Note that the last term of (13) satisfies (11) for arbitrary $\nu_{i_1 \ldots i_{s+1}}$.

We shall assume that $\nu_{i_1 \ldots i_r}, r = 1 \ldots s$, is symmetric:

$$\nu_{i_1 \ldots i_k \ldots i_r}, \nu_{i_1 \ldots i_k \ldots i_r} \in \Upsilon_{p+1-r}, \quad p \geq s. \quad (14)$$

Substituting (13) into (11) and using (14) we get

$$[\varphi_j, g] + \nu_{i_1}, [\varphi_j, \varphi_{i_1}] + \sum_{r=2}^{s} \frac{1}{(r-1)!} \left( [\varphi_j, \nu_{i_1 \ldots i_{r-1}}] + \nu_{i_1 \ldots i_r}[\varphi_j, \varphi_{i_r}] \right) \varphi_{i_1} \ldots \varphi_{i_{r-1}} \in \Upsilon_s.$$
It is easy to see that a solution to these equations is
\[ \nu_{i_1 \ldots i_r} = (-1)^r D_{i_1} \ldots D_{i_r} \bar{g}, \quad r = 1 \ldots s. \] (15)

Here \( D_i = c_{ij}[\varphi_j, \cdot] \).

One can check that \( D_i \) satisfy the commutator relations
\[ D_i D_j - D_j D_i = [c_{ij}, \cdot]_{D} \] (16)
and for \( u \in \Upsilon_r \)
\[ D_i u \in \Upsilon_{r-1}. \] (17)

Now let us consider equations (14). It is sufficient to find a solution to these equations for \( a = k + 1, b = k, k = 1 \ldots r - 1 \). Substituting (15) into (14) and using (16) we have
\[ D_{i_r} \ldots D_{i_{k+2}} [c_{i_{k+1}k}, D_{i_{k-1}} \ldots D_{i_1} \bar{g}]_{D} \in \Upsilon_{p+1-r}, \quad k = 1 \ldots r - 1. \] (18)

A solution to these equations is given by
\[ c_{ij} = \psi_{ij}(\varphi) + v_{ij}, \quad v_{ij}(\eta) \in \Upsilon_{p-1}. \] (19)

Here \( \psi_{ij} \) is a function of the constraints \( (\varphi_j) \) only.

To check that \( c_{ij} \) satisfy equations (18) observe that for \( f \in A \)
\[ [c_{ij}, f]_{D} \in \Upsilon_{p-1} \]
and due to (17)
\[ D_{i_r} \ldots D_{i_{k+2}} [c_{i_{k+1}k}, D_{i_{k-1}} \ldots D_{i_1} \bar{g}]_{D} \in \Upsilon_{p+k-r} \subset \Upsilon_{p+1-r} \]
for all \( k = 1 \ldots r - 1 \). Thus for \( c_{ij} \) (19) expressions (13, 15) give us a solution to equations (11) with the initial condition (12).

Let now \( \tilde{g}' \) be another solution to equations (11) with the same initial condition \( \tilde{g}' \in \{g\} \). Then \( \sigma = \tilde{g} - \tilde{g}' \) is a solution to (11)
\[ [\varphi_j, \sigma] \in \Upsilon_{s} \] (20)
and \( \sigma = \sigma_i \varphi_i \) for some \( \sigma_i = \sigma_i(\eta) \).
From (20) it follows
\[ [\varphi_{j_1}, \ldots, [\varphi_{j_{m-1}}, [\varphi_j, \sigma]]] \in \Upsilon_{s-m+1}. \quad (21) \]

Assume that
\[ \sigma = \sigma_{i_1 \ldots i_m}(\eta)\varphi_{i_1} \ldots \varphi_{i_m}. \quad (22) \]

Substituting (22) into (21) for \( m \leq s \) we get \( \sigma_{i_1 \ldots i_m} |_{\varphi=0} = 0 \) and hence \( \sigma = \sigma_{i_1 \ldots i_{m+1}}(\eta)\varphi_{i_1} \ldots \varphi_{i_{m+1}}. \)
For \( m = s \)
\[ \sigma = \sigma_{i_1 \ldots i_{s+1}}(\eta)\varphi_{i_1} \ldots \varphi_{i_{s+1}}. \quad (23) \]

We have proved the proposition:

**PROPOSITION 3.1.** For \( c_{ij} \) (19) and \( g \in A \) the set \( \{g\} \cap \Omega_{s+1}, \quad s = 1 \ldots p, \) consists of all the expressions
\[ \hat{g} = g + \sum_{r=1}^{s} \frac{(-1)^r}{r!} (D_{i_r} \ldots D_{i_1} g) \varphi_{i_1} \ldots \varphi_{i_r} + \nu_{i_1 \ldots i_{s+1}} \varphi_{i_1} \ldots \varphi_{i_{s+1}}, \quad (24) \]

where \( \nu_{i_1 \ldots i_{s+1}}(\eta) \) are arbitrary functions.

In what follows we shall assume that \( c_{ij} \) is given by (19) and \( 1 \leq s \leq p. \)

It is convenient to introduce the notation
\[ L_s(g) = g + \sum_{r=1}^{s} \frac{(-1)^r}{r!} (D_{i_r} \ldots D_{i_1} g) \varphi_{i_1} \ldots \varphi_{i_r}. \]

The hamiltonian in \( \Omega_{s+1} \) is
\[ \tilde{H} = L_s(H) + u, \quad u \in \Upsilon_{s+1} \quad (25) \]

It can be represented in the form (2), satisfies equation (3) and hence belongs to the family of admissible hamiltonians.
4 Algebraic properties of $\Omega_{s+1}$

PROPOSITION 4.1. $\Omega_{s+1}$ is an algebra and $\Upsilon_{s+1}$ is an ideal of $\Omega_{s+1}$ with respect to the original Poisson bracket, Dirac bracket and pointwise multiplication.

The proof is straightforward.

Due to this proposition $\Omega_{s+1}, \Upsilon_{s+1}$ and $\Omega_{s+1}/\Upsilon_{s+1}$ are Poisson algebras with respect to $[\cdot,\cdot]_D$ as well as $[\cdot,\cdot]$.

Let $\tilde{g}_a = L_s(g_a) + u_a$, $u_a \in \Upsilon_{s+1}$, (26)

$a = 1, 2$, be some elements of $\Omega_{s+1}$ and let $\{\tilde{g}_a\}_s$ be some elements of $\Omega_{s+1}/\Upsilon_{s+1}$ be the coset represented by $\tilde{g}_a \in \Omega_{s+1}$.

PROPOSITION 4.2. For $\tilde{g}_1, \tilde{g}_2$ (26) one has

$$[\tilde{g}_1, \tilde{g}_2] = L_s([g_1, g_2]_D) + \tilde{u}_{12}, \quad [\tilde{g}_1, \tilde{g}_2]_D = L_s([g_1, g_2]_D) + \tilde{v}_{12},$$

$$\tilde{g}_1 \tilde{g}_2 = L_s(g_1 g_2) + \tilde{w}_{12}, \quad \tilde{u}_{12}, \tilde{v}_{12}, \tilde{w}_{12} \in \Upsilon_{s+1}.$$

Proof. One can check that $[\tilde{g}_1, \tilde{g}_2]$ satisfies equations (11) with the initial condition $[\tilde{g}_1, \tilde{g}_2] \in \{[g_1, g_2]_D\}$. Due to results of the previous section one has

$$[\tilde{g}_1, \tilde{g}_2] = L_s([g_1, g_2]_D) + \tilde{u}_{12}, \quad \tilde{u}_{12} \in \Upsilon_{s+1}.$$

Other statements of the proposition are proved by using similar arguments.

COROLLARY 4.3. The Dirac bracket algebra $\Omega_{s+1}/\Upsilon_{s+1}$ is isomorphic to the algebra $\Omega_{s+1}/\Upsilon_{s+1}$ with respect to the original Poisson bracket.

Proof. From equations (27) we have

$$[[\tilde{g}_1]_s, [\tilde{g}_2]_s] = [[\tilde{g}_1]_s, [\tilde{g}_2]_s]_D = \{L_s([g_1, g_2]_D)\}_s.$$

□
5 New realizations of observables

THEOREM 5.1.

(i) The Dirac bracket algebra $A/\Phi$ is isomorphic to the algebra $\Omega_{s+1}/\Upsilon_{s+1}$ with respect to the original Poisson bracket.

(ii) $A/\Phi$ and $\Omega_{s+1}/\Upsilon_{s+1}$ are isomorphic with respect to the pointwise multiplication.

Proof.

Let us define the linear function $T_s : \Omega_{s+1}/\Upsilon_{s+1} \rightarrow A/\Phi$

$$T_s(\{g\}_s) = \{g\}.$$ 

Each function $g' \in \{g\} \cap \Omega_{s+1}$ can be written in the form (24). Hence the inverse function $T^{-1}_s : A/\Phi \rightarrow \Omega_{s+1}/\Upsilon_{s+1}$ is given by

$$T^{-1}_s(\{g\}) = \{L_s(g)\}_s.$$ 

Computations show that $T_s$ is the homomorphism

$$T_s([\{g\}_s, \{f\}_s]) = [T_s(\{g\}_s), T_s(\{f\}_s)]_D$$

and hence $A/\Phi$ and $\Omega_{s+1}/\Upsilon_{s+1}$ are isomorphic.

To prove the second statement we observe that $T_s$ is the homomorphism with respect to the pointwise multiplication:

$$T_s(\{g\}_s \cdot \{f\}_s) = T_s(\{gf\}_s) = \{gf\} = \{g\} \cdot \{f\} = T_s(\{g\}_s) \cdot T_s(\{f\}_s).$$

\[\square\]

COROLLARY 5.1. $\Omega_{s+1}/\Upsilon_{s+1}, s = 1 \ldots p$, are isomorphic to each other as Poisson algebras.

The function $T_{s+k,s}$ which defines isomorphism between $\Omega_{s+k+1}/\Upsilon_{s+k+1}, k \geq 0$, and $\Omega_{s+1}/\Upsilon_{s+1}$ is given by

$$T_{s+k,s}(\{g\}_{s+k}) = \{g\}_s.$$ 

Theorem 5.1. gives us new realizations of the algebra of observables $A/\Phi$ through the original Poisson bracket. For a given system we have $p$ realizations, where $p$ is defined by the form of $c_{ij}$ (19).
For $p = 1$ the matrix $c_{ij}$ is arbitrary and there is only one realization $\Omega_2/\Upsilon_2 = \Omega/\Upsilon$.

For $p = 2$

$$c_{ij} = \psi_{ij}(\varphi) + v_{ijk}\varphi_k, \quad v_{ijk} = v_{ijk}(\eta).$$

In this case the observables can be realized by $\Omega/\Upsilon$ or $\Omega_3/\Upsilon_3$. When $c_{ij} = \psi_{ij}(\varphi)$ there is an infinite series of such realizations. Number $p$ can be used for classification of second class constraints.

According to (4) and (3) the Hamilton equation in $\Omega_{s+1}/\Upsilon_{s+1}$ is

$$\frac{d}{dt}\{f\}_s = \{\{f\}_s, \tilde{H}_T\}_s.$$

Here $\tilde{H}_T$ is given by (25).

## 6 Conclusion

In the present article we have obtained new realizations of observables in dynamical systems with second class constraints. The observables are realized as Poisson algebras with respect to the original bracket. We have found the restrictions which are imposed on constraints by construction of such algebras. The number of possible realizations of the observables for a given system can be used for classification of second class constraints. We have obtained explicit expressions for generators and brackets of all the algebras under consideration.

**Acknowledgements**

Author thanks I.V. Tyutin for reading the manuscript and helpful comments. The research was supported in part by RFBR grant 03-02-96521.

**References**


